Modeling and Optimization within Interacting Systems

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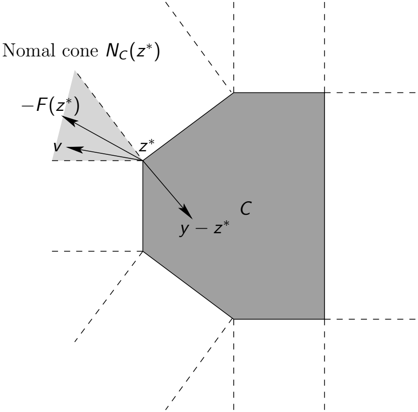
KKT conditions and Normal Cones

If $\mathcal{C} = \{z : g(z) \leq 0\}$, g convex, (with CQ)

$$x^* \text{ solves } \min_{x \in \mathcal{C}} f(x)$$

$$\iff 0 = \nabla f(x^*) + \nabla g(x^*)\lambda,$$
$$0 \leq -g(x^*) \perp \lambda \geq 0$$

$$\iff 0 \in \nabla f(x^*) + N_{\mathcal{C}}(x^*)$$



Many applications where F is not the derivative of some f

Variational Inequality (replace $\nabla f(z)$ with $F(z)$)

- $F : \mathbb{R}^n \to \mathbb{R}^n$
- Ideally: $\mathcal{C} \subseteq \mathbb{R}^n$ – constraint set; Often: $\mathcal{C} \subseteq \mathbb{R}^n$ – simple bounds

$$VI(F, \mathcal{C}) : \quad 0 \in F(z) + N_{\mathcal{C}}(z)$$

- model vi / F, g /;
  empinfo: vi F z g
- VI generalizes many problem classes
- Nonlinear Equations: $F(z) = 0$ set $\mathcal{C} \equiv \mathbb{R}^n$
- Convex optimization: $F(z) = \nabla f(z)$
- For MCP (rectangular VI), set $\mathcal{C} \equiv [l, u]^n$.
- Can now do MPEC (as opposed to MPCC)!
- Projection algorithms, robustness (evaluate F only at points in $\mathcal{C}$)

The PIES Model (Hogan) - Optimal Power Flow (OPF)

$$\begin{aligned} \min_{x} \; & c(x) && \text{cost} \\ \text{s.t. } \; & Ax \geq q && \text{balance} \\ & Bx = b, x \geq 0 && \text{technical constr} \end{aligned}$$

The PIES Model (Hogan) - Optimal Power Flow (OPF)

$$\min_x \; c(x) \qquad \text{cost}$$
$$\text{s.t. } Ax \geq d(\pi) \qquad \text{balance}$$
$$Bx = b, x \geq 0 \qquad \text{technical constr}$$

- $q = d(\pi)$: issue is that π is the multiplier on the "balance" constraint
- Such multipliers (LMP's) are critical to operation of market
- Can solve the problem iteratively or by writing down the KKT conditions of this QP, forming an LCP and exposing π to the model
- Existence, uniqueness, stability from variational analysis
- EMP does this automatically from the annotations

Reformulation details

$$
\begin{aligned}
0 &\leq Ax - d(\pi) && \perp \quad \mu \geq 0 \\
0 &= Bx - b && \perp \quad \lambda \\
0 &\leq \nabla c(x) - A^T\mu - B^T\lambda && \perp \quad x \geq 0
\end{aligned}
$$

- empinfo: dualvar π balance
- replaces $\mu \equiv \pi$

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\end{aligned}
$$

- empinfo: dualvar π balance
- replaces $\mu \equiv \pi$
- LCP/MCP is then solvable using PATH

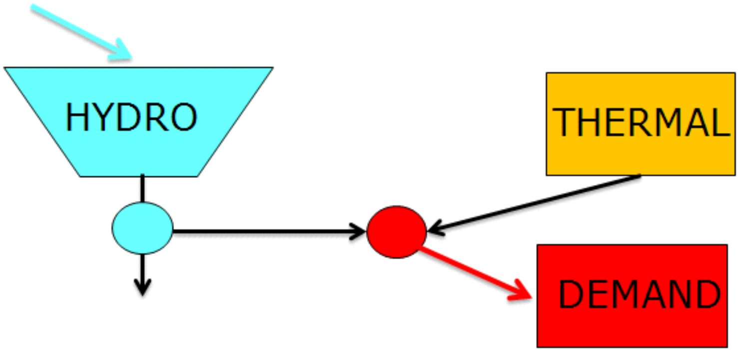
$$
z = \begin{bmatrix} \pi \\ \lambda \\ x \end{bmatrix}, \quad F(z) = \begin{bmatrix} & & A \\ & & B \\ -A^T & -B^T & \end{bmatrix} z + \begin{bmatrix} -d(\pi) \\ -b \\ \nabla c(x) \end{bmatrix}
$$

Other applications of complementarity

- Economics: Walrasian equilibrium (supply equals demand), taxes and tariffs, computable general equilibria, option pricing (electricity market), airline overbooking
- Transportation: Wardropian equilibrium (shortest paths), selfish routing, dynamic traffic assignment
- Applied mathematics: Free boundary problems
- Engineering: Optimal control (ELQP)
- Mechanics: Structure design, contact problems (with friction)
- Geology: Earthquake propogation

Good solvers exist for large scale instances of VI

Hydro-Thermal System (Philpott/F./Wets)



- Competing agents (consumers, or generators in energy market)
- Each agent minimizes objective independently (cost)
- Market prices are function of all agents activities

Simple electricity "system optimization" problem

$$\begin{aligned} \text{SO:} \quad \max_{d_k,u_i,v_j,x_i \geq 0} \quad & \sum_{k\in\mathcal{K}} W_k(d_k) - \sum_{j\in\mathcal{T}} C_j(v_j) + \sum_{i\in\mathcal{H}} V_i(x_i) \\ \text{s.t.} \quad & \sum_{i\in\mathcal{H}} U_i(u_i) + \sum_{j\in\mathcal{T}} v_j \geq \sum_{k\in\mathcal{K}} d_k, \\ & x_i = x_i^0 - u_i + h_i^1, \quad i \in \mathcal{H} \end{aligned}$$

- u_i water release of hydro reservoir $i \in \mathcal{H}$
- v_j thermal generation of plant $j \in \mathcal{T}$
- x_i water level in reservoir $i \in \mathcal{H}$
- prod fn U_i (strictly concave) converts water release to energy
- $C_j(v_j)$ denote the cost of generation by thermal plant
- $V_i(x_i)$ future value of terminating with storage x (assumed separable)
- $W_k(d_k)$ utility of consumption d_k

SO equivalent to CE (price takers)

$$\text{Consumers } k \in \mathcal{K} \text{ solve CP}(k): \max_{d_k \geq 0} \quad W_k(d_k) - \pi^T d_k$$

$$\text{Thermal plants } j \in \mathcal{T} \text{ solve TP}(j): \max_{v_j \geq 0} \quad \pi^T v_j - C_j(v_j)$$

$$\text{Hydro plants } i \in \mathcal{H} \text{ solve HP}(i): \max_{u_i, x_i \geq 0} \quad \pi^T U_i(u_i) + V_i(x_i)$$

$$\text{s.t.} \quad x_i = x_i^0 - u_i + h_i^1$$

Perfectly competitive (Walrasian) equilibrium is a MOPEC

$$\begin{aligned}
\text{CE:} \qquad d_k &\in \arg\max \text{CP}(k), && k \in \mathcal{K}, \\
v_j &\in \arg\max \text{TP}(j), && j \in \mathcal{T}, \\
u_i, x_i &\in \arg\max \text{HP}(i), && i \in \mathcal{H}, \\
0 \leq \pi \perp \sum_{i \in \mathcal{H}} U_i(u_i) &+ \sum_{j \in \mathcal{T}} v_j \geq \sum_{k \in \mathcal{K}} d_k.
\end{aligned}$$

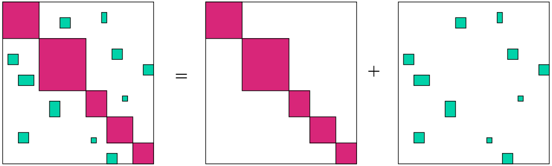
MOPEC

$$\min_{x_i} \theta_i(x_i, x_{-i}, \pi) \text{ s.t. } g_i(x_i, x_{-i}, \pi) \leq 0, \forall i$$

$$\pi \text{ solves } \mathrm{VI}(h(x,\cdot), C)$$

```
equilibrium
min theta(1) x(1) g(1)
...
min theta(m) x(m) g(m)
vi h pi cons
```

- (Generalized) Nash
- Reformulate optimization problem as first order conditions (complementarity)
- Use nonsmooth Newton methods to solve
- Solve overall problem using "individual optimizations"?



Perfect competition

$$\max_{x_i} \pi^T x_i - c_i(x_i) \qquad \text{profit}$$

$$\text{s.t. } B_i x_i = b_i, x_i \geq 0 \qquad \text{technical constr}$$

$$0 \leq \sum_i x_i - d(\pi) \perp \pi \geq 0$$

- When there are many agents, assume none can affect π by themselves
- Each agent is a price taker
- Two agents, $d(\pi) = 24 - \pi$, $c_1 = 3$, $c_2 = 2$
- KKT(1) + KKT(2) + Market Clearing gives Complementarity Problem
- $x_1 = 0$, $x_2 = 22$, $\pi = 2$

Cournot: two agents (duopoly)

$$\max_{x_i} p(\sum_j x_j)^T x_i - c_i(x_i) \qquad \text{profit}$$

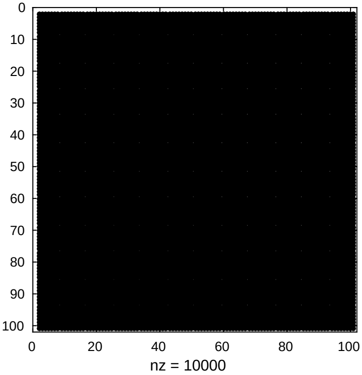
$$\text{s.t. } B_i x_i = b_i, x_i \geq 0 \qquad \text{technical constr}$$

- Cournot: assume each can affect p by choice of x_i
- Two agents, same data
- KKT(1) + KKT(2) gives Complementarity Problem
- $x_1 = 20/3$, $x_2 = 23/3$, $\pi = 29/3$
- Exercise of market power (some price takers, some Cournot)

Computational issue: PATH

- Cournot model: $|\mathcal{A}| = 5$
- Size $n = |\mathcal{A}| * N_a$

Size (n)	Time (secs)
2,500	48.431
5,000	570.214

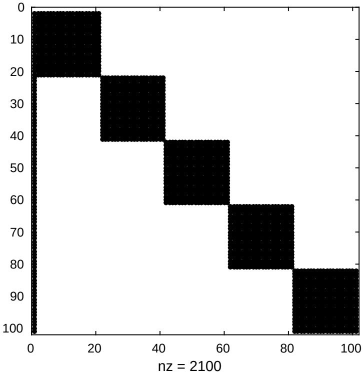


Jacobian nonzero pattern
$n = 100$

Computation: implicit functions

- Use implicit fn: $z(x) = \sum_j x_j$
- Generalization to $F(z, x) = 0$ (via adjoints)
- empinfo: implicit z F

Size (n)	Time (secs)
2,500	0.696
5,000	1.408
10,000	2.780
50,000	17.856
100,000	41.440



Jacobian nonzero pattern
$n = 100$

Other specializations and extensions

$$\min_{x_i} \theta_i(x_i, x_{-i}, \pi) \text{ s.t. } g_i(x_i, x_{-i}, \pi) \leq 0, \forall i$$

$$\pi \text{ solves } \mathrm{VI}(h(x, \cdot), C)$$

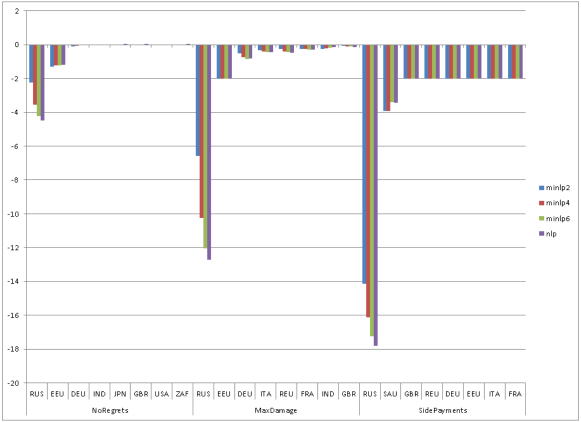
- NE: Nash equilibrium (no VI coupling constraints, $g_i(x_i)$ only)
- GNE: Generalized Nash Equilibrium (feasible sets of each players problem depends on other players variables)
- Shared constraints: some g_i's are known to all (many) players
- Force all shared constraints to have same dual variable (VI solution)
- Can use EMP to write all these problems, and convert to MCP form
- Research topic: exploit structure in solution

Optimal Sanctions (Boehringer/F./Rutherford)

- Sanctions can be modeled using similar formulations used for tariff calculations
- Model as a Nash equilibrium with players being countries (or a coalition of countries)
- Demonstrate the actual effects of different policy changes and the power of different economic instruments
- GTAP global production/trade database: 113 countries, 57 goods, 5 factors
- Coalition members strategically choose trade taxes to
  1. optimize their welfare (trade war) or
  2. *minimize* Russian welfare
- Russia chooses trade taxes to *maximize* Russian welfare in response
- Impose (QS) constraints that limit the number of instruments used for each country

Optimal Sanctions: Results

- Resulting Nash equilibrium with trade war, maximize damage, side payments - all have big impact on Russia
- Restricting instruments can change effects (these are the different colored bars)
- Collective (coalition) action significantly better



Same model can used to determine effects of Russian trade sanctions on Turkey

Stochastic: Agents have recourse?

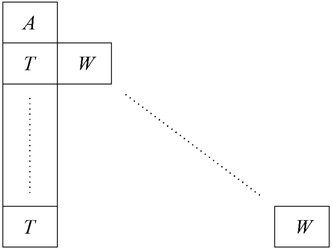
- Agents face uncertainties in reservoir inflows
- Two stage stochastic programming, x^1 is here-and-now decision, recourse decisions x^2 depend on realization of a random variable
- ρ is a risk measure (e.g. expectation, CVaR)

$$\text{SP: } \min \quad c(x^1) + \rho[q^T x^2]$$

$$\text{s.t.} \quad Ax^1 = b, \quad x^1 \geq 0,$$

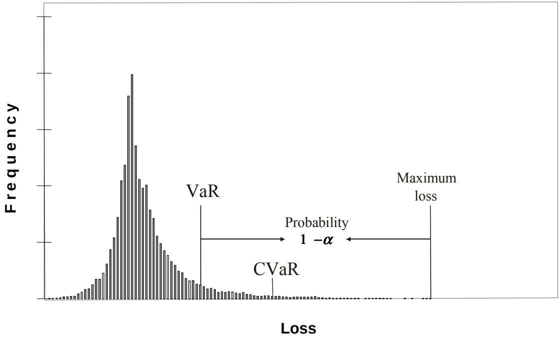
$$T(\omega)x^1 + W(\omega)x^2(\omega) \geq d(\omega),$$

$$x^2(\omega) \geq 0, \forall \omega \in \Omega.$$



Risk Measures

- Modern approach to modeling risk aversion uses concept of risk measures
- $\overline{CVaR}_\alpha$: mean of upper tail beyond α-quantile (e.g. $\alpha = 0.95$)



- mean-risk, mean deviations from quantiles, VaR, CVaR
- Much more in mathematical economics and finance literature
- Optimization approaches still valid, different objectives, varying convex/non-convex difficulty

Dual Representation of Risk Measures

- Dual representation (of coherent r.m.) in terms of risk sets

$$\rho(Z) = \sup_{\mu \in \mathcal{D}} \mathbb{E}_\mu[Z]$$

- If $\mathcal{D} = \{p\}$ then $\rho(Z) = \mathbb{E}[Z]$
- If $\mathcal{D}_{\alpha,p} = \{\lambda : 0 \leq \lambda_i \leq p_i/(1-\alpha), \sum_i \lambda_i = 1\}$, then

$$\rho(Z) = \overline{CVaR}_\alpha(Z)$$

- Special case of a Quadratic Support Function

$$\rho(y) = \sup_{u \in U} \langle u, By + b \rangle - \frac{1}{2} \langle u, Mu \rangle$$

- EMP allows any Quadratic Support Function to be defined and facilitates a model transformation to a tractable form for solution

Addition: compose equilibria with QS functions

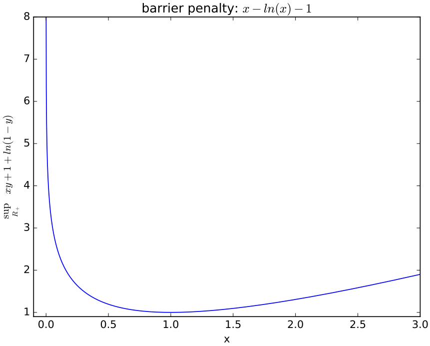
- Add soft penalties to objectives and/or within constraints:

$$\min_x \theta(x) + \rho_O(F(x))$$
$$\text{s.t. } \rho_C(g(x)) \le 0$$

```
QS g rhoC udef B M
...

QSF cvarup F rhoO theta p
```

- $batinclude QSprimal modname using emp min obj
- Allow modeler to compose QS functions automatically
- Can solve using MCP or primal reformulations
- More general conjugate functions also possible:



The link to MOPEC

$$\min_{x \in X} \theta(x) + \rho(F(x))$$

$$\rho(y) = \sup_{u \in U} \langle u, y \rangle - \frac{1}{2} \langle u, Mu \rangle$$

$$0 \in \partial\theta(x) + \nabla F(x)^T \partial\rho(F(x)) + N_X(x)$$

$$0 \in \partial\theta(x) + \nabla F(x)^T u + N_X(x)$$

$$0 \in -u + \partial\rho(F(x)) \iff 0 \in -F(x) + Mu + N_U(u)$$

This is a MOPEC, and we have multiple copies of this for each agent

$$\text{CP:} \quad \min_{d^1 \geq 0} \quad p^1 d^1 - W(d^1)$$

$$\text{TP:} \quad \min_{v^1 \geq 0} \quad C(v^1) - p^1 v^1$$

$$\text{HP:} \quad \min_{u^1, x^1 \geq 0} \quad -p^1 U(u^1)$$

$$\text{s.t.} \quad x^1 = x^0 - u^1 + h^1,$$

$$0 \leq p^1 \perp U(u^1) + v^1 \geq d^1$$

Two stage stochastic MOPEC (1,1,1)

$$\text{CP:} \quad \min_{d^1, d^2_\omega \geq 0} \quad p^1 d^1 - W(d^1) + \rho_C \left[p^2_\omega d^2_\omega - W(d^2_\omega) \right]$$

$$\text{TP:} \quad \min_{v^1, v^2_\omega \geq 0} \quad C(v^1) - p^1 v^1 + \rho_T \left[C(v^2_\omega) - p^2_\omega v^2(\omega) \right]$$

$$\text{HP:} \quad \min_{\substack{u^1, x^1 \geq 0 \\ u^2_\omega, x^2_\omega \geq 0}} \quad -p^1 U(u^1) + \rho_H \left[-p^2(\omega) U(u^2_\omega) - V(x^2_\omega) \right]$$

$$\text{s.t.} \quad x^1 = x^0 - u^1 + h^1,$$

$$x^2_\omega = x^1 - u^2_\omega + h^2_\omega$$

$$0 \leq p^1 \perp U(u^1) + v^1 \geq d^1$$

Two stage stochastic MOPEC (1,1,1)

$$\text{CP:} \quad \min_{d^1, d^2_\omega \geq 0} \quad p^1 d^1 - W(d^1) + \rho_C \left[p^2_\omega d^2_\omega - W(d^2_\omega) \right]$$

$$\text{TP:} \quad \min_{v^1, v^2_\omega \geq 0} \quad C(v^1) - p^1 v^1 + \rho_T \left[C(v^2_\omega) - p^2_\omega v^2(\omega) \right]$$

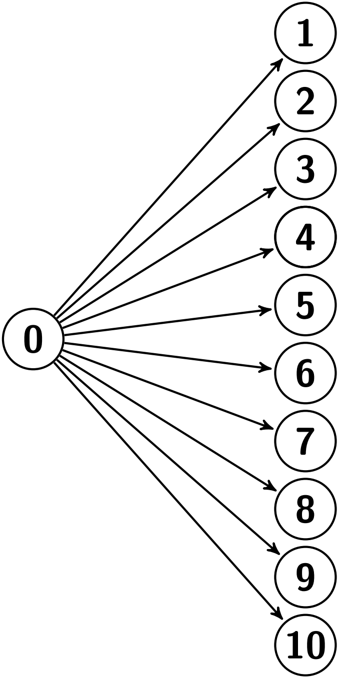
$$\text{HP:} \quad \min_{\substack{u^1, x^1 \geq 0 \\ u^2_\omega, x^2_\omega \geq 0}} \quad - p^1 U(u^1) + \rho_H \left[-p^2(\omega) U(u^2_\omega) - V(x^2_\omega) \right]$$

$$\text{s.t.} \quad x^1 = x^0 - u^1 + h^1,$$

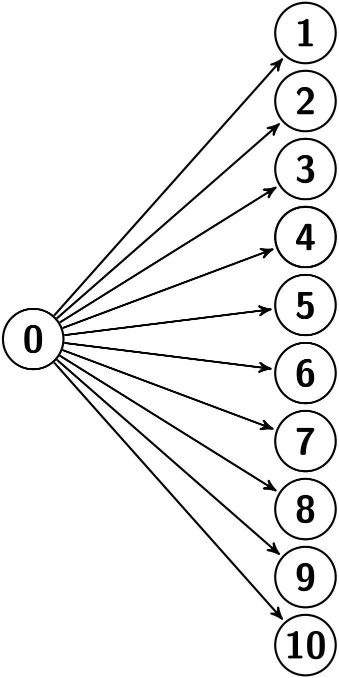
$$x^2_\omega = x^1 - u^2_\omega + h^2_\omega$$

$$0 \leq p^1 \perp U(u^1) + v^1 \geq d^1$$

$$0 \leq p^2_\omega \perp U(u^2_\omega) + v^2_\omega \geq d^2_\omega, \forall \omega$$



- Single hydro, thermal and representative consumer
- Initial storage 10, inflow of 4 to 0, equal prob random inflows of i to node i
- Risk neutral: SO equivalent to CE (key point is that each risk set is a singleton, and that is the same as the system risk set)



- Single hydro, thermal and representative consumer
- Initial storage 10, inflow of 4 to 0, equal prob random inflows of i to node i
- Risk neutral: SO equivalent to CE (key point is that each risk set is a singleton, and that is the same as the system risk set)
- Each agent has its own risk measure, e.g. $0.8EV + 0.2CVaR$
- Is there a system risk measure?
- Is there a system optimization problem?

$$\min \sum_i C(x_i^1) + \rho_i \left(C(x_i^2(\omega)) \right) ????$$

Equilibrium or optimization?

Theorem

If (d, v, u, x) solves (risk averse) SO, then there exists a probability distribution σ_k and prices p so that (d, v, u, x, p) solves (risk neutral) CE(σ)

(Observe that each agent must maximize their own expected profit using probabilities σ_k that are derived from identifying the worst outcomes as measured by SO. These will correspond to the worst outcomes for each agent only under very special circumstances)

- High initial storage level (15 units)
  - Worst case scenario is 1: lowest system cost, smallest profit for hydro
  - SO equivalent to CE
- Low initial storage level (10 units)
  - Different worst case scenarios
  - SO different to CE (for large range of demand elasticities)
- Attempt to construct agreement on what would be the worst-case outcome by trading risk

Contracts in MOPEC (Philpott/F./Wets)

- Can we modify (complete) system to have a social optimum by trading risk?
- How do we design these instruments? How many are needed? What is cost of deficiency?
- Facilitated by allowing contracts bought now, for goods delivered later (e.g. Arrow-Debreu Securities)
- Conceptually allows to transfer goods from one period to another (provides wealth retention or pricing of ancilliary services in energy market)
- Can investigate new instruments to mitigate risk, or move to system optimal solutions from equilibrium (or market) solutions

Theory and Observations

- agent problems are multistage stochastic optimization models
- perfectly competitive partial equilibrium still corresponds to a social optimum when all agents are risk neutral and share common knowledge of the probability distribution governing future inflows
- situation complicated when agents are risk averse
  - utilize stochastic process over scenario tree
  - under mild conditions a social optimum corresponds to a competitive market equilibrium if agents have time-consistent dynamic coherent risk measures and there are enough traded market instruments (over tree) to hedge inflow uncertainty
- Otherwise, must solve the stochastic equilibrium problem
- Research challenge: develop reliable algorithms for large scale decomposition approaches to MOPEC

What is EMP?

Annotates existing equations/variables/models for modeler to provide/define additional structure

- equilibrium
- vi (agents can solve min/max/vi)
- bilevel (reformulate as MPEC, or as SOCP)
- dualvar (use multipliers from one agent as variables for another)
- QS functions (both in objectives and constraints)
- implicit functions and shared constraints

- Currently available within GAMS
- Some solution algorithms implemented in modeling system - limitations on size, decomposition and advanced algorithms