Risk Sensitive Inventory Management with Financial Hedging

Süleyman Özekici
Koç University
Department of Industrial Engineering
Sarıyer, İstanbul

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Introduction and Motivation
Literature
Static Financial Hedging Model
Dynamic Financial Hedging Model
Numerical Illustration
Future Research
Demand and supply uncertainties are primary sources of risk for inventory managers.

Price volatility is also a major source of risk.

Commodity-based raw material prices, exchange rate fluctuations are typical examples.

Exposure to these risks have significant impacts on input costs, sales prices and volume.

Successful inventory management can create even more value in fluctuating price environments.

One can use financial markets to cope with price and exchange rate fluctuations.
Typical models are based on adjusting the replenishment policy to optimize a certain measure such as
  - the mean of the cash flow
  - the variance of the cash flow
  - utility of the decision maker
  - probability of achieving a certain target profit

**Expected Utility Models:** Boukazis and Sobel [OR’92], Agrawal and Seshandi [MSOM’00], Chen et al. [OR’07]

**Mean-Variance Models:** Lau [JORS’80], Berman and Schnabel [IJSS’86], Chen and Federgruen [WP’00], Wu et al. [OMEGA’09]

**Value-at-Risk Models:** Luciano et al. [IJPE’03], Tapiero [EJOR’05], Özler et al. [IJPE’09]

**Minimum-Variance Models:** Okyay et al. [ORS’14], Tekin and Özekici [IIETr’15]
The use of financial markets to reduce the price and inventory risk has gained importance.

A financial hedge is an investment position to reduce the effect of potential losses/gains incurred from another investment.

Hedges can be constructed using stocks, indices, futures, options, swaps, etc.

Future contracts are most widely used to hedge the risks in commodity prices, energy prices, foreign currencies, interest rates, etc.

Gaur and Seshadri [MSOM’05], Caldentey and Haugh [MOR’06], Chod et al. [MS’10], Okyay et al. [ORS’14], Tekin and Özekici [IIETr’15]
The Model

- Single-period inventory model with fluctuating sales prices and modulated demand
- Sales period is \([0, T]\)
- Stochastic price process \(P = \{P_t; t \geq 0\}\) (compounded to time \(T\))
- \(P_0\): Purchase price at time 0
- Sales price at time \(t\) is \(f(P_t)\) (for example, \(f(p) = \alpha p\), where \(\alpha > 1\) is the markup)
- The customer arrival process is \(N = \{N_t : t \geq 0\}\) with intensity process \(\Lambda = \{\Lambda_t = \lambda(P_t); t \geq 0\}\) (Doubly stochastic Poisson process)
- \(h(p)\): holding cost, \(b(p)\): backordering cost

Cash flow at time \(T\) under order-up-to decision \(y\) is

\[
CF(y, N, P) = -P_0 y + \sum_{j=1}^{NT} f(P_{T_j}) - [b(P_T)(NT - y)^+ + h(P_T)(y - NT)^+]
\]
Price and Arrival Processes

Inventory Levels

Market Prices

Customer Arrivals
Price related risks: sales price and demand (also purchase price in multiperiod inventory model)
We assume that there are $M$ financial securities which are correlated with the price process $P$

$S^{(i)} = \left\{ S_t^{(i)} ; t \geq 0 \right\}$ : The price process for security $i$ (compounded to time $T$)

$S = (S^{(1)}, S^{(2)}, ..., S^{(M)})$ : The vector of security price processes

$T = (t_0, t_1, t_2, ..., t_{n-1})$ : Prespecified trading times ($t_0 = 0, t_n = T$)

$\theta_k = (\theta_k^{(1)}, \theta_k^{(2)}, ..., \theta_k^{(M)})$ : Portfolio decision at time $t_k$

$\theta = (\theta_0, \theta_1, ..., \theta_{n-1})$ : Financial hedging strategy or portfolio
The final payoff of the financial portfolio at time $T$ as

$$G(\theta, S) = \sum_{i=1}^{M} \sum_{k=0}^{n-1} \theta_k^{(i)} \left( S_{t_{k+1}}^{(i)} - S_{t_k}^{(i)} \right) = \sum_{k=0}^{n-1} \theta_k^T \triangle S_k = \theta^T \triangle S$$

$\triangle S_k = S_{t_{k+1}} - S_{t_k}$: Vector of net payoffs for holding one unit of each security during $(t_k, t_{k+1})$

$\theta_k, \triangle S_k$ are $M \times 1$ column vectors

$\theta, \triangle S$ are $Mn \times 1$ column vectors
The Hedged Cash Flow

- Total hedged cash flow at time $T$ is

$$HCF (\theta, y, N, P, S) = CF (y, N, P) + G (\theta, S)$$

- The objective of the inventory manager is to solve

$$\max_{y \geq 0} E [HCF (\theta (y), y, N, P, S)]$$

subject to

$$\theta (y) = \arg \min_{\theta} \text{Var} (HCF (\theta, y, N, P, S))$$
Static Model: Minimum-Variance Portfolio

- Portfolio chosen once at the beginning only \((n = 1, t_0 = 0)\)
- Covariance matrix
  \[
  C_{ij} = \text{Cov} \left( S_T^{(i)}, S_T^{(j)} \right)
  \]

- Covariance vector
  \[
  \mu_i (y) = \text{Cov} \left( CF (y, N, P), S_T^{(i)} \right)
  = \text{Cov} \left( \sum_{j=1}^{N_T} f \left( P_{T_j} \right), S_T^{(i)} \right) - \text{Cov} \left( h \left( P_T \right) (y - N_T)^+ , S_T^{(i)} \right)
  - \text{Cov} \left( b \left( P_T \right) (N_T - y)^+ , S_T^{(i)} \right)
  \]

Theorem

\( \text{Var} \left( \theta, y, N, P, S \right) \) is convex in \( \theta \) and minimum-variance portfolio for order quantity \( y \) is given by

\[
\theta^* (y) = -C^{-1} \mu (y)
\]
Static Model: Optimal Base-Stock Level

Assumption

The function

\[ E \left[ (h(P_T) + b(P_T)) 1_{\{N_T \leq y\}} \right] - Cov \left( (h(P_T) + b(P_T)) 1_{\{N_T \leq y\}}, S_T \right)^T C^{-1} E[\Delta S] \]

is increasing in \( y \).

Theorem

The optimal order quantity that maximizes the expected cash flow using the minimum-variance portfolio \( \theta^*(y) = -C^{-1} \mu(y) \) is

\[ y^* = \inf \{ y \geq 0; E \left[ (h(P_T) + b(P_T)) 1_{\{N_T \leq y\}} \right] - Cov \left( (h(P_T) + b(P_T)) 1_{\{N_T \leq y\}}, S_T \right)^T C^{-1} E[\Delta S] \} \]

\[ \geq E[b(P_T)] - P_0 - Cov(b(P_T), S_T)^T C^{-1} E[\Delta S] \]
If \( E[\Delta S] = 0 \) (or security prices are martingales), then Assumption 1 is always satisfied.

In this case, optimal order-up-to level is

\[
y^* = \inf\{y \geq 0; E[(h(P_T) + b(P_T))1_{\{N_T \leq y\}}] \geq E[b(P_T)] - P_0\}
\]

This is not the newsvendor solution since \( N \) and \( P \) are dependent.

We obtain the newsvendor solution

\[
y^* = \inf\left\{y \geq 0; P\{N_T \leq y\} \geq \frac{b - c}{h + b}\right\}
\]

if \( P_0 = c, h(p) = h \) and \( b(p) = b \)
If \( N \) is a Poisson process with rate \( \lambda \) and independent of \( P \), then Assumption 1 is always satisfied. In this case

\[
\mu(y) = \lambda \int_0^T Cov(f(P_t), S_T) \, dt - E[(y - N_T)^+] \, Cov(h(P_T), S_T) - E[(N_T - y)^+] \, Cov(b(P_T), S_T)
\]

and the optimal order-up-to level is

\[
y^* = \inf \left\{ y \geq 0 : P\{N_T \leq y\} \geq \frac{E[b(P_T)] - P_0 - Cov(b(P_T), S_T)^T C^{-1} E[\Delta S]}{E[h(P_T)] + E[b(P_T)] - Cov(h(P_T) + b(P_T), S_T)^T C^{-1} E[\Delta S]} \right\}
\]

If \( E[\Delta S] = 0 \), then

\[
y^* = \inf \left\{ y \geq 0 : P\{N_T \leq y\} \geq \frac{E[b(P_T)] - P_0}{E[h(P_T)] + E[b(P_T)]} \right\}
\]
If $S$ is a future written on $P_T$, then $S_0 = P_0$ and $S_T = P_T$

Let us assume that $b(P_T) = b + P_T$, $h(P_T) = h - \gamma P_T$

In this case, optimal order-up-to level is

$$y^* = \inf \left\{ y \geq 0 : P \{ N_T \leq y \} \geq \frac{b + \gamma (E[P_T] - P_0)}{b + h + (1 - \gamma)P_0} \right\}$$

and the minimum-variance portfolio is

$$\theta^* = E \left[ (N_T - y)^+ \right] - \gamma E \left[ (y - N_T)^+ \right] - \lambda \int_0^T \beta_t dt$$

where

$$\beta_t = \frac{Cov \left( f(P_t), P_T \right)}{Var \left( P_T \right)}$$
Dynamic Model

- Trading times are $t_0 = 0, t_1, t_2, \ldots, t_{n-1}$
- Assumption: Security prices are martingales
- In this case, the minimum-variance problem for every $y$ decision is reduced to

$$\min_\theta E \left[ \left( CF (y, N, \mathcal{P}) + \theta^T \triangle S \right)^2 \right]$$

- This objective is separable in terms of dynamic programming
Dynamic Programming: State Transitions

- We use four states $X, W, P, S$
- Inventory level transitions

\[ X_{k+1} = X_k - N_{t_k, t_{k+1}} \]
\[ X_0 = y \]

- Wealth level transitions

\[ W_{k+1} = W_k + R_{t_k, t_{k+1}} + \theta_k^T \triangle S_k \]
\[ W_0 = 0 \]

- Operational revenue during $[t_k, t_{k+1}]$

\[ R_{[t_k, t_{k+1}]} = \sum_{j=1}^{N_{[t_k, t_{k+1}]} f \left( P_{T_j + t_k} \right) } \]
Dynamic Programming Formulation

- Objective function is

\[
E \left[ \left( CF (y, N, P) + \sum_{k=0}^{n-1} \theta_k^T \triangle S_k \right)^2 \right] = E \left[ (W_n - [b (P_{t_n}) (-X_n)^+ + h (P_{t_n}) X_n^+])^2 \right]
\]

- DP formulation is

\[
V_k (x, w, p, s) = \min_{\theta_k} E \left[ V_{k+1} (X_{t_k} - N_{[t_k,t_{k+1}]}, W_{t_k} + R_{[t_k,t_{k+1}]} + \theta_k \triangle S_k, P_{t_{k+1}}, S_{t_{k+1}}) \mid X_{t_k} = x, W_{t_k} = w, P_{t_k} = p, S_{t_k} = s \right]
\]

with boundary condition

\[
V_n (x, w, p, s) = (w - b (p) (-x)^+ - h (p) x^+)^2
\]

- Here \( V_k \) is the value function at trading time \( t_k \)
Notation

\[ C_k(s)_{ij} = \text{Cov} \left( S^{(i)}_{tk+1}, S^{(j)}_{tk+1} \mid S^i_{tk} = s^i, S^j_{tk} = s^j \right) \]

\[ R_{[tk,tn]} = \sum_{j=k}^{n-1} R_{[tj,tj+1]} \]

\[ \mu_k(x,p,s)_{j} = \text{Cov} \left( R_{[tk,tn]} - b(Ptn) \left( N_{[tk,tn]} - x \right)^+ - h(Ptn) \left( x - N_{[tk,tn]} \right)^+, S^{(j)}_{tk+1} \mid P_{tk} = p, S^j_{tk} = s^j \right) \]

\[ g_k(x,w,p) = E \left[ \left( w + R_{[tk,tn]} - b(Ptn) \left( N_{[tk,tn]} - x \right)^+ - h(Ptn) \left( x - N_{[tk,tn]} \right)^+ \right)^2 \mid P_{tk} = p \right] \]

\[ h_k(x,p,s) = -\mu_k(x,p,s)^T C_k(s)^{-1} \mu_k(x,p,s) + E \left[ h_{k+1} (x - N_{[tk,tk+1]}, P_{tk+1}, S_{tk+1}) \mid P_{tk} = p, S_{tk} = s \right] \]
The Optimal Solution

Theorem

Value function for period $k$ is

$$V_k(x, w, p, s) = g_k(x, w, p) + h_k(x, p, s)$$

and the minimum-variance portfolio is

$$\theta_k^*(x, p, s) = -C_k(s)^{-1} \mu_k(x, p, s)$$

Theorem

Optimal order-up-to level that maximizes the expected hedged cash flow is

$$y^* = \inf \{ y \geq 0; E \left[ (h(P_T) + b(P_T)) 1_{N_T \leq y} \right] \geq E[b(P_T)] - P_0 \}$$

- Risk-neutral solution
- Martingale security price processes
Schwartz and Smith [MS’00] uses a price model that describes long and short term behaviours of commodities

\[ P_t = e^{\chi_t + \xi_t} \]

where

\[ d\chi_t = -\kappa \chi_t dt + \sigma_\chi dW_t^{(\chi)} \]

is an Ornstein-Uhlenbeck process that models the short-term deviations (mean reverting to zero) and

\[ d\xi_t = \mu_\xi dt + \sigma_\xi dW_t^{(\xi)} \]

is a geometric Brownian motion that models the long-term equilibrium \( (dW_t^{(\chi)} dW_t^{(\xi)} = \rho dt) \)

We use the risk-neutral version where

\[ dP_t = (\sigma_\xi + \sigma_\chi \rho) P_t dW_t^1 + \sigma_\chi \sqrt{1 - \rho^2} P_t dW_t^2 \]
\begin{itemize}
  \item $T = 1$, $f(p) = 2p$, $b(p) = 4 + p$, $h(p) = 1$, $\lambda_t = (90 - 1.4P_t)^+$
  \item $P_0 = 20$, $\sigma_\chi = 0.25$, $\sigma_\xi = 0.15$, $\rho = 0.3$
  \item $S_t^{(1)} = P_t$ : Future
  \item $S_t^{(2)} = E[(P_T - 20)^+ | P_t]$ : Call option
\end{itemize}
Effect of Financial Hedging on Risk Reduction

Mean $\sim 2000$
Financial hedging of an inventory system where a stochastic price process modulates the demand and sales prices

We characterize the static and dynamic financial hedging policies

We also analyzed the multi-period inventory version

Different objective functions (mean-variance, utility functions)

Continuous-time versions

Budget constraint