Reliability

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OUTLINE

- Basics in reliability
- Repairable systems
  - Renewal process
  - Poisson process
- Case studies
  - Gas escapes
  - Train doors' failures
  - Software reliability

WHAT IS RELIABILITY?

- Probability that a system operates correctly, under specified conditions, for a given time
- Quality over time (Condra, 1993)
- $P(T \geq t)$ (reliability function), with T failure time, nonnegative r.v.

RELIABILITY IN OUR LIFE

- Alarm clock
- Coffee machine
- Car/bus/train
- PC/machinery/phone
- Car/bus/train
- Oven
- Shower
- TV set
- Bed

RELIABILITY IN OUR SOCIETY

- Transportation (cars, airplanes, ships)
- Bridges and roads
- Buildings
- Dams
- Health devices (cardiac valves)
- Nuclear plants
- Chemical plants
- Missiles
- Appliances

SOME ISSUES IN RELIABILITY

- *System* performance
- Monetary costs
- Social costs
- Warranty (length, cost, forecast)
- Inventory of spare parts
- Maintenance and replacement policy
- Product testing
- Degradation up to failure
- Safety and security

RELIABILITY DATA

The next figures are taken from

W. Meeker and L. Escobar (1998),
Statistical Methods for Reliability Data, Wiley.

FAILURE DATA REPRESENTATION

Table 1.1. Ball Bearing Failure Times in Millions of Revolutions

17.88	28.92	33.00	41.52	42.12	45.60
48.40	51.84	51.96	54.12	55.56	67.80
68.64	68.64	68.88	84.12	93.12	98.64
105.12	105.84	127.92	128.04	173.40	

Data from Lawless (1982).

FAILURE DATA REPRESENTATION

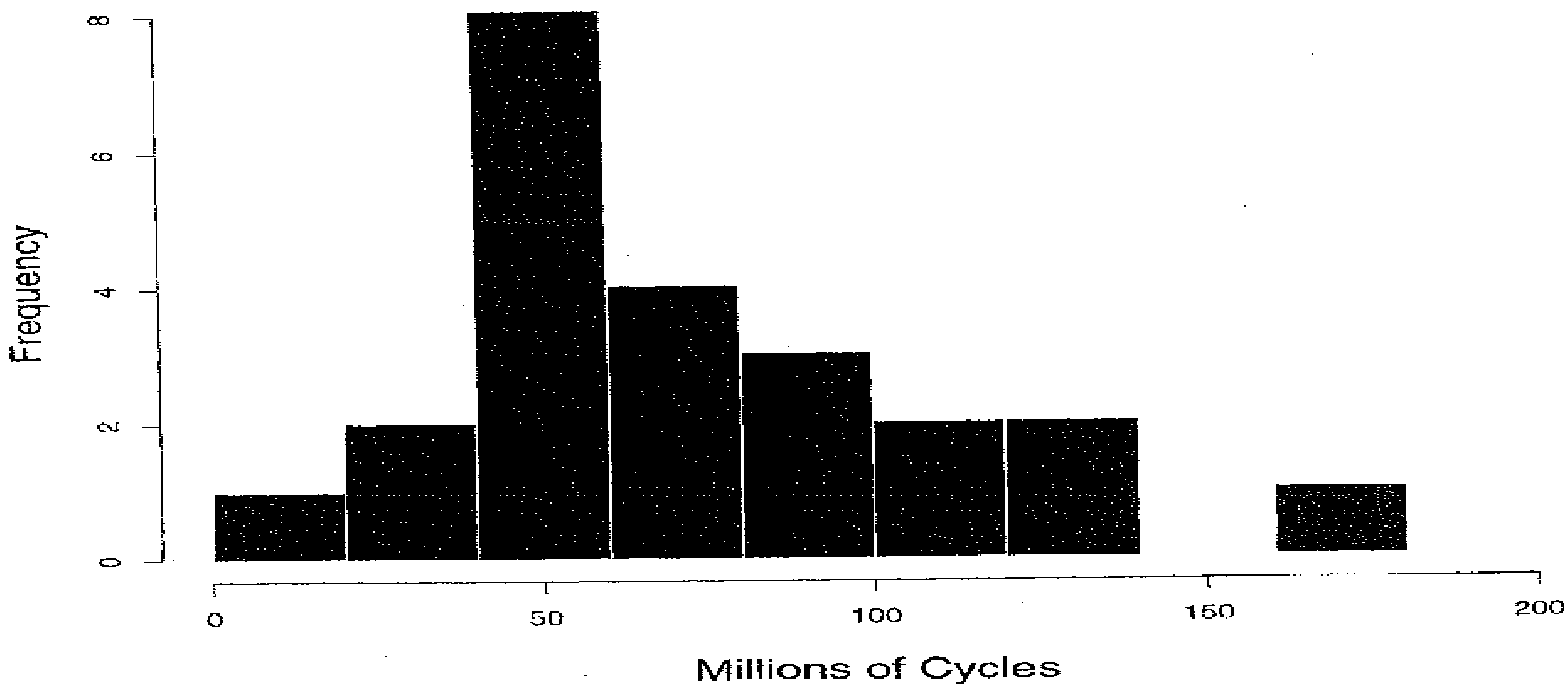


Figure 1.1. Histogram of the ball bearing failure data.

FAILURE DATA REPRESENTATION

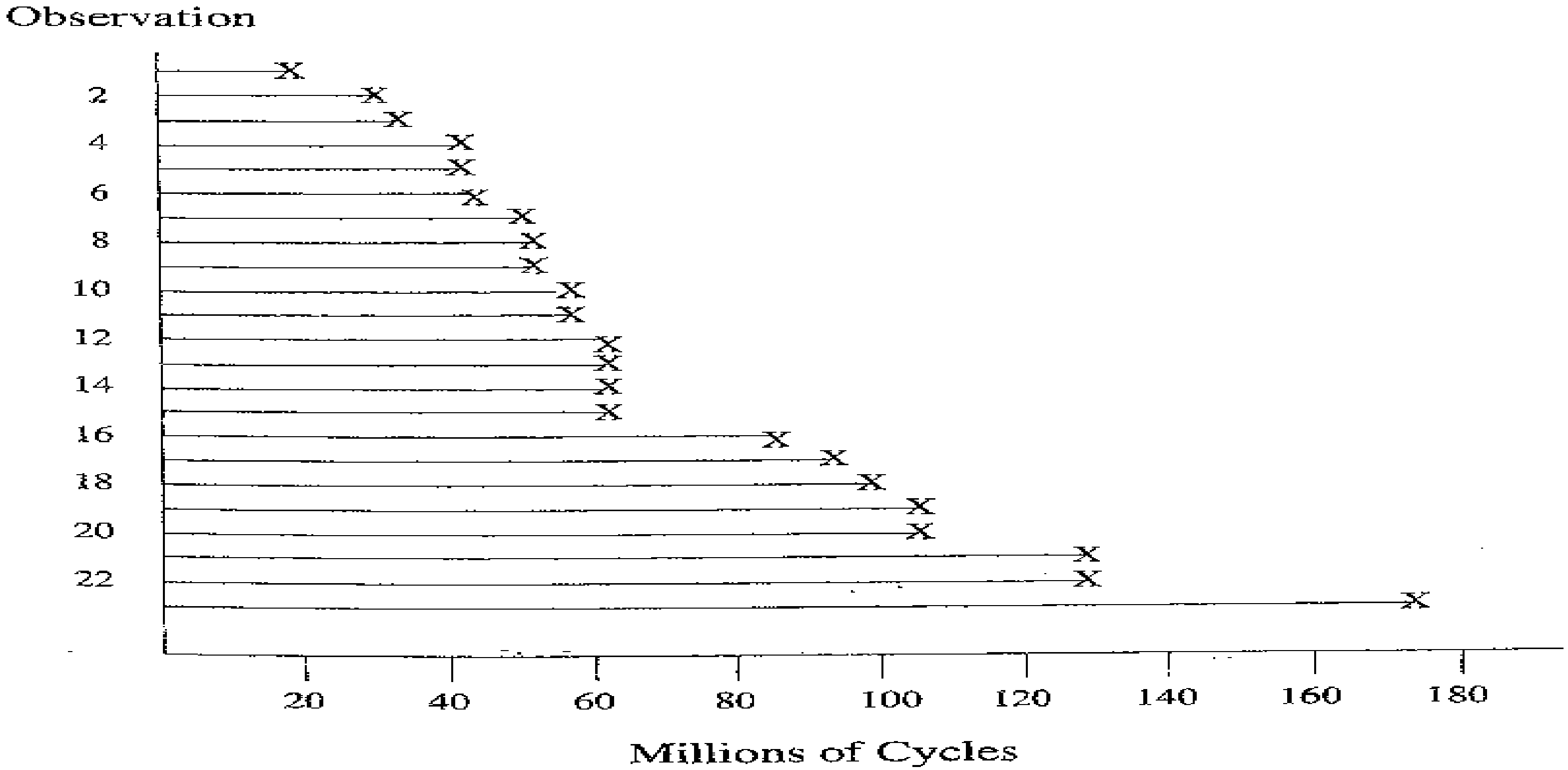


Figure 1.2. Display of the ball bearing failure data.

DIFFERENT TIMES (OPERATING)

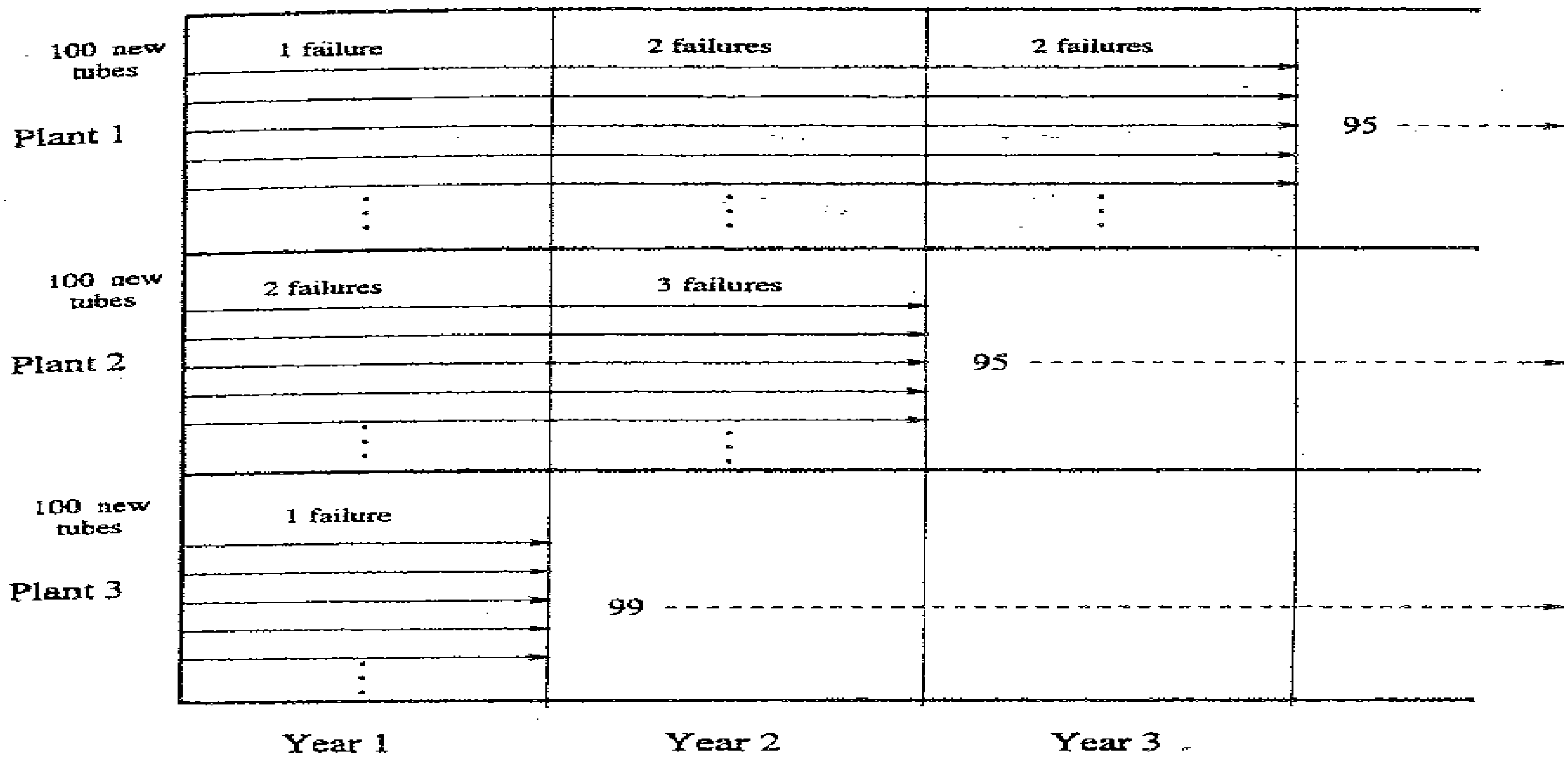


Figure 1.7. Heat exchanger tube crack inspection data in operating time.

DIFFERENT TIMES (CALENDAR)

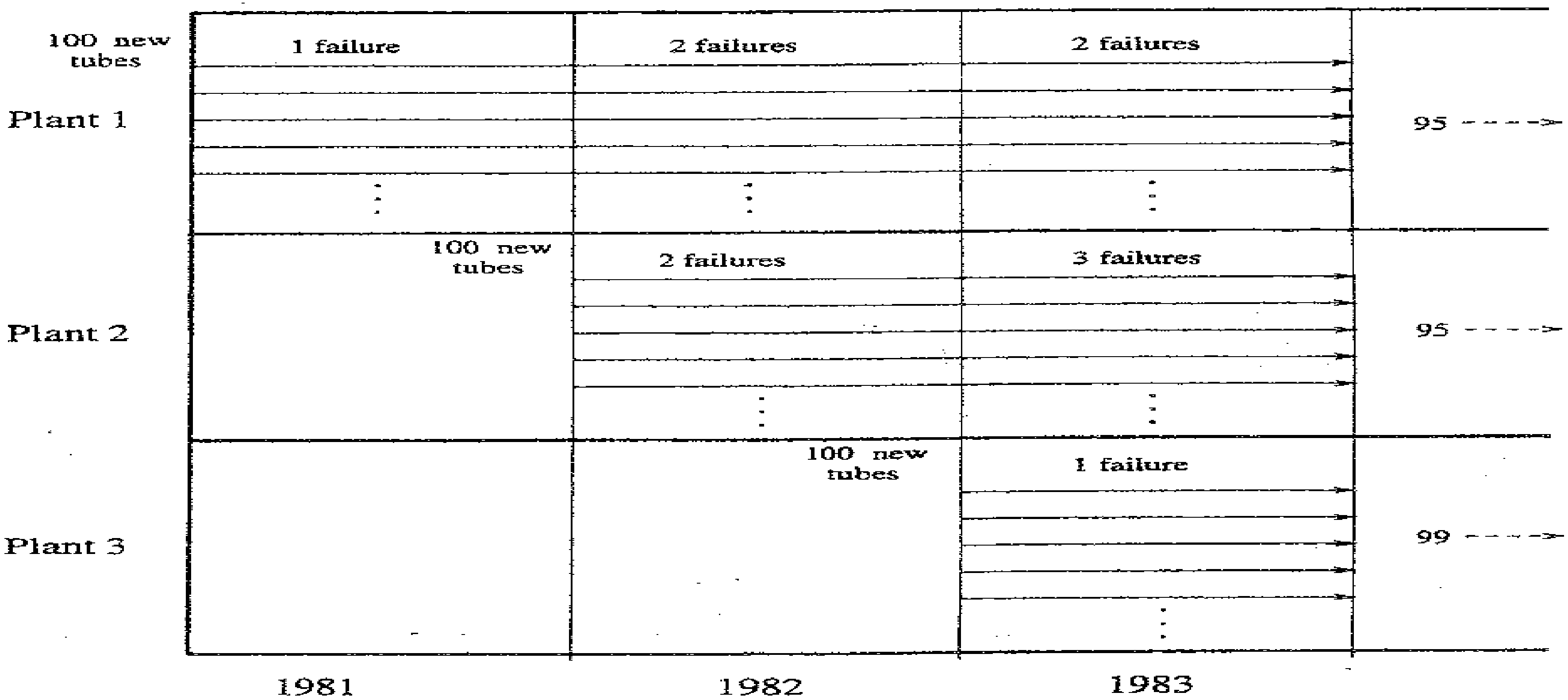


Figure 1.6. Heat exchanger tube crack inspection data in calendar time.

CENSORED DATA

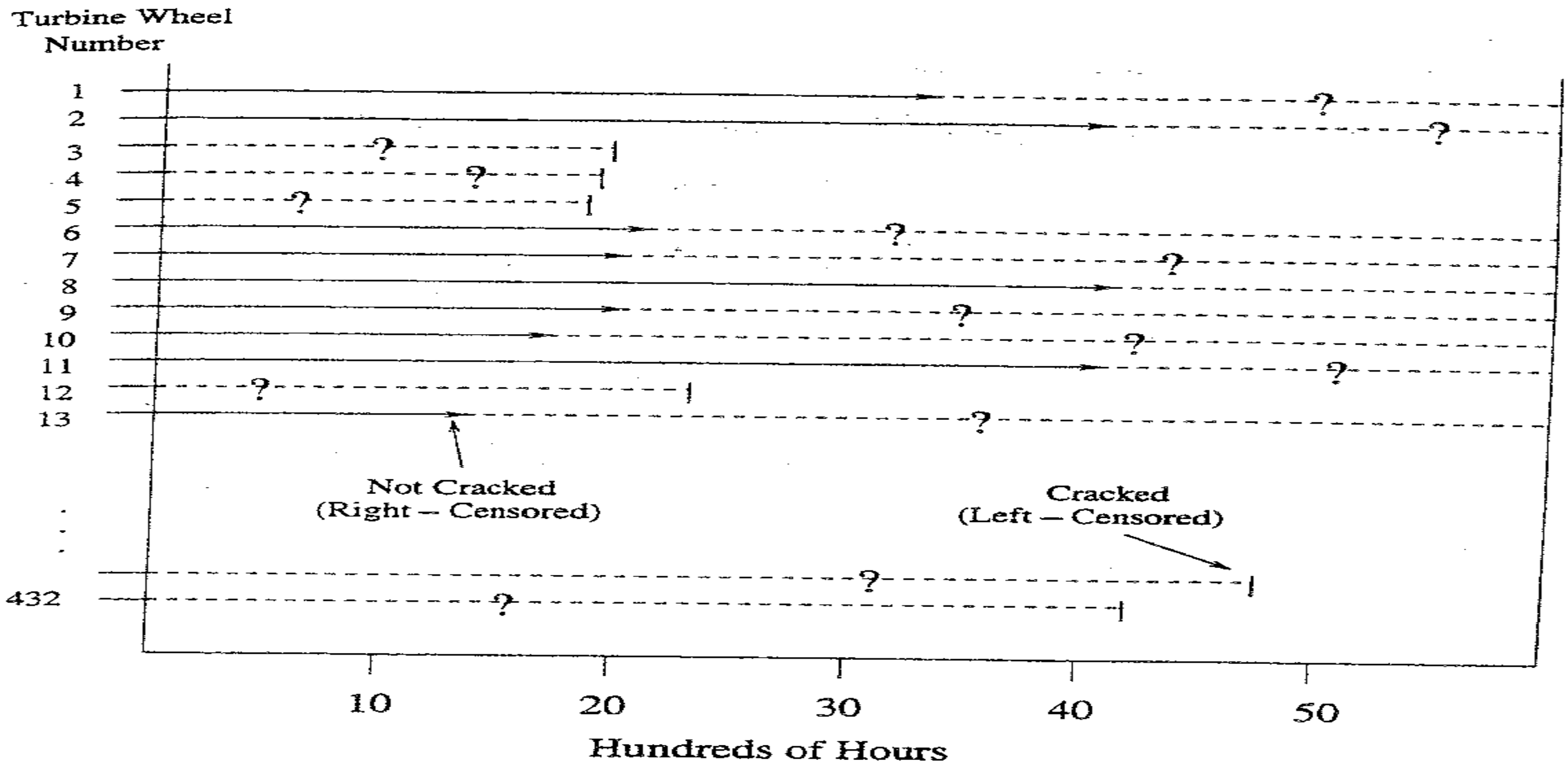


Figure 1.8. Turbine wheel inspection data summary at time of study.

ACCELERATED TEST

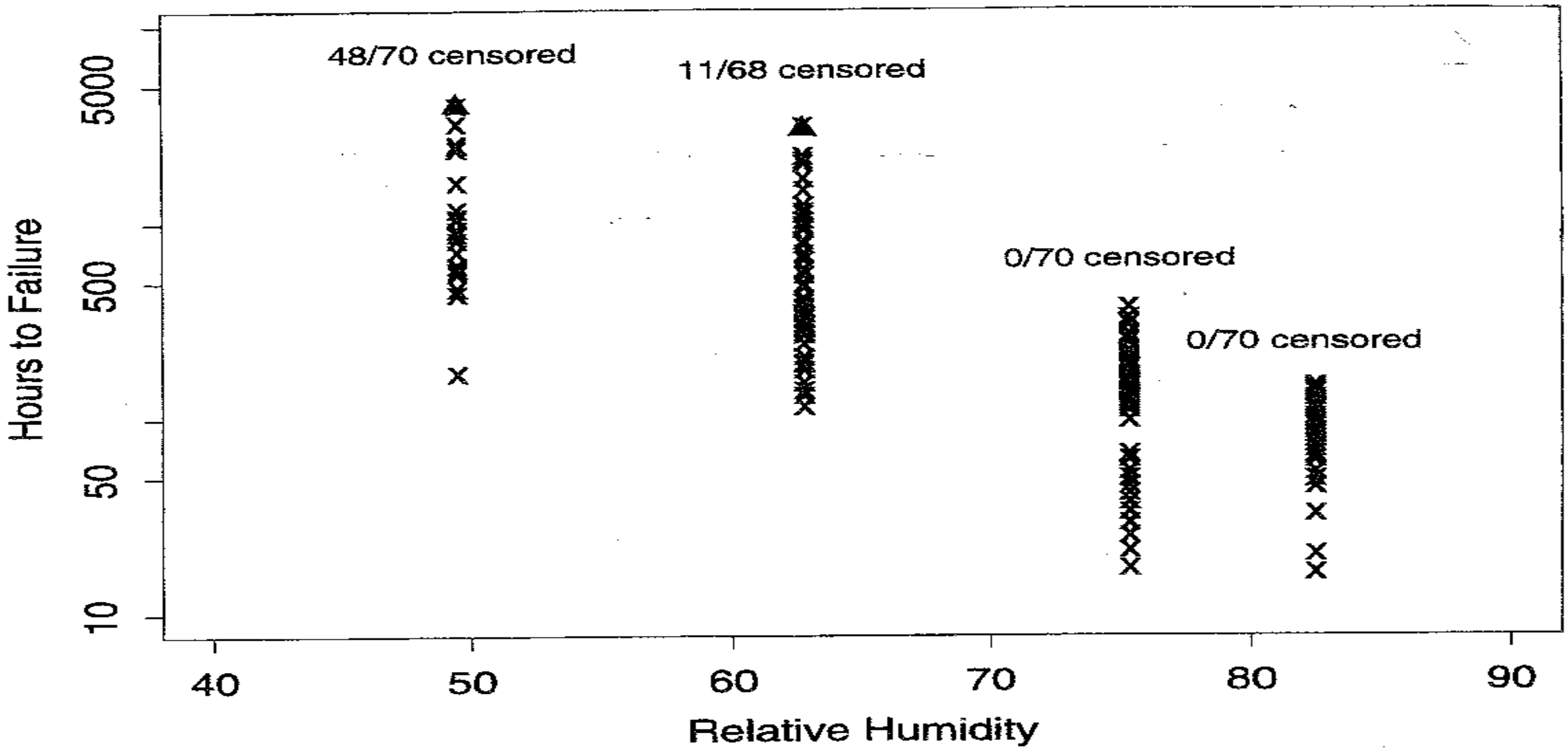


Figure 1.9. Scatter plot of printed circuit board accelerated life test data.

DEGRADATION

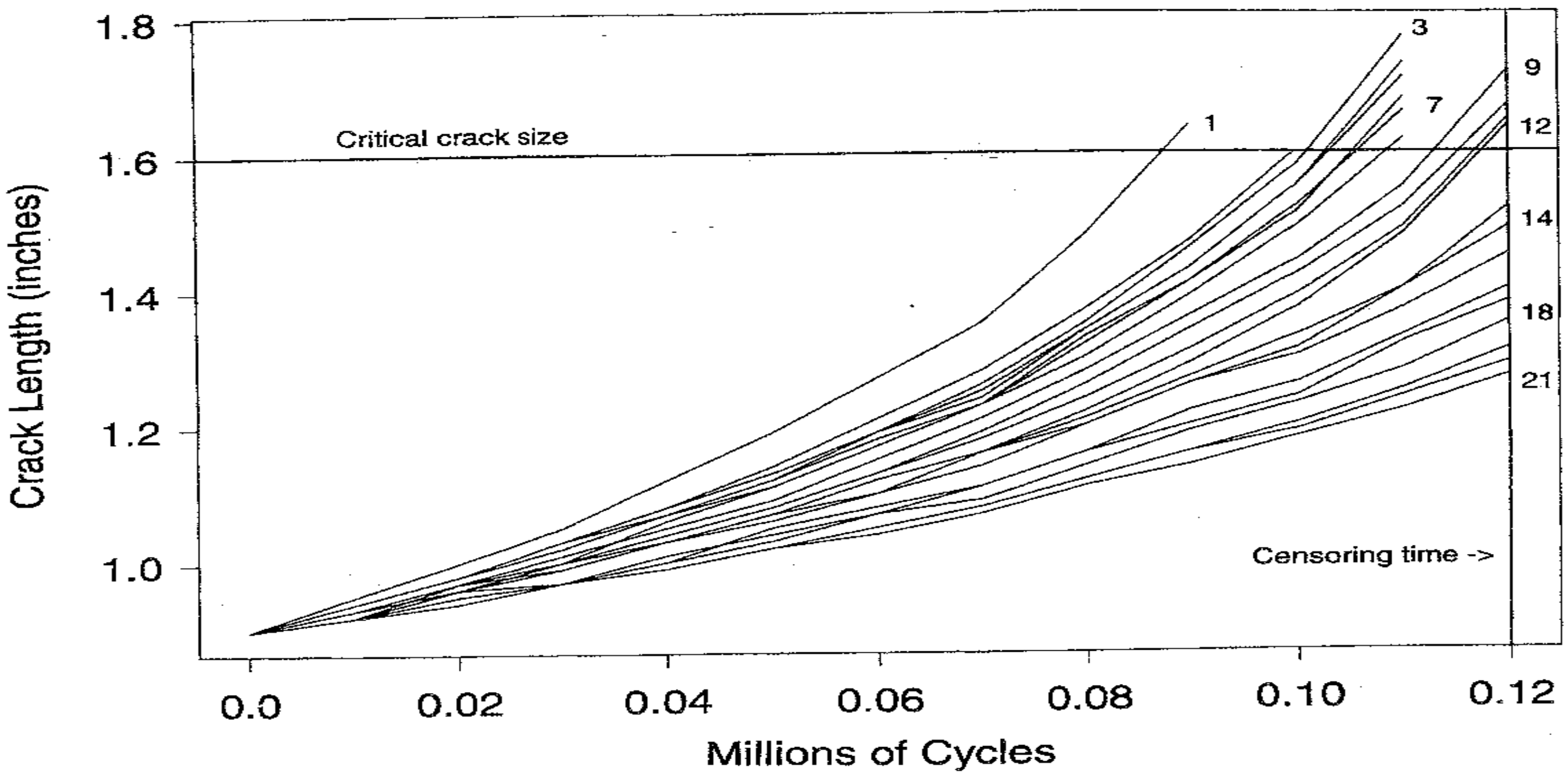


Figure 1.10. Alloy-A fatigue crack size as a function of number of cycles.

BURN-IN TESTING

Table 1.3. Failure Data from a Circuit Pack Field Tracking Study

Operating Hours Interval Endpoint		Number Failing	
Lower	Upper	Vendor 1	Vendor 2
0	1	10	unknown
1	2	1	unknown
2	5	3	unknown
5	10	1	unknown
10	20	2	unknown
20	50	6	unknown
50	100	3	unknown
100	200	2	unknown
200	500	8	unknown
500	1,000	4	unknown
1,000	2,000	5	2
2,000	5,000	6	5
5,000	6,000	3	6
6,000	7,000	9	11
7,000	8,000	10	7
8,000	9,000	16	14
9,000	10,000	7	10
10,000	11,000	unknown	14

After 10,000 hours of operation, there were 4897 unfailed packs for Vendor 1 and after 11,000 hours of operation there were 4924 unfailed packs for Vendor 2.

MAIN FEATURES OF RELIABILITY DATA

- Non negative
  - usually continuous (lifetime of light bulb)
  - sometimes integer (# cycles of washing machine)
- Truncated (censored)
  - left: missing starting date or unrecorded failures before observation starts (gas escapes)
  - right: tests stopped before all systems fail
- Extrapolated
  - w.r.t. time: e.g. $\%$ of population failing after 900 hours, based on a 400 hours test
  - w.r.t. operating conditions: e.g. time for 1% of population to fail at 50°, based on a 85° test

MAIN FEATURES OF RELIABILITY DATA

Some differences

- Time scale: e.g. time vs. kilometers from first train ride (or both)
- Start point: e.g. construction time or first train ride
- Failure: e.g. broken vs. reduced capacity (60% w.r.t. initial one) of light bulb
- Repair strategy: repairable vs. non repairable systems (or mixed)

BASIC DEFINITIONS AND PROPERTIES

- Failure time T, with pdf $f(t)$ and cdf $F(t)$, $t \geq 0$
- Reliability function: $S(t) = \mathbb{P}(T > t) = \int_t^\infty f(x)dx$
- Properties of $S(t)$:
  - Non decreasing monotone
  - Continuous
  - $S(0) = 1$
  - $\lim_{t\to\infty} S(t) = 0$
- Mean time to failure $MTTF = \int_0^\infty tf(t)dt = \int_0^\infty S(t)dt$
- Mean time between failures $MTBF$

BASIC DEFINITIONS AND PROPERTIES

- Hazard function (hazard rate, failure rate):

$$
\begin{aligned}
h(t) &= \lim_{\Delta t \to 0} \frac{\mathbb{P}(t \le T < t + \Delta t | T \ge t)}{\Delta t} \\
&= \frac{1}{\mathbb{P}(T \ge t)} \lim_{\Delta t \to 0} \frac{\mathbb{P}(t \le T < t + \Delta t)}{\Delta t} \\
&= \frac{f(t)}{S(t)}
\end{aligned}
$$

- $h(t)$ is not a density, but

$$h(t)\Delta t \approx \mathbb{P}(t \le T < t + \Delta t | T \ge t)$$

BASIC DEFINITIONS AND PROPERTIES

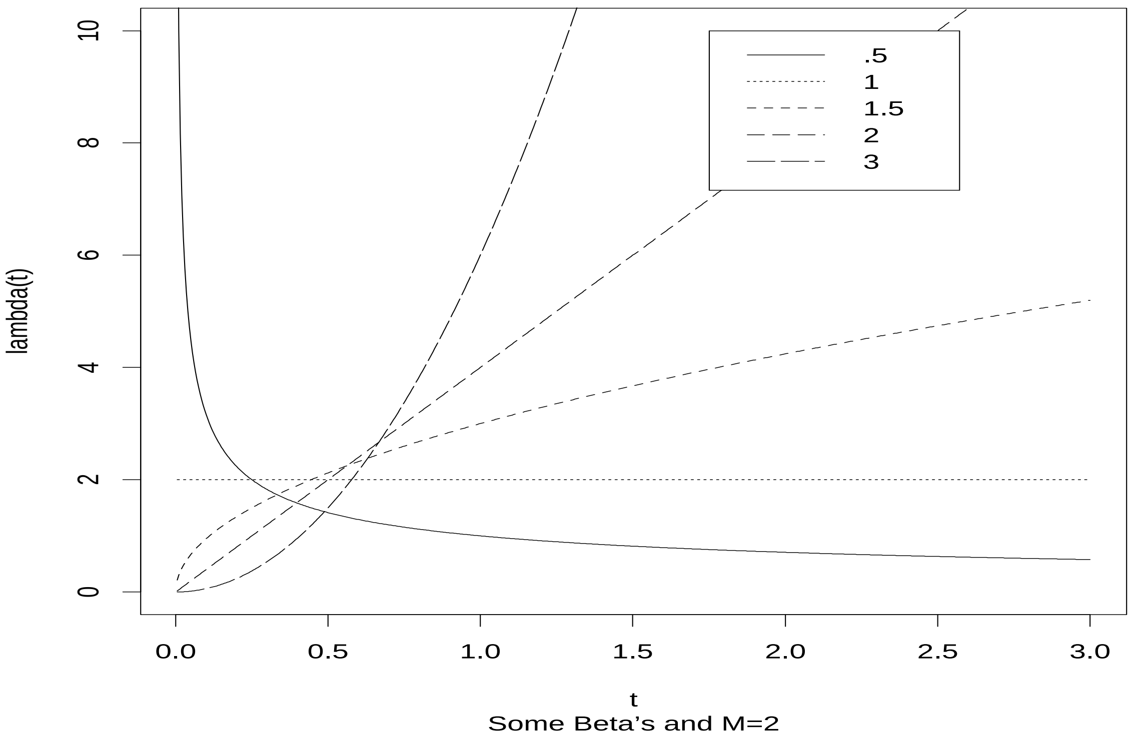
- $h(t) = \frac{f(t)}{S(t)}$
- $$\begin{aligned} f(t) = -S'(t) \quad &\Rightarrow \quad h(t) = \frac{-S'(t)}{S(t)} \\ &\Rightarrow \quad h(t) = \frac{-d \log S(t)}{dt} \\ &\Rightarrow \quad \log S(x) \,|_0^t = -\int_0^t h(x)dx \\ &\Rightarrow \quad \log S(t) = -\int_0^t h(x)dx \\ &\Rightarrow \quad S(t) = e^{-\int_0^t h(x)dx} \end{aligned}$$
- $f(t) = h(t)S(t) = h(t)e^{-\int_0^t h(x)dx}$

Example. *(Weibull failure time)*

- $T \sim \mathcal{W}(1, \beta)$ with $f(t) = \beta t^{\beta-1} e^{-t^{\beta}}$, $\beta > 0$, $t > 0$
- Starting from $f(t)$
  - $S(t) = \int_t^{\infty} \beta x^{\beta-1} e^{-x^{\beta}} = -e^{-x^{\beta}} |_t^{\infty} = e^{-t^{\beta}}$
  - $\Rightarrow h(t) = \dfrac{f(t)}{S(t)} = \dfrac{\beta t^{\beta-1} e^{-t^{\beta}}}{e^{-t^{\beta}}} = \beta t^{\beta-1}$
- Starting from $h(t) = \beta t^{\beta-1}$
  - $\Rightarrow f(t) = h(t) e^{-\int_0^t h(x)dx} = \beta t^{\beta-1} e^{-\int_0^t \beta x^{\beta-1} dx} = \beta t^{\beta-1} e^{-t^{\beta}}$
  - $\Rightarrow S(t) = e^{-\int_0^t \beta x^{\beta-1} dx} = e^{-t^{\beta}}$
  - $\Rightarrow T \sim \mathcal{W}(1, \beta)$

BASIC DEFINITIONS AND PROPERTIES

Weibull hazard rate for different β's



REPAIRABLE SYSTEMS

Failure of a water pump in a car

- Water pump $\Rightarrow$ non-repairable system
- Car $\Rightarrow$ repairable system

Most common models for repairable systems:

- Renewal Process *("Good as new")*
  - sequence of i.i.d. r.v.'s denoting time between two failures
- Non-homogeneous Poisson Process (NHPP) *("Bad as old")*

Both models have drawbacks:

Repair $\Rightarrow$ reliability growth but not "as new"

Different models used in disjoint time intervals

RENEWAL PROCESS

- Sequence of failure times $X_0 = 0 \leq X_1 \leq X_2 \leq \cdots$
- Interfailure times $T_i = X_i - X_{i-1}$ for $i = 1, 2, \ldots$
- If $T_1, T_2, \ldots$ sequence of i.i.d. random variables
- $\Rightarrow \{T_i\}$ stochastic process called *renewal process*
- HPP renewal process since interfailure times are i.i.d. exponential random variables

RENEWAL PROCESS

- Light bulbs replaced upon failure
- Replacement time negligible w.r.t. bulb lifetime
- All the bulbs have the same characteristics and operate under identical conditions
- $\Rightarrow$ assume interfailure times, $T_1, \ldots, T_n$, a sequence of i.i.d. Ex(λ) r.v.'s
- Likelihood $l(\lambda|data) = \lambda^n e^{-\lambda \sum_{i=1}^n T_i}$
- MLE: $\dfrac{n}{\sum_{i=1}^n T_i}$
- Conjugate gamma Ga(α, β) prior chosen for λ, with prior mean α/β
- $\Rightarrow$ Posterior gamma Ga$\big(\alpha + n, \beta + \sum_{i=1}^n T_i\big)$
- Posterior mean: $\dfrac{\alpha + n}{\beta + \sum_{i=1}^n T_i}$

RENEWAL PROCESS

- Posterior predictive density $f_{n+k}(x)$ of the $(n+k)$-th failure time $X_{n+k} = \sum_{i=1}^{n+k} T_i$, for any integer $k > 0$
  - Sum of k i.i.d. Ex(λ) random variables $\Rightarrow$ Ga(k, λ)
  - for $x > 0$,

$$f_{n+k}(x + \sum_{i=1}^{n} T_i | T_1, \ldots, T_n) = \int f_k(x|\lambda) f(\lambda | T_1, \ldots, T_n) d\lambda$$

$$\begin{aligned} &= \int \frac{\lambda^k}{\Gamma(k)} x^{k-1} e^{-\lambda x} \cdot \frac{(\beta + \sum_{i=1}^n T_i)^{\alpha+n}}{\Gamma(\alpha+n)} \lambda^{\alpha+n-1} e^{-(\beta + \sum_{i=1}^n T_i)\lambda} d\lambda \\ &= x^{k-1} \frac{\Gamma(\alpha+n+k)}{\Gamma(k)\Gamma(\alpha+n)} \frac{(\beta + \sum_{i=1}^n T_i)^{\alpha+n}}{(\beta + x + \sum_{i=1}^n T_i)^{\alpha+n+k}}. \end{aligned}$$

  - One-step-ahead predictive distribution is given by

$$f_{n+1}(x|T_1, \ldots, T_n) = (\alpha+n) \frac{(\beta + \sum_{i=1}^n T_i)^{\alpha+n}}{(\beta + x + \sum_{i=1}^n T_i)^{\alpha+n+1}}$$

FEATURES OF A NHPP

- NHPP used to model reliability growth/decay
- NHPP good for
  - prototype testing
  - repair of small components in complex systems
- Repair strategies in a NHPP:
  - instantaneous
  - minimal repair ($\Rightarrow$ back to previous reliability)

Repairs could worsen the reliability

NONHOMOGENEOUS POISSON PROCESS

- $N_t, t \geq 0$ # events by time t
- $N(y, s)$ # events in $(y, s]$
- $\Lambda(t) = \mathcal{E}N_t$ mean value function
- $\Lambda(y, s) = \Lambda(s) - \Lambda(y)$ expected # events in $(y, s]$

$N_t, t \geq 0$, NHPP with intensity function $\lambda(t)$ *iff*

1. $N_0 = 0$
2. independent increments
3. $\mathcal{P}\{\# \text{ events in } (t, t+h) \geq 2\} = o(h)$
4. $\mathcal{P}\{\# \text{ events in } (t, t+h) = 1\} = \lambda(t)h + o(h)$

$$\Rightarrow \mathcal{P}\{N(y, s) = k\} = \frac{\Lambda(y, s)^k}{k!} e^{-\Lambda(y,s)}, \forall k \in \mathcal{N}$$

NONHOMOGENEOUS POISSON PROCESS

$\lambda(t) \equiv \lambda \; \forall t \Rightarrow$ HPP

- $\lambda(t)$: intensity function of N_t
- $\lambda(t) := \lim_{\Delta \to 0} \frac{\mathcal{P}\{N(t, t+\Delta] \geq 1\}}{\Delta}, \; \forall t \geq 0$
- $\mu(t) := \frac{d\Lambda(t)}{dt}$: Rocof (rate of occurrence of failures)

Property 3. $\Rightarrow \mu(t) = \lambda(t)$ a.e. $\Rightarrow \Lambda(y, s) = \int_y^s \lambda(t)dt$

NONHOMOGENEOUS POISSON PROCESS

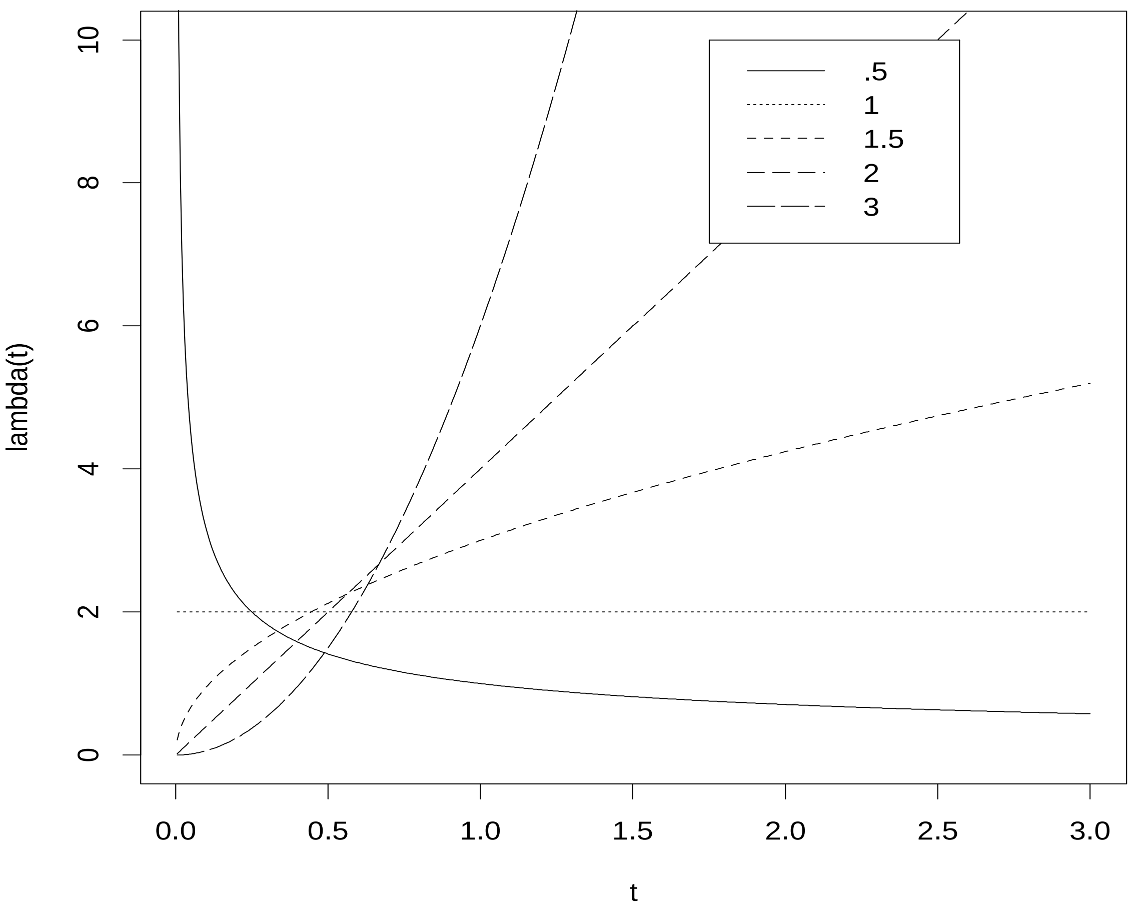
- Intensity function $\lambda(t; \theta) \Rightarrow$ inference on θ

- Failures $\underline{T} = (t_1, \ldots, t_n)$ in $(0, y] \Rightarrow$ likelihood $L(\theta \mid \underline{T}) = \prod_{i=1}^{n} \lambda(t_i) e^{-\Lambda(y)}$

- N_t Power Law Process (PLP)
  - $\lambda(t) = M\beta t^{\beta-1},\ M, \beta, t > 0$ or $\lambda(t) = (\beta/\alpha)(t/\alpha)^{\beta-1},\ \alpha, \beta, t > 0$
  - $\Lambda(t) = Mt^{\beta}$ or $\Lambda(t) = (t/\alpha)^{\beta}$
  - $\beta > 1 \Rightarrow$ reliability decay
    $\beta < 1 \Rightarrow$ reliability growth
    $\beta = 1 \Rightarrow$ constant reliability

- $L(\theta \mid \underline{T}) = M^n \beta^n \prod_{i=1}^{n} t_i^{\beta-1} e^{-My^{\beta}}$
  $L(\theta \mid \underline{T}) = M^n e^{-My}$ ($\beta = 1 \Rightarrow$ HPP with parameter M)

POWER LAW PROCESS



Some Beta's and M=2

FREQUENTIST ANALYSIS

Failures $\underline{T} = (t_1, \ldots, t_n) \Rightarrow$ likelihood

$$L(\alpha, \beta \mid \underline{T}) = (\beta/\alpha)^n \prod_{i=1}^{n} (t_i/\alpha)^{\beta-1} e^{-(y/\alpha)^\beta}$$

- *Failure truncation* $\Rightarrow y = t_n$

  MLE: $\widehat{\beta} = n / \sum_{i=1}^{n-1} \log(t_n/t_i)$ and $\widehat{\alpha} = t_n / n^{1/\widehat{\beta}}$

  C.I. for β : $\left(\widehat{\beta}\chi^2_{\gamma/2}(2n-2)/(2n), \widehat{\beta}\chi^2_{1-\gamma/2}(2n-2))/(2n)\right)$

- *Time truncation* $\Rightarrow y = t$

  MLE: $\widehat{\beta} = n / \sum_{i=1}^{n} \log(t/t_i)$ and $\widehat{\alpha} = t / n^{1/\widehat{\beta}}$

  C.I. for β : $\left(\widehat{\beta}\chi^2_{\gamma/2}(2n)/(2n), \widehat{\beta}\chi^2_{1-\gamma/2}(2n)/(2n)\right)$

Unbiased estimators, $\widehat{\lambda}(t)$, approx. C.I., hypothesis testing, goodness-of-fit, etc.

BAYESIAN ANALYSIS

Failure truncation $\equiv$ Time truncation

$$L(\alpha, \beta \mid \underline{T}) = (\beta/\alpha)^n \prod_{i=1}^{n} (t_i/\alpha)^{\beta-1}\, e^{-(y/\alpha)^\beta}$$

- $\pi(\alpha, \beta) \propto (\alpha\beta^\gamma)^{-1} \quad \alpha > 0, \beta > 0, \gamma = 0, 1 \Rightarrow \beta|\underline{T} \sim \widehat{\beta}\chi^2(2(n-\gamma))/(2n)$
- $\pi(\alpha) \propto \alpha^{-1}$ and $\beta \sim \mathcal{U}(\beta_1, \beta_2) \Rightarrow \pi(\beta|\underline{T}) \propto \beta^{n-1} w^\beta I_{[\beta_1,\beta_2]}(\beta)$
- $\pi(\alpha|\beta) \propto \beta s^{a\beta} \alpha^{-a\beta-1} e^{-b(s/\alpha)^\beta} \quad a, b, s > 0$ and $\beta \sim \mathcal{U}(\beta_1, \beta_2)$

$$\Rightarrow \pi(\beta|\underline{T}) \propto \beta^n \prod_{i=1}^{n} \left(\frac{t_i}{s}\right)^\beta \left[(\frac{t_n}{s})^\beta + b\right]^{-n-a} I_{[\beta_1,\beta_2]}(\beta)$$

$\Rightarrow \alpha|\underline{T}$ numerically

Other parametrisation: $M, \beta \sim \mathcal{G}$
$\Rightarrow M|\beta, \underline{T} \sim \mathcal{G}$ and $\beta|M, \underline{T}$ known apart from a constant
$\Rightarrow$ Gibbs sampling with Metropolis step

Interest in posterior $\mathcal{E}\beta, \mathcal{P}\{\beta < 1\}$, modes, C.I.'s, $\mathcal{E}M$ (for $\lambda(t) = M\beta t^{\beta-1}$)

RELIABILITY MEASURES

- System reliability
  - Data on the same system ($y = t$ or t_n):

    $$R(y, s) = \mathcal{P}\{N(y, s) = 0|\alpha, \beta\} = e^{-\left(\frac{s}{\alpha}\right)^{\beta} + \left(\frac{y}{\alpha}\right)^{\beta}}$$

  - Data on equivalent system:

    $$R(s) = \mathcal{P}\{N_s = 0\} = e^{-(s/\alpha)^{\beta}}$$

- Expected number of failures in future intervals
  - Same system: $\mathcal{E}[N(y, s)|\alpha, \beta] = (s/\alpha)^{\beta} - (y/\alpha)^{\beta}$
  - Equivalent system: $\mathcal{E}[N_s|\alpha, \beta] = (s/\alpha)^{\beta}$

- Intensity function at y:

  Reliability growth models without further improvements $\Rightarrow$ constant intensity $\lambda(y)$

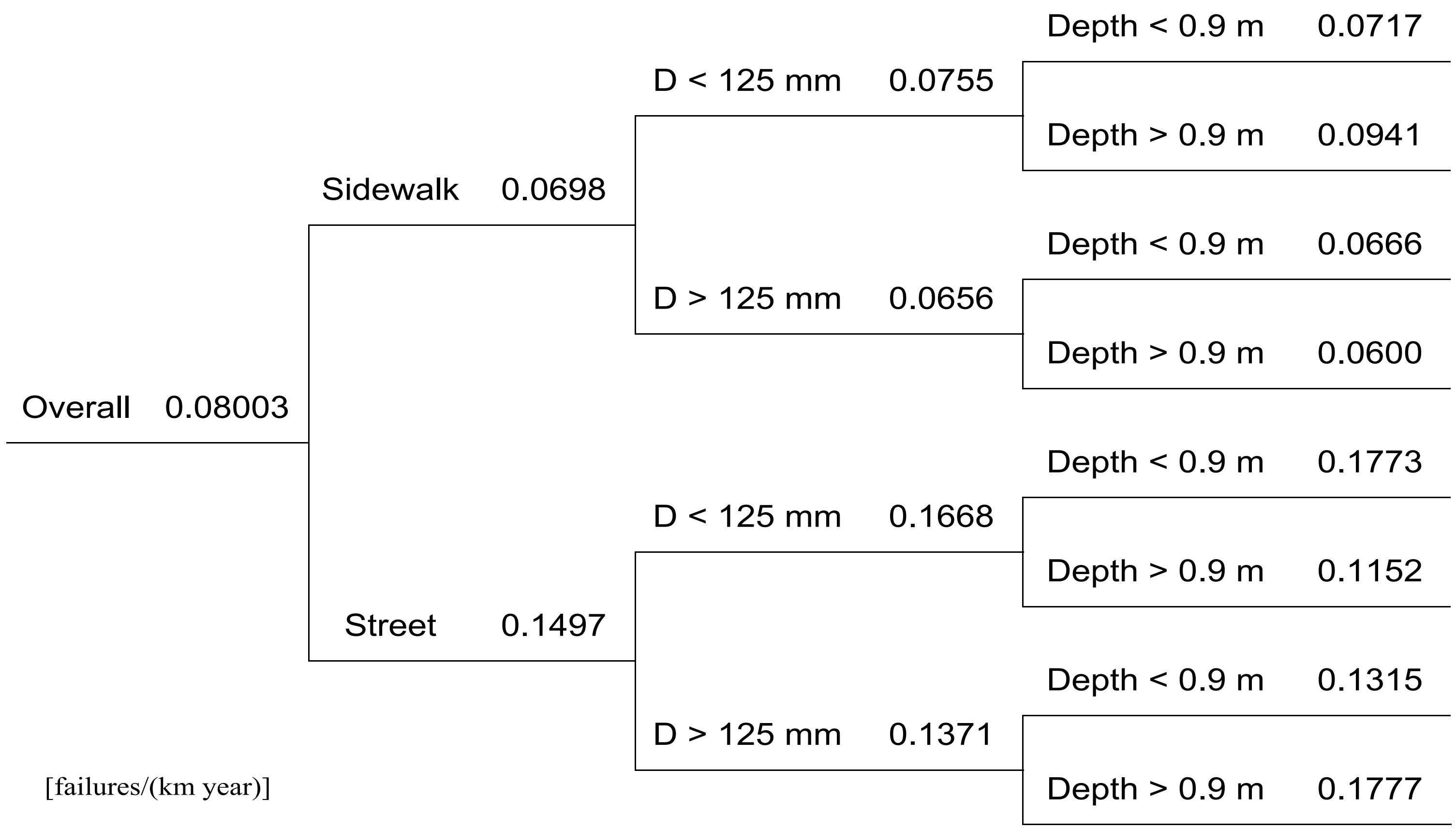
CASE STUDY 1: GAS ESCAPES IN A CITY NETWORK

- **Deciding** an efficient replacement policy to reduce (monetary and social) **risks** in a large metropolitan gas distribution network developed in the last century
- Assessment of failure rate of the pipelines, different for materials (cast iron, steel, polyethylene, etc.) and conditions (diameter, laying depth, etc.)
- **Decision:** Change of pipelines with highest failure rate
- (Traditional) cast iron pipelines with higher failure rate than other materials
  - not subject to corrosion (aging)
  - propensity-to-failure in a unit time period or unit length does not vary significantly with time and space
  - rare events occurring *randomly* and not affecting the next ones
  - $\Rightarrow$ HPP
- EDA identified diameter, laying depth and location as the most significant factors

FAILURES IN CAST-IRON PIPES

- HPP with parameter λ (unit failure rate in time and space)
- n failures in $[0, T] \times \mathcal{S}, \Rightarrow L(\lambda|n, T, \mathcal{S}) = (\lambda s T)^n e^{-\lambda s T}$, with $s = meas(\mathcal{S})$
- Data: $n = 150$ failures in $T = 6$ years on a net $\approx s = 312$ Km long
- $\Rightarrow L(\lambda|n, T, \mathcal{S}) = (1872\lambda)^{150} e^{-1872\lambda}$
- MLE $\widehat{\lambda} = n/(sT) = 150/1872 = 0.080$
- $\lambda \sim \mathcal{G}(\alpha, \beta) \Rightarrow \lambda|n, T, \mathcal{S} \sim \mathcal{G}(\alpha + n, \beta + sT)$
- Consider 8 classes determined by two levels of relevant covariates: diameter, location and depth

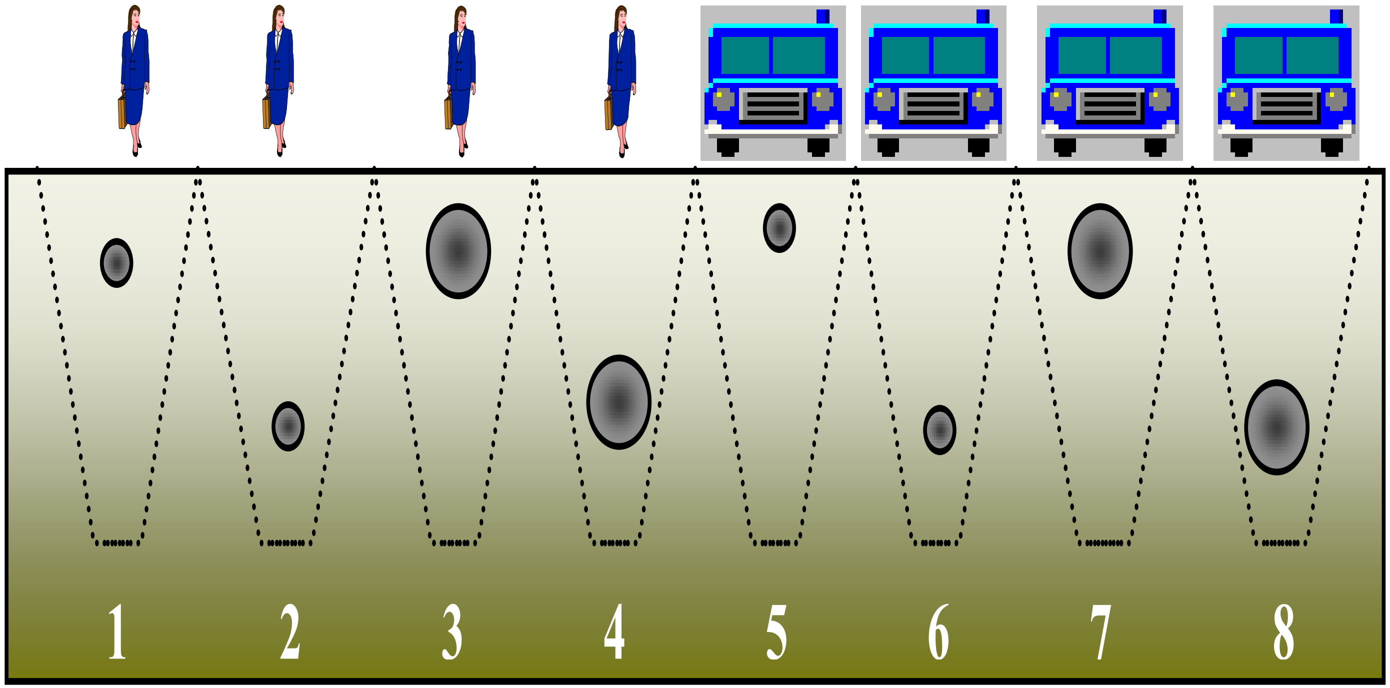
FAILURES IN CAST-IRON PIPE



ELICITATION OF EXPERTS' OPINIONS

- A questionnaire was given to 26 experts from different areas within the company
- Interviewees were unable to say how many failures they expected to see on a kilometer of a given kind of pipe in a year (even upper and lower bounds on them!)
- The experts had great difficulty in saying how and how much a factor influenced the failure and expressing opinions directly on the model parameters while they were able to compare the performance against failure of different pipeline classes
- To obtain such a propensity-to-failure index, each expert was asked to compare the pipeline classes pairwise. In a pairwise comparison the judgement is the expression of the relation between two elements that is given, for greater simplicity, in a linguistic shape
- The linguistic judgement scale is referred to a numerical scale (Saaty's proposal: Analytic Hierarchy Process) and the numerical judgements can be reported in a single matrix of pairwise comparisons

ELICITATION OF EXPERTS' OPINIONS



ANALYTIC HIERARCHY PROCESS

- Two alternatives A and B

B	"equally likely as"	A $\rightarrow$ 1
B	"a little more likely than"	A $\rightarrow$ 3
B	"much more likely than"	A $\rightarrow$ 5
B	"clearly more likely than"	A $\rightarrow$ 7
B	"definitely more likely than"	A $\rightarrow$ 9

- Pairwise comparison for alternatives $A_1, \ldots, A_n$
- $\Rightarrow$ square matrix of size n
- $\Rightarrow$ (normalized) eigenvector associated with the largest eigenvalue
- $\Rightarrow (P(A_1), \ldots, P(A_n))$
- **Question**: if a gas escape occurs, where do think it will occur if you have to choose between subnetwork A and subnetwork B?

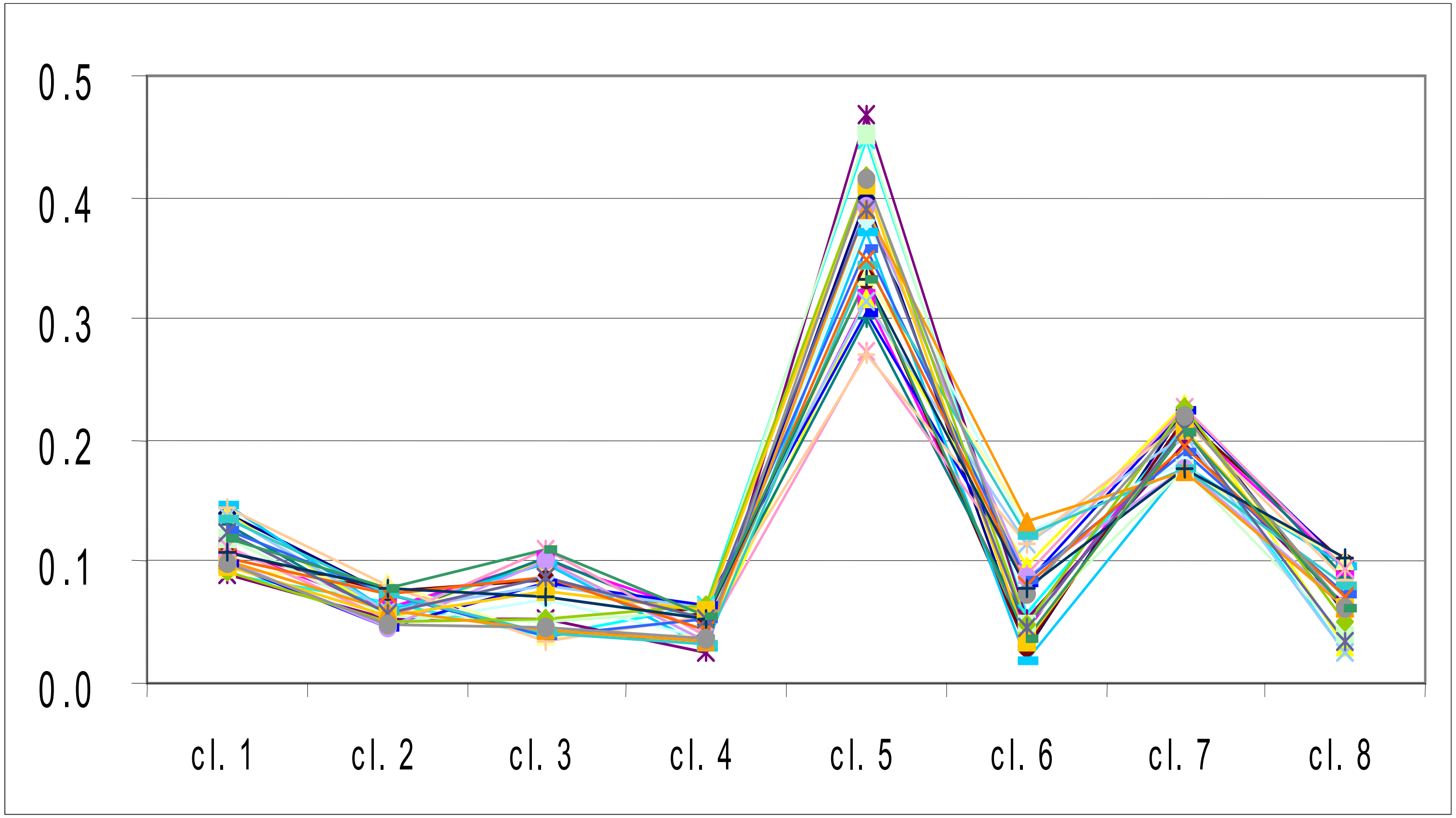
ANALYTIC HIERARCHY PROCESS

An expert's opinion on propensity to failure of cast-iron pipes

Class	1	2	3	4	5	6	7	8
1	1	3	3	3	1/6	1	1/6	3
2	1/3	1	1/4	2	1/6	1/2	1/5	1
3	1/3	4	1	1	1/4	1	1/6	2
4	1/3	1/2	1	1	1/5	1	1/5	1
5	6	6	4	5	1	4	4	5
6	1	2	1	1	1/4	1	1/6	1
7	6	5	6	5	1/4	6	1	4
8	1/3	1	1/2	1	1/5	1	1/4	1

ELICITATION OF EXPERTS' OPINIONS

Values elicited by experts ⇒ similar opinions



MODELS FOR CAST-IRON PIPES

Independent classes $A_i, i = 1, 8$, given by 3 covariates (diameter, location and depth)
$\Rightarrow$ find the “most risky” class

- Failures in the network occur at rate λ and allocated to class A_i with probability $P(A_i) \Rightarrow$ failures in class A_i occur at rate $\lambda_i = \lambda P(A_i)$ (*Coloring Theorem*)
- $P(A_i)$ given by AHP for any expert
- Choice of $\lambda \Rightarrow$ *critical*
  - Proper way to proceed:
    - Use experts’ opinions through AHP to get a Dirichlet prior on $p_i = P(A_i)$
    - Ask the experts about the expected number of gas escapes for given period and length of network $\Rightarrow$ statements on λ, unit failure rate for entire network, and get a gamma prior on it
  - What we did
    - Estimate λ by MLE $\widehat{\lambda}$ with a unique HPP for the network
    - Use experts’ opinions through AHP to get a prior on $\lambda_i = \widehat{\lambda} P(A_i)$

MODELS FOR CAST-IRON PIPES

- Choice of priors
  - Gamma vs. Lognormal [informal sensitivity]
  - For each expert, eigenvector from AHP multiplied by $\widehat{\lambda} \Rightarrow$ *sample* about $(\lambda_1, \ldots, \lambda_8)$
  - Mean and variance of priors on λ_i's estimated from the *sample* of size 26 (number of experts) using the method of moments
- Posterior mean of failure rate λ_i for each class
- Classes ranked according to posterior means (largest $\Rightarrow$ most keen to gas escapes)
- Sensitivity
  - Classes of Gamma priors on λ with mean and/or variance in intervals
  - Quantile class on λ
- Non-dominated (Bayes) actions under classes of priors/losses

ESTIMATES' COMPARISON

- Location: **W** (under walkway) or **T** (under traffic)
- Diameter: **S** (small, $<$ 125 mm) or **L** (large, $\geq$ 125 mm)
- Depth: **N** (not deep, $<$ 0.9 m) or **D** (deep , $\geq$ 0.9 m)

Class	MLE	Bayes ($\mathcal{LN}$)	Bayes ($\mathcal{G}$)
TSN	*.177*	**.217**	**.231**
TSD	*.115*	*.102*	*.104*
TLN	*.131*	*.158*	*.143*
TLD	**.178**	*.092*	*.094*
WSN	.072	.074	.075
WSD	.094	.082	.081
WLN	.066	.069	.066
WLD	.060	.049	.051

Highest value; *2^{nd}-4^{th} values*

- Location is the most relevant covariate
- TLD: 3 failures along 2.8 Km but quite unlikely to fail according to the experts
- $\mathcal{LN}$ and $\mathcal{G}$ $\Rightarrow$ similar answers

GAS ESCAPES IN STEEL PIPES

- Replacements not affecting network reliability $\Rightarrow$ repairable system
- Steel pipes subject to corrosion (aging) $\Rightarrow$ NHPP and relevance of installation date
- Network split into subnetworks based upon year of installation, *as if* pipes were installed on July, 1st each year
- Independent PLP's $N_i(t)$ for each subnetwork, with $\lambda_i(t) = M_i \beta_i t^{\beta_i - 1}$
- *Superposition Theorem*: Sum of independent NHPPs $N_i(t)$ with intensity functions $\lambda_i(t)$ is still a NHPP $N(t)$ with intensity function $\lambda(t) = \sum \lambda_i(t)$
- Characteristics of pipes installed vs. PLP parameters
  - Equal pipes $\Rightarrow$ same M and β for each $N_i(t)$
  - Completely different pipes $\Rightarrow$ different, independent M_i and β_i for each $N_i(t)$
  - Similar pipes $\Rightarrow$ M_i and β_i for each $N_i(t)$ coming from a common distribution (exchangeability)

GAS ESCAPES IN STEEL PIPES

- Experts asked about interval of first gas escape X_1
  - Choice of section of the network (e.g. length l)
  - Choice of time intervals in a list (e.g. $[T_0, T_1]$)
  - Degree of belief on each interval (choice among 95%, 85% and 75%); e.g. for PLP (M, β)
    $\Rightarrow P(X_1 \in [T_0, T_1]) = \text{exp}\{-lMT_0^\beta\} - \text{exp}\{-lMT_1^\beta\} = 0.95$
  - Check for consistency, e.g. $A \subset B \not\Rightarrow P(A) > P(B)$

- Pooling of experts' opinions
  - $\Rightarrow$ *sample* from priors
  - $\Rightarrow$ hyperparameters in priors, matching moments

GAS ESCAPES IN STEEL PIPES

- *Known* length l_s of network installed in year $s = 1, \ldots, r$
- *Known* installation date δ_k of k-th failed pipe
- Likelihood $L(\underline{M}, \underline{\beta}; \underline{t}, \underline{\delta}) = \prod_{k=1}^{n} \beta_{\delta_k} l_{\delta_k} M_{\delta_k} (t_k - \delta_k)^{\beta_{\delta_k} - 1} e^{-\sum_{s=1}^{r} l_s M_s [(T_1 - s)^{\beta_s} - (s \vee T_0 - s)^{\beta_s}]}$
- $M_s \sim \mathcal{E}(\theta_M) \perp \beta_s \sim \mathcal{E}(\theta_\beta)$, $s = 1, \ldots, r$, but exchangeable among themselves
- $\theta_M \sim \mathcal{E}(\tau_M)$ and $\theta_\beta \sim \mathcal{E}(\tau_\beta)$
- Posterior $\pi(\underline{M}, \underline{\beta} | \underline{t}, \underline{\delta})$ obtained integrating out θ_M and θ_β
- $\pi(\underline{M}, \underline{\beta} | \underline{t}, \underline{\delta}) \propto \left(\prod_{s=1}^{r} (l_s M_s \beta_s)^{|I_s|} \right) \left(\prod_{k=1}^{n} (t_k - \delta_k)^{\beta_{\delta_k} - 1} \right) e^{-\sum_{s=1}^{r} l_s M_s [(T_1 - s)^{\beta_s} - (s \vee T_0 - s)^{\beta_s}]}$.

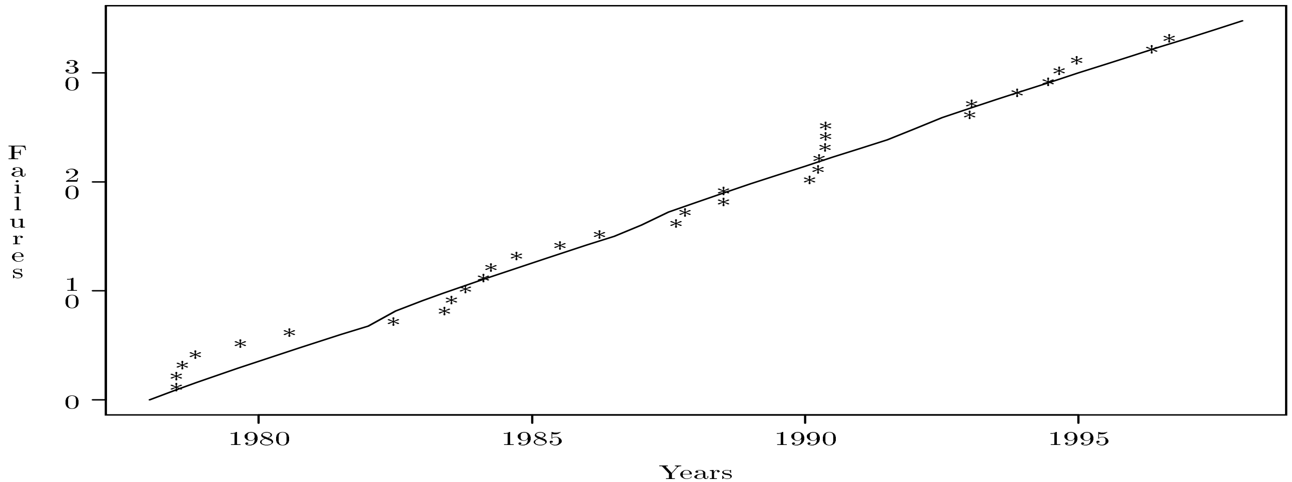
$$\cdot \tau_M \tau_\beta \frac{r!}{\left[\sum_{s=1}^{r} (M_s + \tau_M / r)\right]^{r+1}} \frac{r!}{\left[\sum_{s=1}^{r} (\beta_s + \tau_\beta / r)\right]^{r+1}}$$

MODELS FOR STEEL PIPES

Exchangeable M and β; known installation dates

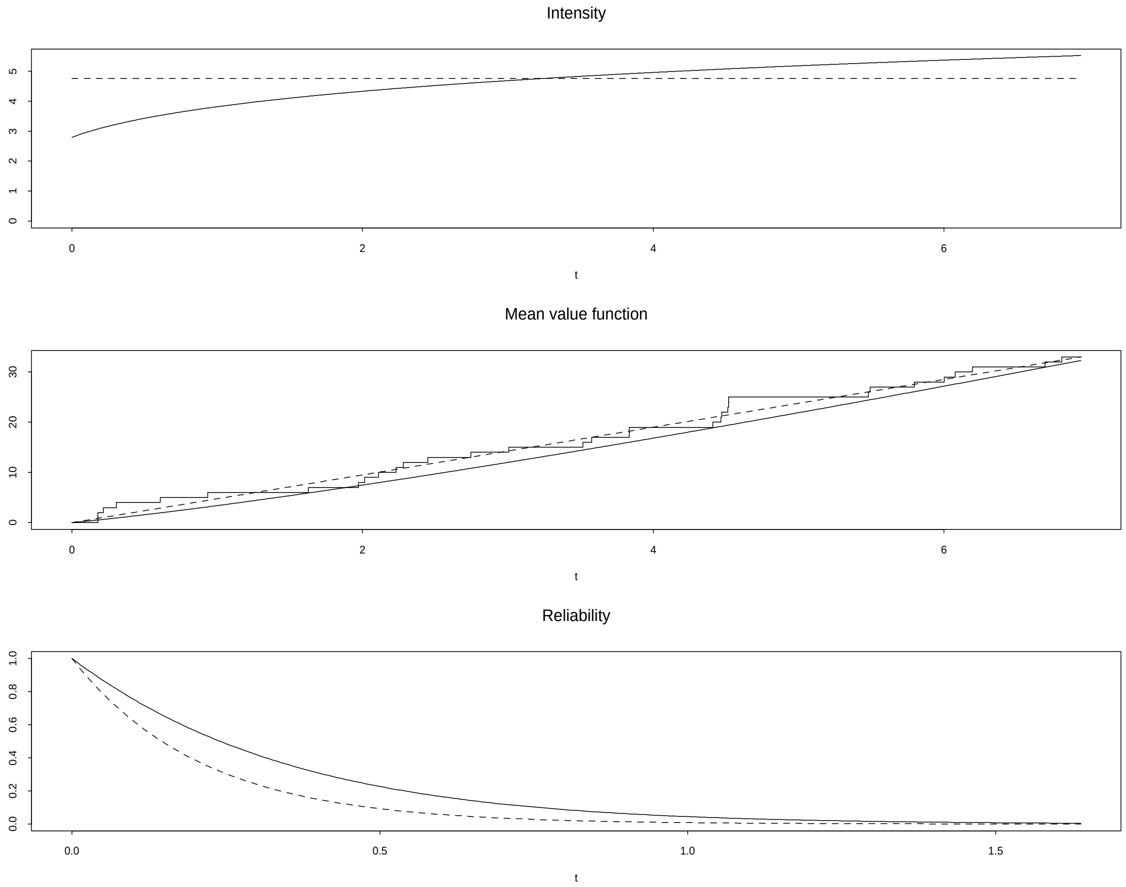
95% credible intervals for reliability measures:

- System reliability over 5 years: $P\{N(1998, 2002) = 0\} \Rightarrow [0.0000964, 0.01]$
- Expected number of failures in 5 years: $EN(1998, 2002) \Rightarrow [4.59, 9.25]$
- Mean value function (solid) vs. cumulative # failures (points)



A VERY SIMPLE NHPP MODEL

MLE (dashed) vs. Bayes (solid) for $\lambda_\theta(t) = a \ln(1 + bt) + c$



NONPARAMETRIC APPROACH

events in $[T_0, T_1] \sim \mathcal{P}(\Lambda[T_0, T_1])$, with $\Lambda[T_0, T_1] = \Lambda(T_1) - \Lambda(T_0)$

Parametric case: $\Lambda[T_0, T_1] = \int_{T_0}^{T_1} \lambda(t) dt$

Nonparametric case: $\Lambda[T_0, T_1] \sim \mathcal{G}(\cdot, \cdot) \Rightarrow \Lambda$ d.f. of the random measure M

Notation: $\mu B := \mu(B)$

Definition 1 *Let α be a finite, σ-additive measure on $(\mathbb{S}, \mathcal{S})$. The random measure μ follows a* **Standard Gamma** *distribution with* shape α *(denoted by $\mu \sim \mathcal{GG}(\alpha, 1)$) if, for any family $\{S_j, j = 1, \ldots, k\}$ of disjoint, measurable subsets of $\mathbb{S}$, the random variables μS_j are independent and such that $\mu S_j \sim \mathcal{G}(\alpha S_j, 1)$, for $j = 1, \ldots, k$.*

Definition 2 *Let β be an α-integrable function and $\mu \sim \mathcal{GG}(\alpha, 1)$. The random measure $M = \beta\mu$, s.t. $\beta\mu(A) = \int_A \beta(x)\mu(dx), \forall A \in \mathcal{S}$, follows a* **Generalised Gamma** *distribution, with* shape α *and* scale β *(denoted by $M \sim \mathcal{GG}(\alpha, \beta)$).*

NONPARAMETRIC APPROACH

Consequences:

- $\mu \sim \mathcal{P}_{\alpha,1}, \mathcal{P}_{\alpha,1}$ unique p.m. on $(\Omega, \mathcal{M})$, space of finite measures on $(\mathbb{S}, \mathcal{S})$, with these finite dimensional distributions

- $M \sim \mathcal{P}_{\alpha,\beta}$, **weighted random measure**, with $\mathcal{P}_{\alpha,\beta}$ p.m. induced by $\mathcal{P}_{\alpha,1}$

- $EM = \beta\alpha$, i.e. $\int_\Omega M(A)\mathcal{P}_{\alpha,\beta}(dM) = \int_A \beta(x)\alpha(dx), \forall A \in \mathcal{S}$

Theorem 1 *Let $\underline{\xi} = (\xi_1, \ldots, \xi_n)$ be n Poisson processes with intensity measure M. If $M \sim \mathcal{GG}(\alpha, \beta)$ a priori, then $M \sim \mathcal{GG}(\alpha + \sum_{i=1}^n \xi_i, \beta/(1 + n\beta))$ a posteriori.*

NONPARAMETRIC APPROACH

Data: $\{y_{ij}, i = 1 \ldots k_j\}_{j=1}^{n}$ from $\underline{\xi} = (\xi_1, \ldots, \xi_n)$

Bayesian estimator of M: measure $\widetilde{M}$ s.t., $\forall S \in \mathcal{S}$,

$$\widetilde{M}S = \int_S \frac{\beta(x)}{1 + n\beta(x)}\alpha(dx) + \sum_{j=1}^{n}\sum_{i=1}^{k_j} \frac{\beta(y_{ij})}{1 + n\beta(y_{ij})}\mathbb{I}_S(y_{ij})$$

Constant $\beta \Rightarrow \widetilde{M}S = \frac{\beta}{1 + n\beta}[\alpha S + \sum_{j=1}^{n}\sum_{i=1}^{k_j} \mathbb{I}_S(y_{ij})]$

Bayesian estimator of reliability R, $RS = P(\xi S = 0)$, $S \in \mathcal{S}$:

$$\widetilde{R}S = \exp\left\{ - \int_S \ln(1 + \frac{\beta(x)}{1 + n\beta(x)})\alpha(dx) - \sum_{j=1}^{n}\sum_{i=1}^{k_j} \ln(1 + \frac{\beta(y_{ij})\mathbb{I}_S(y_{ij})}{1 + n\beta(y_{ij})}) \right\}$$

Constant $\beta \Rightarrow \widetilde{R}S = \left(1 + \frac{\beta}{1+n\beta}\right)^{-(\alpha S + \sum_{j=1}^{n} \xi_j S)}$

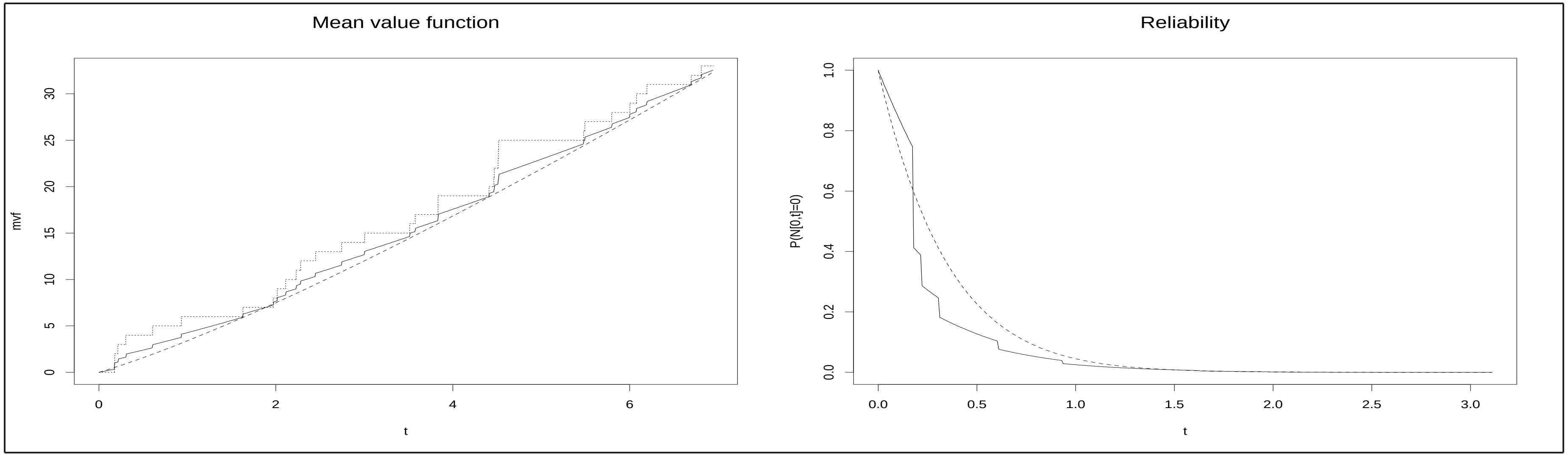
STEEL PIPES

Parametric NHPP: $\widetilde{\Lambda_\theta}(t) = \int_0^t [\tilde{a} \log(1 + \tilde{b}t)]dt + \hat{c}t$

Nonparametric model: $M \sim \mathcal{P}_{\alpha,\beta} : \alpha(ds) := \widetilde{\Lambda_\theta}(s)/\sigma\, ds$, $\beta(s) := \sigma$

$\Rightarrow \mathcal{E}MS = \widetilde{\Lambda_\theta}S$ and $VarMS = \sigma\widetilde{\Lambda_\theta}S$

$\Rightarrow MS$ “centered” at parametric estimator $\widetilde{\Lambda_\theta}S$ and closeness given by σ



Nonparametric (solid) and parametric (dashed) estimators and cumulative $N[0,t]$ *(dotted).*

PARAMETRIC VS. NONPARAMETRIC

$[0,T]$ split into n disjoint I_j, $j = 1, \dots, n$

Data: $\underline{k} = (k_1, \dots, k_n)$, with $k_j = \{\#\text{obs. in } I_j\} \Rightarrow f(\underline{k} \mid \Lambda) = e^{-\Lambda(T)} \prod_{j=1}^{n} \frac{(\Lambda I_j)^{k_j}}{k_j!}$

Parametric: $P(\underline{k} \mid H_P) = \int_{\mathbb{R}_+^3} e^{-\Lambda_\theta(T)} \prod_{j=1}^{n} \frac{[\Lambda_\theta I_j]^{k_j}}{k_j!} \pi(\theta)\, d\theta$

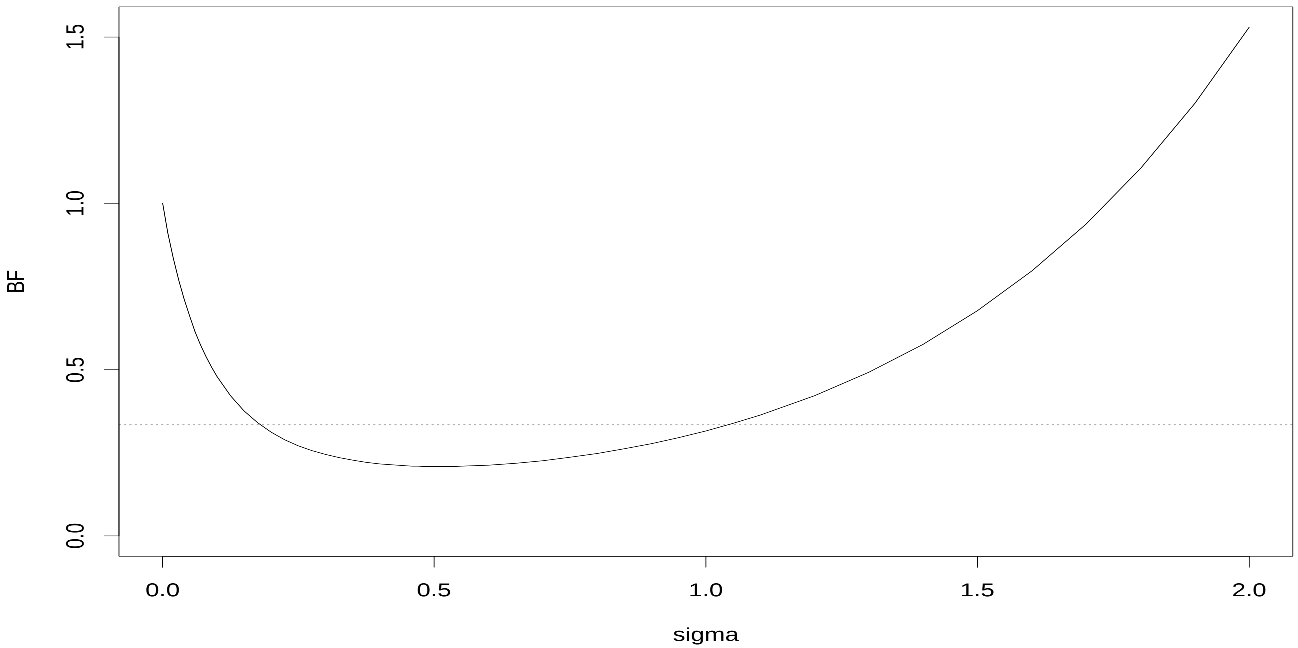
Nonparametric: $\underline{k} \mid M, \theta \sim f(\underline{k} \mid M_\theta)$, $M \mid \theta \sim \mathcal{GG}(\Lambda_\theta/\sigma, \sigma)$ and $\theta \sim \pi$:

$$P(\underline{k} \mid H_N) = \int_{\mathbb{R}_+^3} \prod_{j=1}^{n} \left[\frac{\prod_{i=0}^{k_j-1} (\Lambda_\theta I_j + i\sigma)}{k_j! \exp\left[\left(\frac{\Lambda_\theta I_j}{\sigma} + k_j\right) \ln(1+\sigma)\right]} \right] \pi(\theta)\, d\theta$$

Bayes Factor: $BF_{PN} = \dfrac{P(\underline{k} \mid H_P)}{P(\underline{k} \mid H_N)} = \dfrac{\int_{\mathbb{R}_+^3} e^{-\Lambda_\theta(T)} \prod_{j=1}^{n} (\Lambda_\theta I_j)^{k_j} \pi(\theta)\, d\theta}{\int_{\mathbb{R}_+^3} \prod_{j=1}^{n} \left[(1+\sigma)^{-(\Lambda_\theta I_j/\sigma + k_j)} \prod_{i=0}^{k_j-1} (\Lambda_\theta I_j + i\sigma)\right] \pi(\theta)\, d\theta}$

PARAMETRIC VS. NONPARAMETRIC

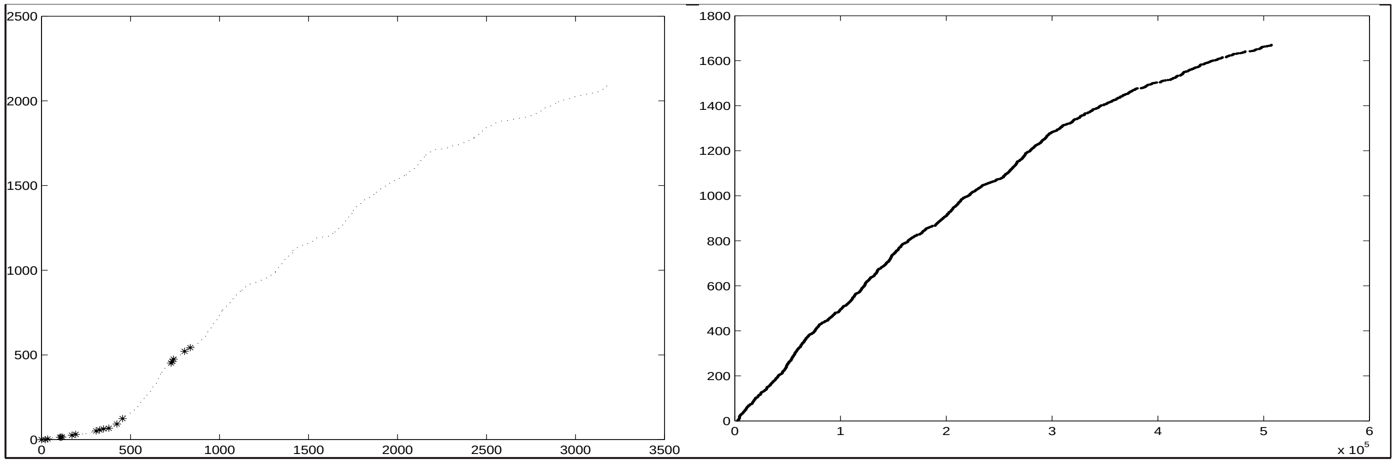
Bayes factor BF_{PN} as a function of σ



CASE STUDY 2: SUBWAY TRAINS' FAILURES

Data: more than 2000 door failures of 40 trains, put on service from 1/4/1990 to 20/7/1992, observed up to 31/12/1998

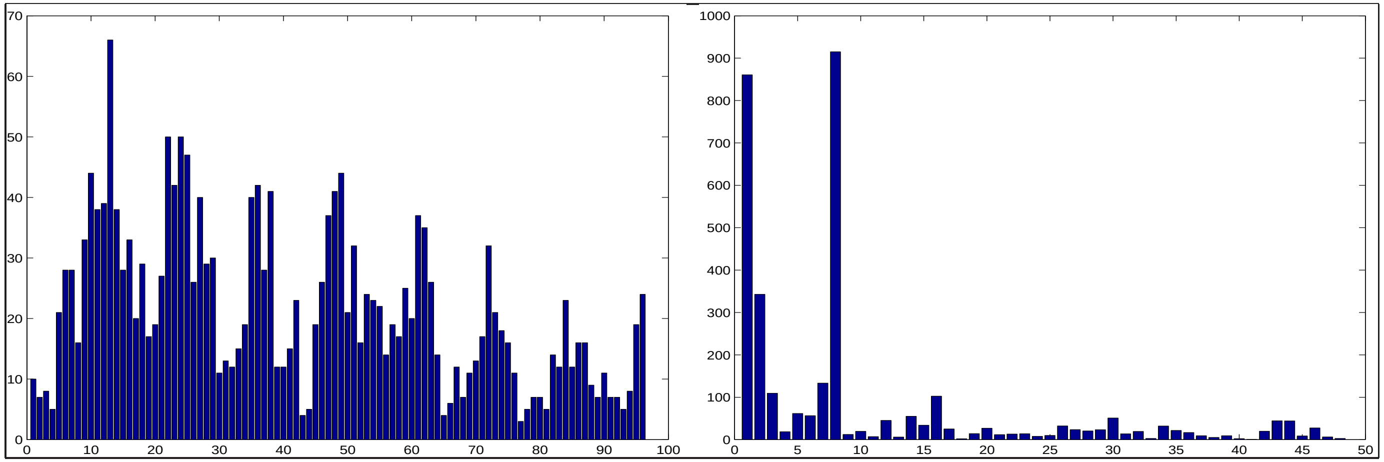
Goal: checking components reliability before warranty's expiration



Failures vs. days (left) and failures vs. kilometers (right)

- Concavity denotes improvement over time
- Oscillations
- Transient behaviour during first 500 days

SEASONALITY



Left: Monthly no. of failures for the 40 trains starting January 1991

Right: Spectrum of the time series of the monthly number of failures from 1991 to 1998

- Decreasing trend
- Periodicity (estimated at 12 months by the spectrum)
- NHPP: $\lambda(t) = \exp\{\alpha + \rho \sin(\omega t + \theta)\}$

MODEL FOR DOORS FAILURES

Marked Poisson process on time scale

$\lambda(t;\theta_1,\theta_2) = \mu(g(t);\theta_1)s(t;\theta_2)$

- $\mu(k;\theta_1) = \beta_0 \dfrac{\log(1+\beta_1 k)}{(1+\beta_1 k)}$
  - $\mu(0;\theta_1) = 0$, maximum at $(e-1)/b_1$ and $\lim_{k\to\infty} u(k;\theta_1) = 0$
  - m.v.f. $\Lambda(k) = \beta_0 \log^2(1+\beta_1 k)/(2\beta_1)$

    suitable for actual cumulative number of failures
- $s(t;\theta_2) = \exp\{\rho\cos(\omega t + \varphi)\}$(periodic component)
- From EDA we could take $k = g(t) = at + bt^2$ and substitute above
- We actually took kilometers $k|t \sim \mathcal{N}(g(t), \sigma^2)$

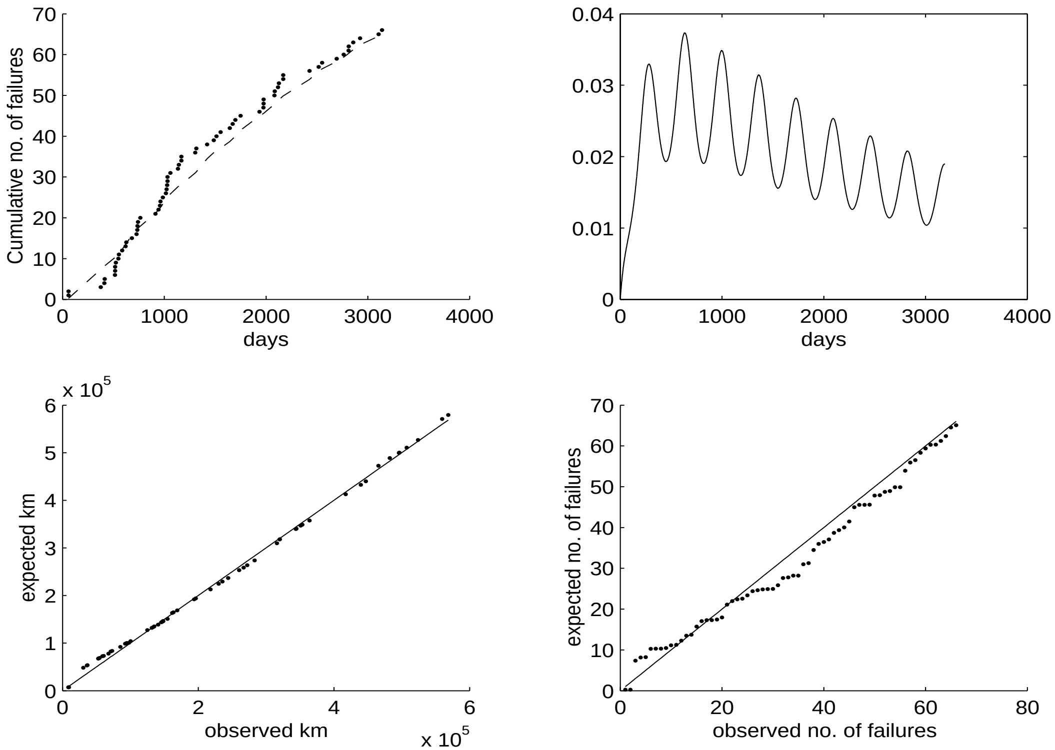
MODEL FOR DOORS FAILURES

- j-th train monitored in $[0, T_j]$
- Failures at times $(t_1, \ldots, t_{n_j}) = \mathbf{t_j}$ and kilometers $(k_1, \ldots, k_{n_j}) = \mathbf{k_j}$
- Likelihood for j-th train

  $L_j(\theta_1, \theta_2) = \prod_{i=1}^{n_j} \mu(g(t_i); \theta_1) s(t_i; \theta_2) \exp \left[- \int_0^{T_j} \mu(g(t); \theta_1) s(t; \theta_2) dt \right]$
- Non-Bayesian analysis

Parameter	MLE	C.I.	Parameter	MLE	C.I.
$a \times 10^{-2}$	1.209	[1.171, 1.247]	$b \times 10^2$	2.025	[1.862, 2.188]
$\sigma^2 \times 10^{-7}$	5.809	[4.214, 8.345]	$\rho \times 10$	3.234	[0.000, 6.779]
$\beta_0 \times 10^2$	7.358	[5.640, 9.076]	$\beta_1 \times 10^5$	2.239	[1.938, 2.540]

DIAGNOSTIC PLOTS FOR ONE TRAIN



Estimated m.v.f. vs. observed failures (top left), estimated intensity function (top right), expected vs. observed odometer readings at failure times (bottom left) and expected vs. observed number of failures (bottom right)

DIAGNOSTIC FOR ONE TRAIN

Theorem 1 *Let* $\Lambda(t)$ *be a continuous nondecreasing function. Then* $T_1, T_2, \ldots$ *are arrival times in a Poisson process* N_t *with m.v.f.* $\Lambda(t)$ *if and only if* $\Lambda(T_1), \Lambda(T_2), \ldots$ *are arrival times in an HPP* H_t *with failure rate one.*

- $\widehat{\Lambda}(t)$ estimated from data $T_1, T_2, \ldots$
- *Suppose* $T_1, T_2, \ldots$ from NHPP with m.v.f. $\widehat{\Lambda}(t)$
- $Y_1 = \widehat{\Lambda}(T_1), Y_2 = \widehat{\Lambda}(T_2), \ldots$ *data* from HPP with rate 1
- Interarrival times $X_i = Y_i - Y_{i-1}$ i.i.d. $\mathcal{E}(1)$
- $U_i = \exp\{-X_i\}$ i.i.d. $\mathcal{U}[0,1]$
- Should $\widehat{\Lambda}(t)$ be the right model, then U_i's should be uniform r.v.'s
- Kolmogorov-Smirnov test to check if data are coming from uniform distribution
- **Unsatisfactory results**

A BAYESIAN MODEL

- Interest in
  - checking if trains fulfill reliability requirements before warranty expiration
  - mathematical model able to predict failures based upon current failure data and knowledge

- $\Rightarrow$ a (more complex) Bayesian model
  - first 2 years of data used to estimate parameters
  - number of failures predicted in the following 1, 2, 3, 4, 5 years (for which observed data are available)
  - compute $\mathcal{E}(N(2,2+i)|N(0,2)) = \int \Lambda((2,2+i)|\theta)\pi(\theta|N(0,2))d\theta$, with 95% credible interval (from simulations), for $i = 1,5$
  - comparison between predicted and actual observed failure data (cumulative number)
  - *good forecast*

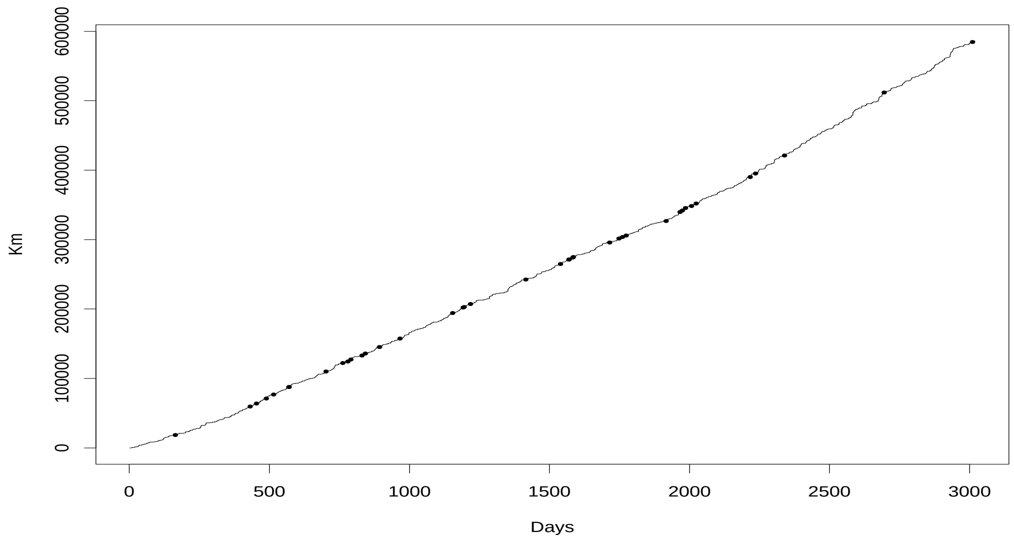
HIERARCHICAL MODEL

- Hierarchical model with $g(t)$ realization of a Gamma process

$$
\begin{aligned}
g(t) &\sim \mathcal{G}(at, b) \\
\theta &\sim \pi(\theta) \\
[\mathbf{t} \mid \mathbf{g}, \theta] &= NHPP\{\mu(g(t); \theta_1) s(t; \theta_2)\} \\
[\mathbf{k} \mid \mathbf{t}, \mathbf{g}] &= \prod_{i=1}^{n} \delta_{g(t_i)}(\cdot)
\end{aligned}
$$

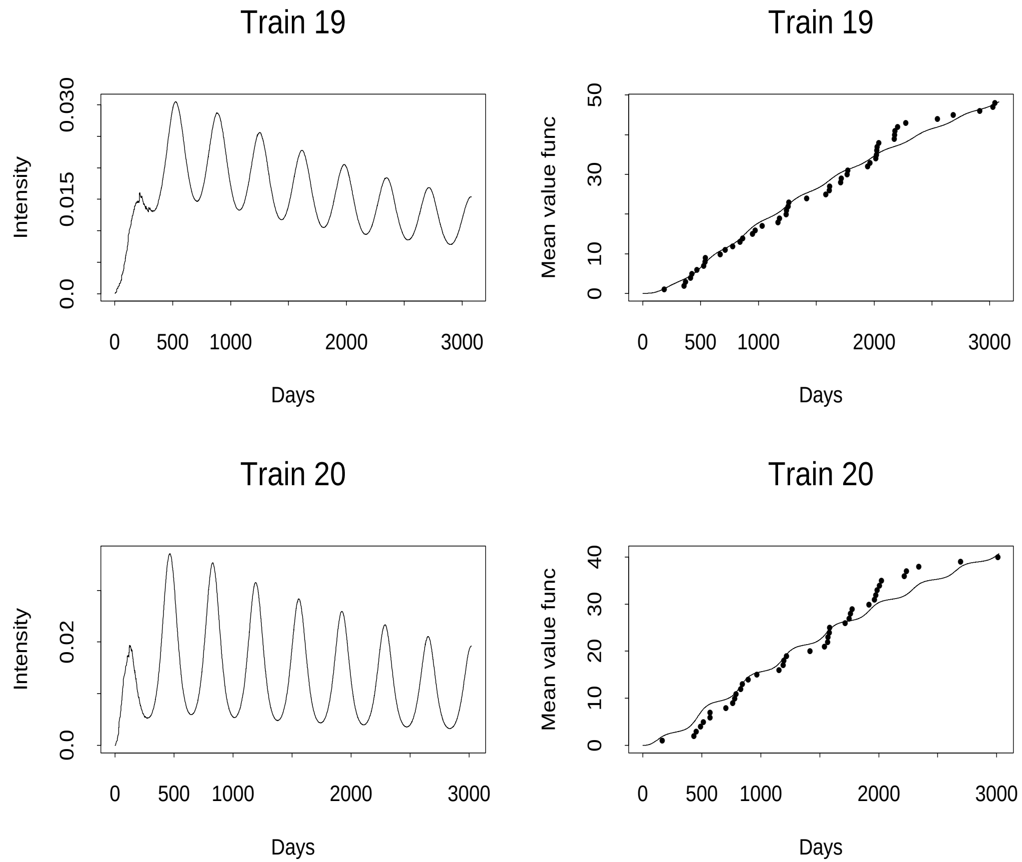
- g needs to go through observed failure data $k_i = g(t_i)$
- link between Dirichlet and Gamma distributions
- $g(t)$ points drawn from the cumulative distribution of a Dirichlet process, multiplied by $g(t_i) - g(t_{i-1})$ and shifted above by $g(t_{i-1})$
- $g(t)$ updated with an acceptance/rejection step

GENERATION OF g

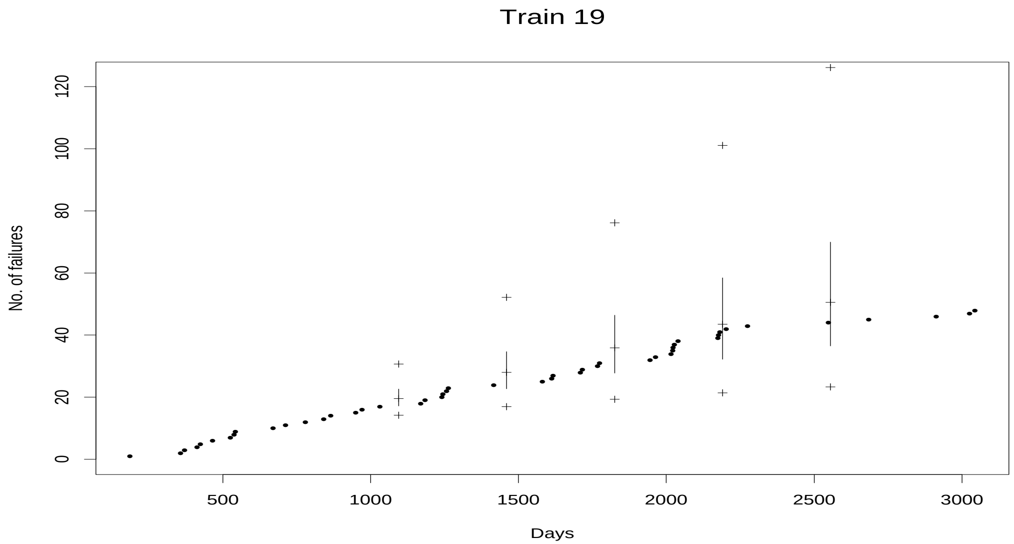


An example of g during the MCMC run

INTENSITY AND MEAN VALUE FUNCTION ESTIMATION



FORECAST



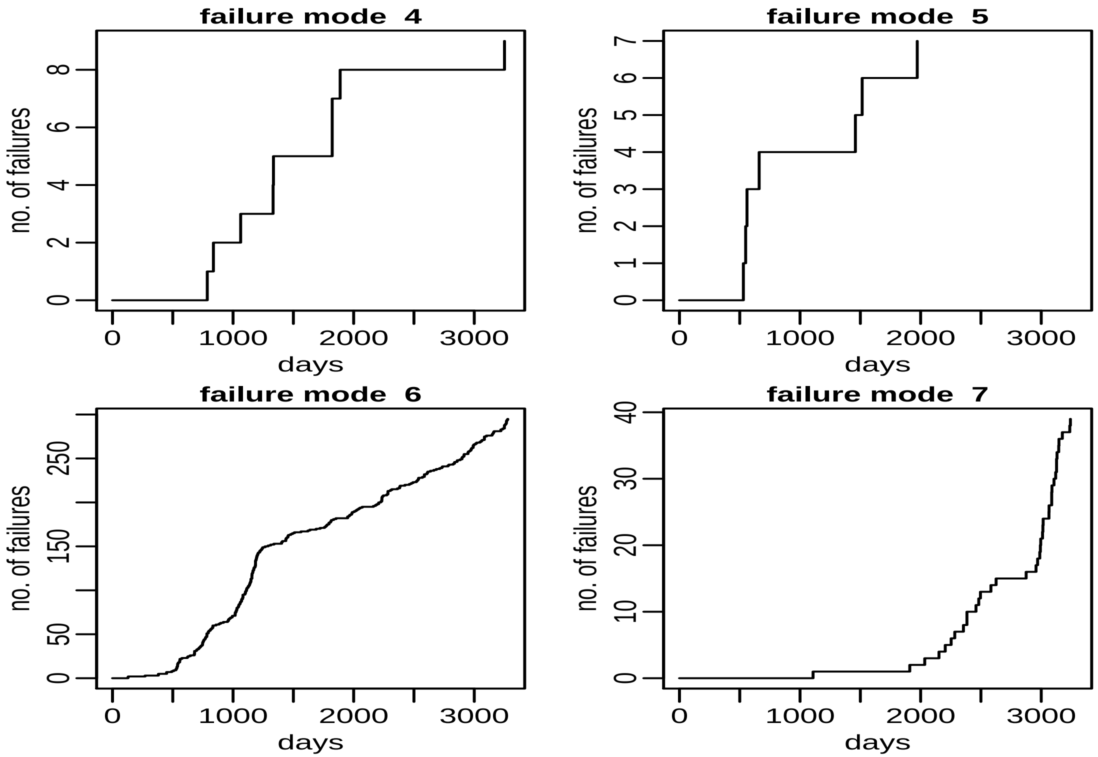
Prediction intervals of the number of failures for train 19 using 730 days (2 years) of observations, up to 5 years ahead. The vertical lines are the interquartile intervals with the posterior median; the plus signs are the extremes of 95% posterior probability intervals

DIFFERENT FAILURE MODES MODEL

Code	Subsystem	No. of parts	Total failures
1	opening commands (electrical)	14	530
2	cables and clamps	4	33
3	mechanical parts	67	1182
4	electrical protections	12	9
5	power supply circuit	2	7
6	pneumatic gear	31	295
7	electro-valves	8	39

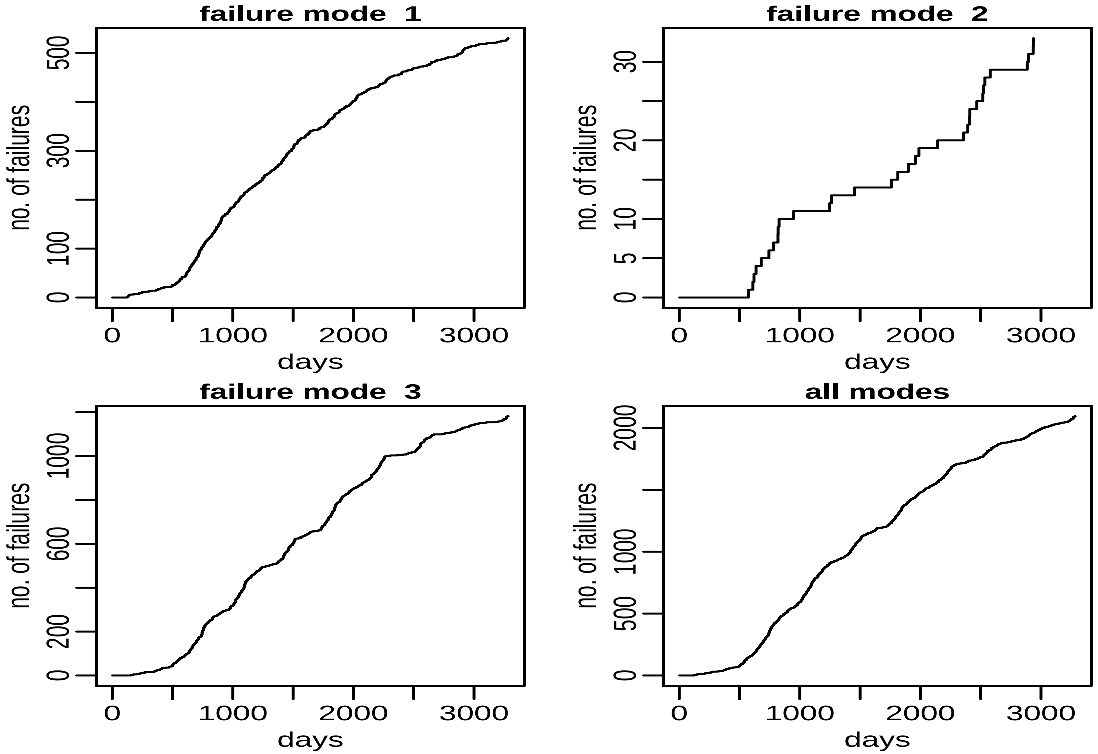
Classification of failure modes and total failures per mode for all trains in nine years

CUMULATIVE NUMBER OF FAILURES



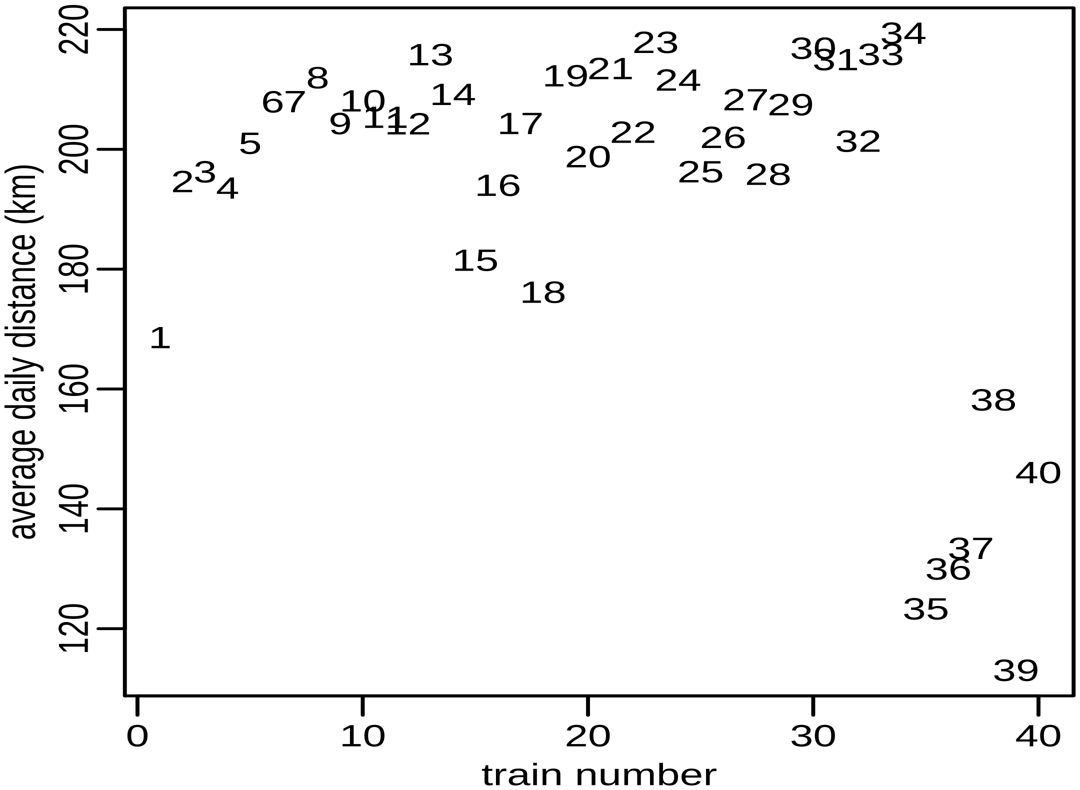
- Failure modes 4 and 5 very rare $\Rightarrow$ not enough information for fitting a stochastic process model
- Failure modes 6 and 7 show change-points

CUMULATIVE NUMBER OF FAILURES



- Failure modes 1, 2 and 3 display a more regular pattern
- Mode 2 failures are only 0.11 per train and per year
- $\Rightarrow$ concentrate on failure modes 1 and 3

AVERAGE DAILY DISTANCE



- Different average daily distance
- More recent trains are used less daily

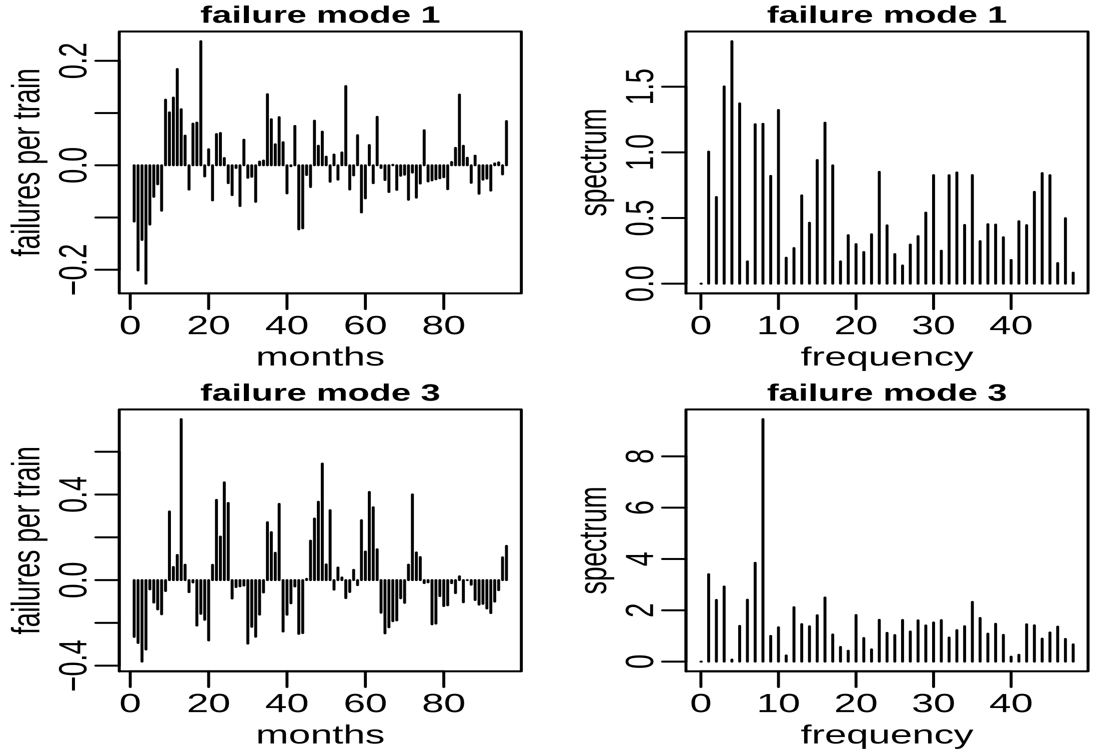
BIVARIATE INTENSITY FUNCTION

For each train i

$$\begin{aligned}\lambda_i(t,s) &= \mu \exp\left\{-\gamma(s-a_i-c_i(t-t_{0i}))^2 w(t-t_{0i})\right\} \cdot \\ &\quad \cdot \exp\left\{A\cos(\omega(t-d))\right\}\lambda_0(t-t_{0i})\end{aligned}$$

- t_{0i} starting operation date
- $a_i + c_i(t-t_{0i})$ expected distance after $(t-t_{0i})$ days in service ((a_i, c_i) different for every train, as seen before)
- $w(\cdot)$ positive weight function, rather close to 0 for $(t-t_{0i}) \approx 0$ and to 1 for $(t-t_{0i})$ large (initial relation between distance and time not linear)
  e.g. $w(z) = \dfrac{\sqrt{1+z}}{1+\sqrt{1+z}}$, bounded between 0.5 and 1
- $\lambda_0(\cdot)$ is a baseline intensity function (depending on time since first ride), common to all trains except for starting point
- exponentiated cosine is a periodic component with phase d (depending on calendar time), common to all trains

PERIODIC COMPONENT



- Periodogram of monthly time series of failure modes 1 and 3 (after detrending)
- No clear frequency for failure mode 1 $\Rightarrow$ omit periodic component in intensity
- 12-month cycle evident for failure mode 3

BASELINE INTENSITY

- $\Lambda_0(u) = Mu^b$ (Power Law process)

- $\Lambda_0(u) = \ln(1 + bu)$ (Reciprocal)

- $\Lambda_0(u) = (1 - e^{-bt})/b$ (Exponential)

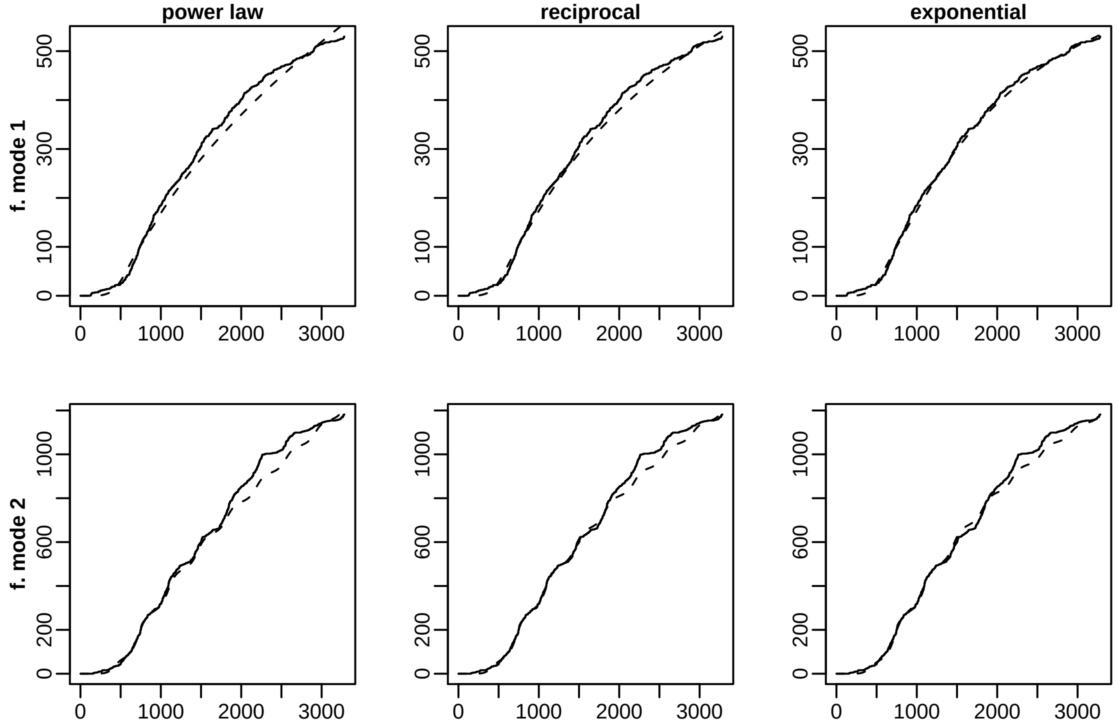
We omit writing likelihood, priors, posterior conditionals and MCMC implementation

ESTIMATE OF MEAN VALUE FUNCTION

- Posterior mean of $\Lambda(t;\theta)$
  - correct one
  - requires numerical integration of $\lambda(t;\theta)$ at each MCMC step

- Plot of $\Lambda(t;\widehat{\theta}) = \sum_{i=1}^{40} \int_{t_{0i}}^{t} \lambda_i(u;\widehat{\theta})\, du, \qquad t = 1, \ldots, 3287$
  - $\widehat{\theta}$ estimate of θ from MCMC run
  - $\lambda_i(t) = \mu \sqrt{\frac{\pi}{\gamma w(t-t_{0i})}}\, \Phi \left\{ (a_i + c_i(t - t_{0i}))\sqrt{2\gamma w(t - t_{0i})} \right\} \cdot$
    $\cdot \exp\{A \cos(\omega\,(t-d))\}\ \lambda_0(t - t_{0i})$
    (marginal of $\lambda_i(t,s)$)
  - not optimal but useful

ESTIMATE OF MEAN VALUE FUNCTION



- Cumulative number of failures for all trains and estimated mean value function (dashed)
- Row 1: failure mode 1; Row 2: failure mode 3
- Each column is for a different baseline (exponential in third column is the best)

FORECAST OF FUTURE FAILURES OF GIVEN MODE

- D_{T_0} data available at day T_0
- $\pi(\cdot \mid D_{T_0})$ posterior density of θ

Predictive distribution

$$P(N_{T_0+u} - N_{T_0} = x \mid D_{T_0}) = \int e^{-\{\Lambda(T_0+u;\theta)-\Lambda(T_0;\theta)\}} \frac{\{\Lambda(T_0+u;\theta) - \Lambda(T_0;\theta)\}^x}{x!} \pi(\theta \mid D_{T_0})\, d\theta$$

Expected value

$$\mathsf{E}(N_{T_0+u} - N_{T_0} \mid D_{T_0}) = \int \{\Lambda(T_0+u;\theta) - \Lambda(T_0;\theta)\}\, \pi(\theta \mid D_{T_0})\, d\theta$$

FORECAST OF FUTURE FAILURES OF MODE 1

end of recording period	forecasting horizon (years)	95% credibility interval	true value	posterior mean
1992	1	(86, 143)	83	114
	2	(79, 140)	72	109
	3	(71, 138)	62	105
1993	1	(69, 124)	72	97
	2	(59, 121)	62	90
	3	(50, 119)	42	85
1994	1	(50, 100)	62	74
	2	(41, 95)	42	66
	3	(32, 91)	35	59
1995	1	(38, 81)	42	59
	2	(30, 74)	35	51
	3	(24, 68)	23	44
1996	1	(27, 60)	35	43
	2	(20, 52)	23	35
1997	1	(19, 46)	23	39

FAILURE FORECAST OF NEW TRAIN

- $N_H(t)$ failure Poisson process for new train
- $\lambda_H(t;\theta)$ intensity function and $\Lambda_H(t;\theta)$ mean value function
- D_t failure data up to time t
- $T_0 = 2$ years

$$\Pr(N_H(T_0) > x_U \mid D_t) = 1 - \int \sum_{x=0}^{x_U} e^{-\Lambda_H(T_0;\theta)} \frac{[\Lambda_H(T_0;\theta)]^x}{x!} \pi(\theta \mid D_t)\, d\theta$$

f. mode 1	x_U	3	4	5	6	7	8	9	10	11	12	13
	prob.	0.82	0.68	0.52	0.36	0.23	0.14	0.07	0.04	0.02	0.01	0.00
f. mode 3	x_U	12	13	14	15	16	17	18	19	20	21	22
	prob.	0.47	0.36	0.26	0.18	0.12	0.08	0.05	0.03	0.01	0.01	0.00

CASE STUDY 3: SOFTWARE RELIABILITY

- Software reliability can be defined as *the probability of failure-free operation of a computer code for a specified mission time in a specified input environment*
- Seminal paper by Jelinski and Miranda (1972)
- More than 100 models after it (Philip Boland, *MMR2002*)
- Many models clustered in few classes
- Search for unifying models (e.g. Self-exciting process, Chen and Singpurwalla, 1997)
- Here interest in bugs detection during software testing

SOFTWARE RELIABILITY: MODELS

Failures at $T_1, T_2, \ldots, T_n$

Inter-failure times $T_i - T_{i-1} \sim \mathcal{E}(\lambda_i)$, independent, $i = 1, \ldots, n$

- $\lambda_i = \phi(N - i + 1)$, $\phi \in \mathbb{R}^+, N \in \mathbb{N}$, *(Jelinski-Moranda, 1972)*
  - Program contains an initial number of bugs N
  - Each bug contributes the same amount to the failure rate
  - After each observed failure, a bug is detected and corrected

Straightforward Bayesian inference with priors $N \sim \mathcal{P}(\nu)$ and $\phi \sim \mathcal{G}(\alpha, \beta)$

SOFTWARE RELIABILITY

- Bugs in software induce failures
- Fixing current bugs sometimes implies introduction of new bugs
- Lack of knowledge about effects of bugs fixing
- $\Rightarrow$ need for models allowing for possible, unobserved introduction of new bugs in a context aimed to reduce bugs
- Software affects our life at a larger extent and its malfunctioning could be very harmful
- Goal: Detecting bad fixing of bugs and reliability level

BUGS INTRODUCTION: MODELS

Failures at $T_1, T_2, \ldots, T_n$

Inter-failure times $T_i - T_{i-1} \sim \mathcal{E}(\lambda_i)$, independent, $i = 1, \ldots, n$

- $\lambda_{i+1} = \lambda_i e^{-\theta_i}$, $\quad \lambda_i, \theta_i \in \mathbb{R}^+$, independent
  (Gaudoin, Lavergne and Soler, 1994)
  - $\theta_i = 0 \Rightarrow$ no debugging effect
  - $\theta_i > 0 \Rightarrow$ good quality debugging
  - $\theta_i < 0 \Rightarrow$ **bad quality debugging**

BUGS INTRODUCTION: MODELS

Birth-death process *(Kremer, 1983)*

- $p_n(t) = \mathcal{P}r\{X(t) = n\}$
- $\nu(t)$ **birth rate**
- $\mu(t)$ death rate
- a initial population

$$p_n^{'}(t) = (n-1)\nu(t)p_{n-1}(t) - n[\nu(t)+\mu(t)]p_n(t) + (n+1)\mu(t)p_{n+1}(t), n \geq 0$$

with $p_{-1} \equiv 0$ and $p_n(0) = \mathbb{1}_{\{n=a\}}$

HIDDEN MARKOV MODEL

- Failure times $t_1 < t_2 < \ldots < t_n$ in $(0, y]$
- Y_t latent process describing *reliability status* of software at time t (e.g. growing, decreasing and constant)
- Y_t changing only after a failure $\Rightarrow Y_t = Y_m$ for $t \in (t_{m-1}, t_m]$, $m = 1, \ldots, n+1$, with $t_0 = 0$, $t_{n+1} = y$ and $Y_{t_0} = Y_0$
- $\{Y_n\}_{n \in \mathbb{N}}$ Markov chain with
  - discrete state space E
  - transition matrix $\mathbb{P}$ with rows $\mathbb{P}_i = (P_{i1}, \ldots, P_{ik})$, $i = 1, \ldots, k$

HIDDEN MARKOV MODEL

- Interarrival times of m-th failure $X_m|Y_m = i \sim \mathcal{E}(\lambda(i)), i = 1, \ldots, k, m = 1, \ldots, n$
- X_m's independent given $Y \Rightarrow f(X_1, \ldots, X_n|Y) = \prod_{m=1}^{n} f(X_m|Y)$
- $\mathbb{P}_i \sim \mathcal{D}ir(\alpha_{i1}, \ldots, \alpha_{ik}), \forall i \in E$, i.e. $\pi(\mathbb{P}_i) \propto \prod_{j=1}^{k} P_{ij}^{\alpha_{ij}-1}$
- Independent $\lambda(i) \sim \mathcal{G}(a(i), b(i)), \forall i \in E$
- Interest in posterior distribution of $\Theta = (\lambda^{(k)}, \mathbb{P}, Y^{(n)})$
  - $\lambda^{(k)} = (\lambda(1), \ldots, \lambda(k))$
  - $Y^{(n)} = (Y_1, \ldots, Y_n)$

HIDDEN MARKOV MODEL: CRITICAL ISSUES

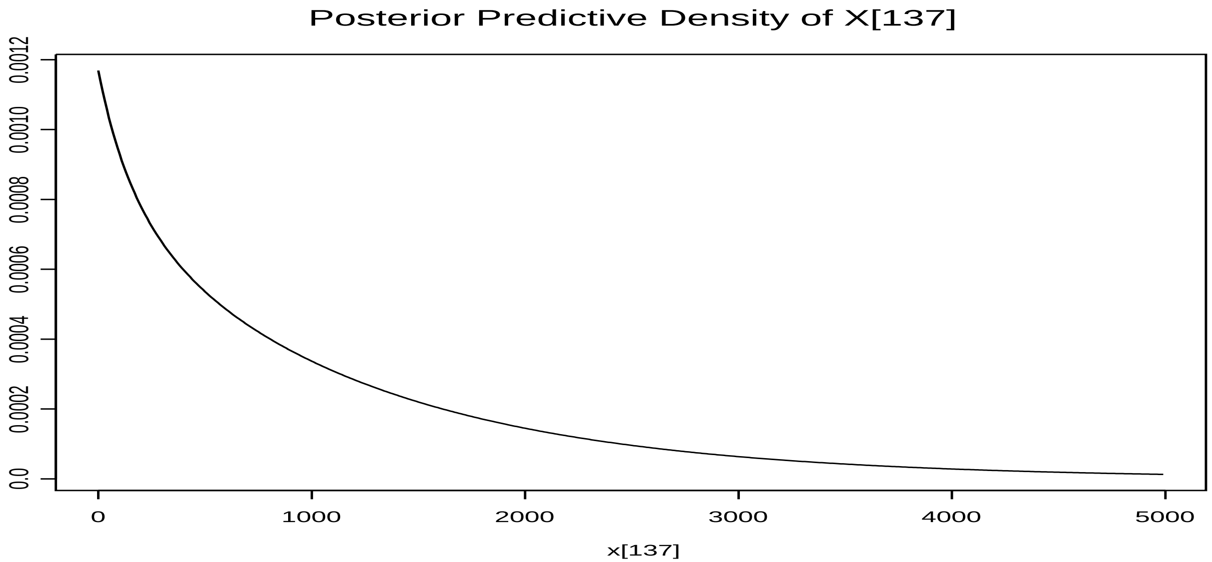
- Ranking of reliability states through ordered λ's
  - Order preserving prior leads to unjustified (by data) equality of adjacent λ's
  - Label switching when considering independent λ's

- Number of states K
  - Reversible jump MCMC
  - Bayes factor out of MCMC (Chib's method)

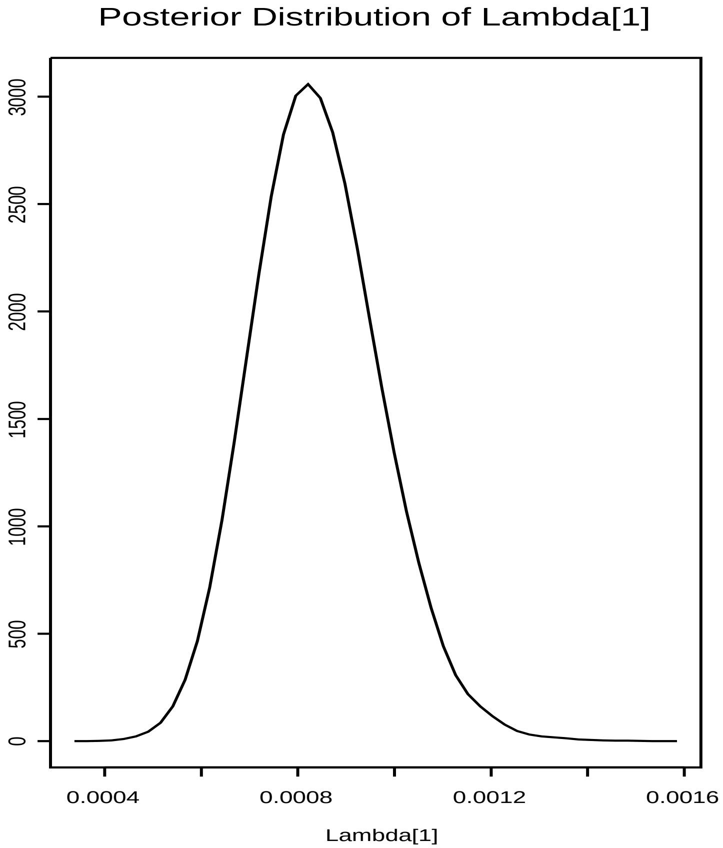
SOFTWARE RELIABILITY - MUSA DATA

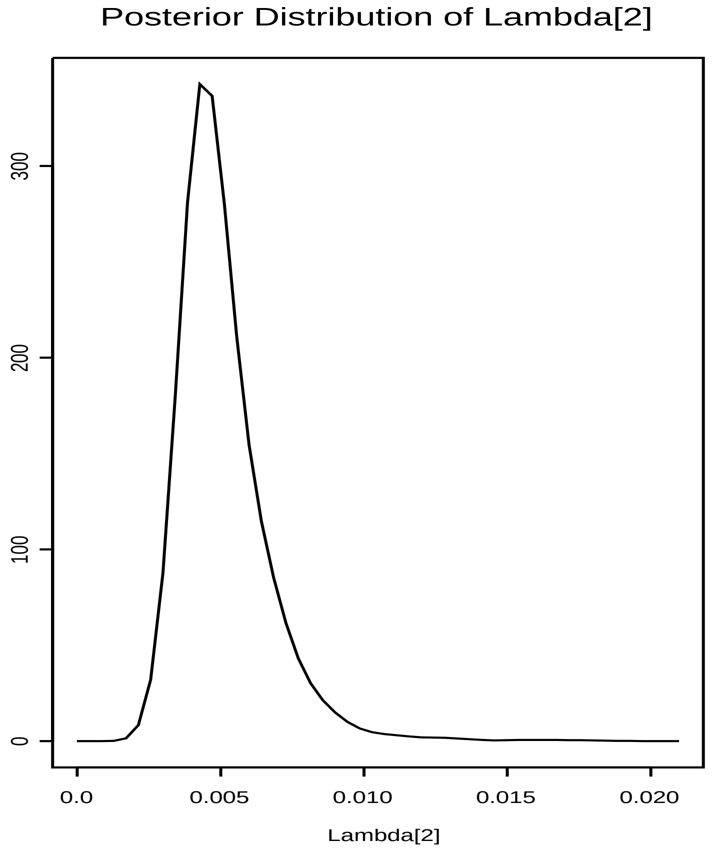
- Musa System 1 data: 136 software failure times

- Hidden Markov model with 2 unknown states

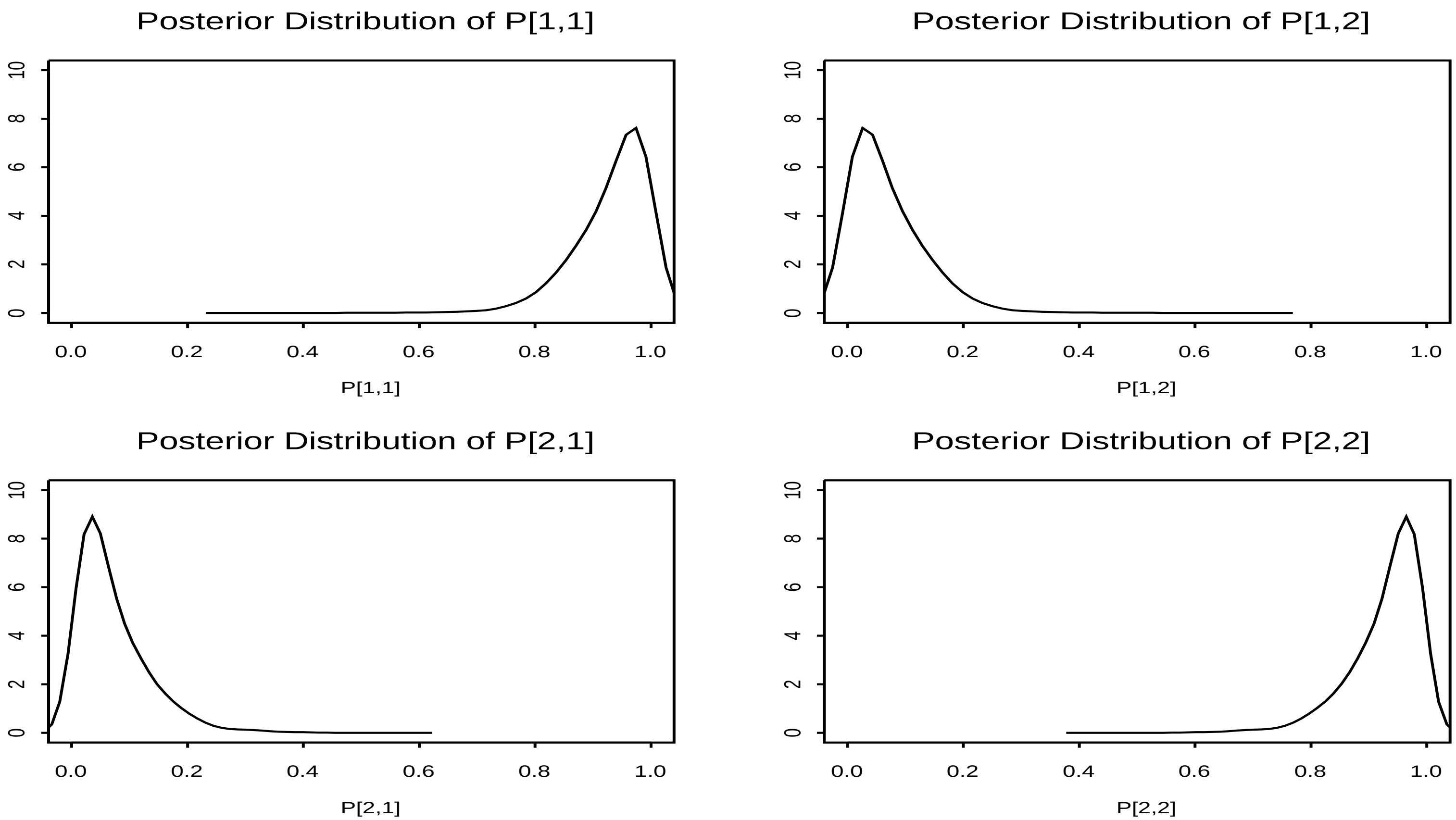


SOFTWARE RELIABILITY - MUSA DATA

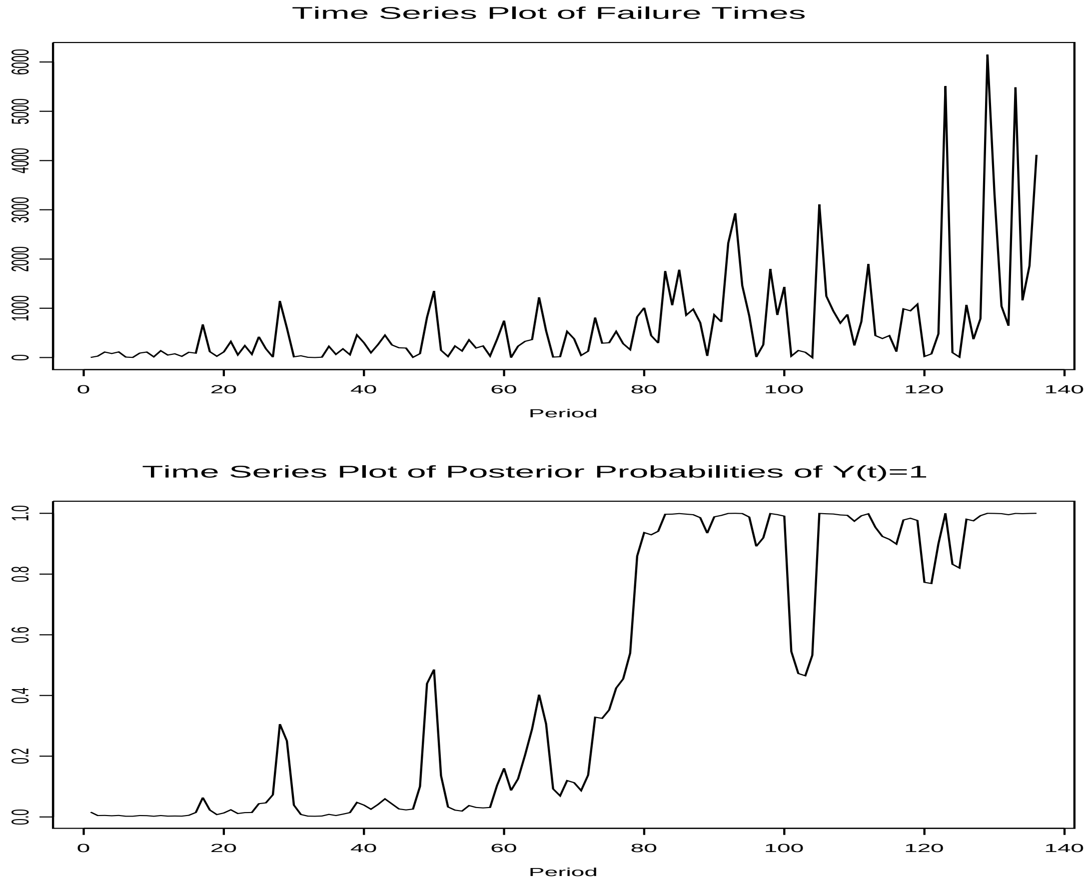




SOFTWARE RELIABILITY - MUSA DATA



SOFTWARE RELIABILITY - MUSA DATA



Longer failure times $\Rightarrow$ higher Bayes estimator of probability of "good" state