Risk Sensitive Inventory Management with Financial Hedging[1]

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Outline

- Introduction and Motivation
- Literature
- Static Financial Hedging Model
- Dynamic Financial Hedging Model
- Numerical Illustration
- Future Research

Introduction and Motivation

- Demand and supply uncertainties are primary sources of risk for inventory managers
- Price volatility is also major source of risk
- Commodity based raw material prices, exchange rate fluctuations are typical examples
- Exposure to these risks have significant impacts on input costs, sales prices and volume
- Successful inventory management can create even more value in fluctuating price environments
- One can use financial markets to cope with price and exchange rate fluctuations

Risk-Sensitive Inventory Management

- Typical models are based on adjusting the replenishment policy to optimize a certain measure such as
  - the mean of the cash flow
  - the variance of the cash flow
  - utility of the decision maker
  - probability of achieving a certain target profit
- **Expected Utility Models:** Boukazis and Sobel [OR'92], Agrawal and Seshandi [MSOM'00], Chen et al. [OR'07]
- **Mean-Variance Models:** Lau [JORS'80], Berman and Schnabel [IJSS'86], Chen and Federgruen [WP'00], Wu et al. [OMEGA'09]
- **Value-at-Risk Models:** Luciano et al. [IJPE'03], Tapiero [EJOR'05], Özler et al. [IJPE'09]
- **Minimum-Variance Models**: Okyay et al. [ORS'14], Tekin and Özekici [IIETr'15]

Financial Hedging

- The use of financial markets to reduce the price and inventory risk has gained importance
- A financial hedge is an investment position to reduce the effect of potential losses/gains incurred from another investment
- Hedges can be constructed using stocks, indices, futures, options, swaps, etc.
- Future contracts are most widely used to hedge the risks in commodity prices, energy prices, foreign currencies, interest rates, etc.
- Gaur and Seshadri [MSOM'05], Caldentey and Haugh [MOR'06], Chod et al. [MS'10], Okyay et al. [ORS'14], Tekin and Özekici [IIETr'15]

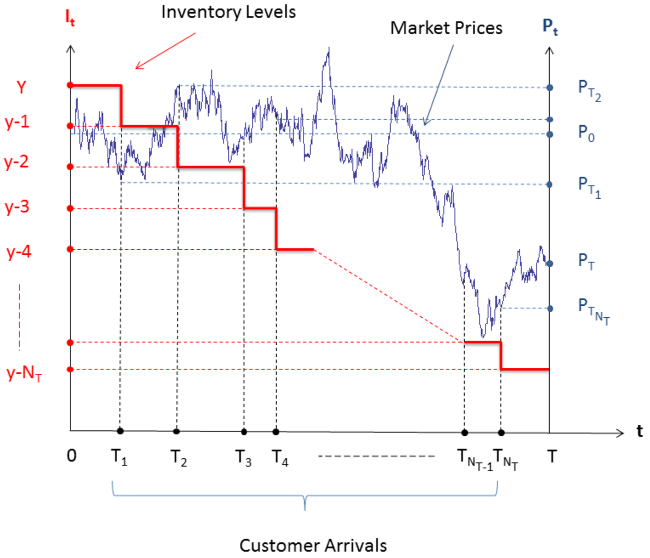
The Model

- Single-period inventory model with fluctuating sales prices and modulated demand
- Sales period is $[0, T]$
- Stochastic price process $P = \{P_t; t \geq 0\}$ (compounded to time T)
- P_0 : Purchase price at time 0
- Sales price at time t is $f(P_t)$ (for example, $f(p) = \alpha p$, where $\alpha > 1$ is the markup)
- The customer arrival process is $N = \{N_t : t \geq 0\}$ with intensity process $\Lambda = \{\Lambda_t = \lambda(P_t); t \geq 0\}$ (Doubly stochastic Poisson process)
- $h(p)$: holding cost, $b(p)$: backordering cost

  Cash flow at time T under order-up-to decision y is

$$CF(y, \mathcal{N}, \mathcal{P}) = -P_0 y + \sum_{j=1}^{N_T} f\left(P_{T_j}\right) - \left[b(P_T)(N_T - y)^+ + h(P_T)(y - N_T)^+\right]$$

Price and Arrival Processes



Security Prices and Trading Times

- Price related risks: sales price and demand (also purchase price in multiperiod inventory model)
- We assume that there are M financial securities which are correlated with the price process P
- $S^{(i)} = \left\{ S_t^{(i)}; t \geq 0 \right\}$: The price process for security i (compounded to time T)
- $S = \left(S^{(1)}, S^{(2)}, ..., S^{(M)} \right)$: The vector of security price processes
- $\mathcal{T} = (t_0, t_1, t_2, ..., t_{n-1})$: Prespecified trading times ($t_0 = 0$, $t_n = T$)
- $\theta_k = \left(\theta_k^{(1)}, \theta_k^{(2)}, ..., \theta_k^{(M)} \right)$: Portfolio decision at time t_k
- $\theta = (\theta_0, \theta_1, ..., \theta_{n-1})$: Financial hedging strategy or portfolio

Financial Cash Flow

- The final payoff of the financial portfolio at time T as

$$G(\theta, \mathcal{S}) = \sum_{i=1}^{M} \sum_{k=0}^{n-1} \theta_k^{(i)} \left(S_{t_{k+1}}^{(i)} - S_{t_k}^{(i)} \right) = \sum_{k=0}^{n-1} \theta_k^T \triangle S_k = \theta^T \triangle S$$

- $\triangle S_k = S_{t_{k+1}} - S_{t_k}$: Vector of net payoffs for holding one unit of each security during (t_k, t_{k+1})
- $\theta_k, \triangle S_k$ are $M \times 1$ column vectors
- $\theta, \triangle S$ are $Mn \times 1$ column vectors

The Hedged Cash Flow

- Total hedged cash flow at time T is

$$HCF\left(\theta, y, \mathcal{N}, \mathcal{P}, S\right) = CF\left(y, \mathcal{N}, \mathcal{P}\right) + G\left(\theta, S\right)$$

- The objective of the inventory manager is to solve

$$\max_{y \geq 0} \quad E\left[HCF\left(\theta\left(y\right), y, \mathcal{N}, \mathcal{P}, S\right)\right]$$

subject to

$$\theta\left(y\right) = \arg\min_{\theta} Var\left(HCF\left(\theta, y, \mathcal{N}, \mathcal{P}, S\right)\right)$$

Static Model: Minimum-Variance Portfolio

- Portfolio chosen once at the beginning only $(n = 1, t_0 = 0)$
- Covariance matrix

$$C_{ij} = Cov\left(S_T^{(i)}, S_T^{(j)}\right)$$

- Covariance vector

$$\begin{aligned}\mu_i(y) &= Cov\left(CF(y, \mathcal{N}, \mathcal{P}), S_T^{(i)}\right) \\ &= Cov\left(\sum_{j=1}^{N_T} f\left(P_{T_j}\right), S_T^{(i)}\right) - Cov\left(h(P_T)(y - N_T)^+, S_T^{(i)}\right) \\ &\quad - Cov\left(b(P_T)(N_T - y)^+, S_T^{(i)}\right)\end{aligned}$$

Theorem

$Var(\theta, y, \mathcal{N}, \mathcal{P}, S)$ is convex in θ and minimum-variance portfolio for order quantity y is given by

$$\theta^*(y) = -C^{-1}\mu(y)$$

Static Model: Optimal Base-Stock Level

Assumption

The function

$$E\left[\left(h\left(P_T\right)+b\left(P_T\right)\right)1_{\{N_T\leq y\}}\right]-Cov\left(\left(h\left(P_T\right)+b\left(P_T\right)\right)1_{\{N_T\leq y\}},S_T\right)^T C^{-1}E\left[\triangle S\right]$$

is increasing in y.

Theorem

The optimal order quantity that maximizes the expected cash flow using the minimum-variance portfolio $\theta^*\left(y\right)=-C^{-1}\mu\left(y\right)$ *is*

$$\begin{aligned} y^* &= \inf\Big\{y\geq 0; E\left[\left(h\left(P_T\right)+b\left(P_T\right)\right)1_{\{N_T\leq y\}}\right] \\ &\quad -Cov\left(\left(h\left(P_T\right)+b\left(P_T\right)\right)1_{\{N_T\leq y\}},S_T\right)^T C^{-1}E\left[\triangle S\right]\Big\} \\ &\quad \geq E\left[b\left(P_T\right)\right]-P_0-Cov\left(b\left(P_T\right),S_T\right)^T C^{-1}E\left[\triangle S\right]\Big\} \end{aligned}$$

Static Model: Zero Expected Financial Gain

- If $E\left[\triangle S\right] = 0$ (or security prices are martingales), then Assumption 1 is always satisfied
- In this case, optimal order-up-to level is

$$y^* = \inf\left\{y \geq 0; E\left[\left(h\left(P_T\right) + b\left(P_T\right)\right)1_{\{N_T \leq y\}}\right] \geq E\left[b\left(P_T\right)\right] - P_0\right\}$$

- This is not the newsvendor solution since N and P are dependent
- We obtain the newsvendor solution

$$y^* = \inf\left\{y \geq 0; P\{N_T \leq y\} \geq \frac{b-c}{h+b}\right\}$$

if $P_0 = c, h\left(p\right) = h$ and $b\left(p\right) = b$

Static Model: Demand is Independent of Prices

- If N is a Poisson process with rate λ and independent of P, then Assumption 1 is always satisfied.
- In this case

$$\mu(y) = \lambda \int_0^T Cov\left(f\left(P_t\right), S_T\right) dt - E\left[(y - N_T)^+\right] Cov\left(h\left(P_T\right), S_T\right) - E\left[(N_T - y)^+\right] Cov\left(b\left(P_T\right), S_T\right)$$

and the optimal order-up-to level is

$$y^* = \inf\left\{y \geq 0; P\left\{N_T \leq y\right\} \geq \frac{E\left[b\left(P_T\right)\right] - P_0 - Cov\left(b\left(P_T\right), S_T\right)^T C^{-1} E\left[\triangle S\right]}{E\left[h\left(P_T\right)\right] + E\left[b\left(P_T\right)\right] - Cov\left(h\left(P_T\right) + b\left(P_T\right), S_T\right)^T C^{-1} E\left[\triangle S\right]}\right\}$$

- If $E\left[\triangle S\right] = 0$, then

$$y^* = \inf\left\{y \geq 0 : P\left\{N_T \leq y\right\} \geq \frac{E\left[b\left(P_T\right)\right] - P_0}{E\left[h\left(P_T\right)\right] + E\left[b\left(P_T\right)\right]}\right\}$$

Static Model: Single Financial Security (Future)

- If S is a future written on P_T, then $S_0 = P_0$ and $S_T = P_T$
- Let us assume that $b(P_T) = b + P_T$, $h(P_T) = h - \gamma P_T$
- In this case, optimal order-up-to level is

$$y^* = \inf\left\{ y \geq 0 : P\{N_T \leq y\} \geq \frac{b + \gamma(E[P_T] - P_0)}{b + h + (1-\gamma)P_0} \right\}$$

and the minimum-variance portfolio is

$$\theta^* = E\left[(N_T - y)^+\right] - \gamma E\left[(y - N_T)^+\right] - \lambda \int_0^T \beta_t dt$$

where

$$\beta_t = \frac{Cov(f(P_t), P_T)}{Var(P_T)}$$

Dynamic Model

- Trading times are $t_0 = 0, t_1, t_2, ..., t_{n-1}$
- Assumption: Security prices are martingales
- In this case, the minimum-variance problem for every y decision is reduced to

$$\min_{\theta} E\left[\left(CF\left(y, N, \mathcal{P}\right) + \theta^T \triangle S\right)^2\right]$$

- This objective is separable in terms of dynamic programming

Dynamic Programming: State Transitions

- We use four states X, W, P, S
- Inventory level transitions

$$\begin{aligned} X_{k+1} &= X_k - N_{[t_k, t_{k+1}]} \\ X_0 &= y \end{aligned}$$

- Wealth level transitions

$$\begin{aligned} W_{k+1} &= W_k + R_{[t_k, t_{k+1}]} + \theta_k^T \triangle S_k \\ W_0 &= 0 \end{aligned}$$

- Operational revenue during $[t_k, t_{k+1}]$

$$R_{[t_k, t_{k+1}]} = \sum_{j=1}^{N_{[t_k, t_{k+1}]}} f\left(P_{T_j + t_k}\right)$$

Dynamic Programming Formulation

- Objective function is

$$E\left[\left(CF\left(y,\mathcal{N},\mathcal{P}\right)+\sum_{k=0}^{n-1}\theta_k^T\triangle S_k\right)^2\right]=E\left[\left(W_n-\left[b\left(P_{t_n}\right)\left(-X_n\right)^+ + h\left(P_{t_n}\right)X_n^+\right]\right)^2\right]$$

- DP formulation is

$$\begin{aligned} V_k\left(x,w,p,s\right) &= \min_{\theta_k} E\left[V_{k+1}\left(X_{t_k}-N_{[t_k,t_{k+1}]},W_{t_k}+R_{[t_k,t_{k+1}]}+\theta_k\triangle S_k,P_{t_{k+1}},S_{t_{k+1}}\right)\right. \\ & \qquad \left. |X_{t_k}=x,W_{t_k}=w,P_{t_k}=p,S_{t_k}=s\right] \end{aligned}$$

with boundary condition

$$V_n\left(x,w,p,s\right)=\left(w-b\left(p\right)\left(-x\right)^+ - h\left(p\right)x^+\right)^2$$

- Here V_k is the value function at trading time t_k

Notation

$$C_k(s)_{ij} = Cov\left(S^{(i)}_{t_{k+1}}, S^{(j)}_{t_{k+1}} \mid S^{(i)}_{t_k} = s^{(i)}, S^{(j)}_{t_k} = s^{(j)}\right)$$

$$R_{[t_k,t_n]} = \sum_{j=k}^{n-1} R_{[t_j,t_{j+1}]}$$

$$\mu_k(x,p,s)_j = Cov\left(R_{[t_k,t_n]} - b(P_{t_n})\left(N_{[t_k,t_n]} - x\right)^+ - h(P_{t_n})\left(x - N_{[t_k,t_n]}\right)^+, S^{(j)}_{t_{k+1}} | P_{t_k} = p, S^{(j)}_{t_k} = s^{(j)}\right)$$

$$g_k(x,w,p) = E\left[\left(w + R_{[t_k,t_n]} - b(P_{t_n})\left(N_{[t_k,t_n]} - x\right)^+ - h(P_{t_n})\left(x - N_{[t_k,t_n]}\right)^+\right)^2 | P_{t_k} = p\right]$$

$$h_k(x,p,s) = -\mu_k(x,p,s)^T C_k(s)^{-1} \mu_k(x,p,s) + E\left[h_{k+1}\left(x - N_{[t_k,t_{k+1}]}, P_{t_{k+1}}, S_{t_{k+1}}\right) \mid P_{t_k} = p, S_{t_k} = s\right]$$

The Optimal Solution

Theorem

Value function for period k is

$$V_k\left(x, w, p, s\right) = g_k\left(x, w, p\right) + h_k\left(x, p, s\right)$$

and the minimum-variance portfolio is

$$\theta_k^*\left(x, p, s\right) = -C_k\left(s\right)^{-1}\mu_k\left(x, p, s\right)$$

Theorem

Optimal order-up-to level that maximizes the expected hedged cash flow is

$$y^* = \inf\left\{y \geq 0; E\left[\left(h\left(P_T\right) + b\left(P_T\right)\right)1_{\{N_T \leq y\}}\right] \geq E\left[b\left(P_T\right)\right] - P_0\right\}$$

- Risk-neutral solution
- Martingale security price processes

Numerical Illustration

- Schwartz and Smith [MS'00] uses a price model that describes long and short term behaviours of commodities

$$P_t = e^{\chi_t + \xi_t}$$

where

$$d\chi_t = -\kappa \chi_t dt + \sigma_\chi dW_t^{(\chi)}$$

is an Ornstein-Uhlenbeck process that models the short-term deviations (mean reverting to zero) and

$$d\xi_t = \mu_\xi dt + \sigma_\xi dW_t^{(\xi)}$$

is a geometric Brownian motion that models the long-term equilibrium ($dW_t^{(\chi)} dW_t^{(\xi)} = \rho dt$)

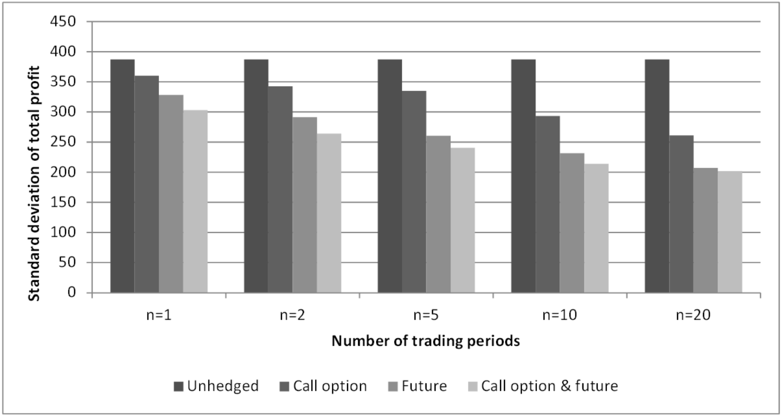
- We use the risk-neutral version where

$$dP_t = (\sigma_\xi + \sigma_\chi \rho) P_t dW_t^1 + \sigma_\chi \sqrt{1 - \rho^2} P_t dW_t^2$$

Financial Securities and Parameters

- $T = 1$, $f(p) = 2p$, $b(p) = 4 + p$, $h(p) = 1$, $\lambda_t = (90 - 1.4P_t)^+$
- $P_0 = 20$, $\sigma_\chi = 0.25$, $\sigma_\xi = 0.15$, $\rho = 0.3$
- $S_t^{(1)} = P_t$: Future
- $S_t^{(2)} = E\left[(P_T - 20)^+ \mid P_t\right]$: Call option

Effect of Financial Hedging on Risk Reduction



Mean ~ 2000

Conclusion and Future Work

- Financial hedging of an inventory system where a stochastic price process modulates the demand and sales prices
- We characterize the static and dynamic financial hedging policies
- We also analyzed the multi-period inventory version
- Different objective functions (mean-variance, utility functions)
- Continuous-time versions
- Budget constraint