The **Hoeffding inequality** is a classic (and often used) tail bound for the sum of a set of independent random variables.

Here is what it says:

Let $X_1, X_2, \ldots, X_N$ be independent (but not necessarily identically distributed) with $\mathrm{E}[X_i] = 0$ and $|X_i| \le a_i$ with probability one for some fixed sequence of real numbers $a_1, a_2, \ldots, a_N$. Then for any $\lambda > 0$,

$$\mathrm{P}\left(\sum_{i=1}^{N} X_i > \lambda\right) \;\le\; 2\exp\left(-\frac{\lambda^2}{2\sum_{i=1}^{N} a_i^2}\right).$$

# Question 1a

Let $z_i = \pm 1$ with equal probability for $i = 1, \ldots, N$. Set

$$S = \sum_{i=1}^{N} z_i$$

It is clear that $|S| \leq N$. But really, how big is $|S|$?

# Question 1b

Let $z_i = \pm 1$ with equal probability for $i = 1, \ldots, N$. Set

$$Q = \langle \boldsymbol{x}, \boldsymbol{Z} \rangle = \sum_{i=1}^{n} z_i x[i]$$

It is clear that $|Q| \leq \|\boldsymbol{x}\|_1$. But really, how big is $|Q|$?

# Question 2

Let $\Phi = \begin{bmatrix} 4 & 1 \end{bmatrix}$ and $y = 1$.

1. Solve

$$\min_{x \in \mathbb{R}^2} \|x\|_2 \quad \text{subject to} \quad \Phi x = y.$$

2. Solve

$$\min_{x \in \mathbb{R}^2} \|x\|_1 \quad \text{subject to} \quad \Phi x = y.$$

(Hint: sketch it)

# Question 3a

I give you a vector $\boldsymbol{y} \in \mathbb{R}^N$. Find the explicit solution to

$$\min_{\boldsymbol{x}} \frac{1}{2}\|\boldsymbol{y} - \boldsymbol{x}\|_2^2 + \tau\|\boldsymbol{x}\|_1$$

Hint: the problem is separable,

$$\frac{1}{2}\|\boldsymbol{y} - \boldsymbol{x}\|_2^2 + \tau\|\boldsymbol{x}\|_1 = \sum_{n=1}^{N} \frac{1}{2}(y[n] - x[n])^2 + \tau|x[n]|$$

so start with the scalar problem

$$\min_{x} \frac{1}{2}(y - x)^2 + \tau|x|$$

# Question 3b

I give you a $K \times N$ matrix $\boldsymbol{Y}$. What do you think the solution to

$$\min_{\boldsymbol{X}} \frac{1}{2} \|\boldsymbol{Y} - \boldsymbol{X}\|_F^2 + \tau \|\boldsymbol{X}\|_*$$

is?