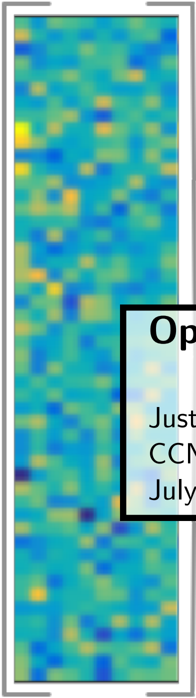
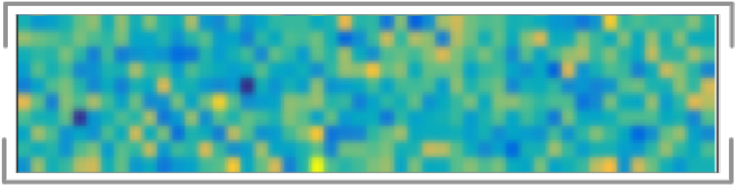




Optimization Basics

Justin Romberg, Georgia Tech ECE

CCNS Summer School, SAMSI, RTP, NC

July 28, 2015

Unconstrained smooth problems

Unconstrained problem, $f(\boldsymbol{x})$ is differentiable:

$$\underset{\boldsymbol{x}}{\text{minimize}} \;\; f(\boldsymbol{x})$$

Necessary and sufficient condition for solution:

$$\nabla f(\boldsymbol{x}^\star) = \mathbf{0}$$

Subgradients

When $f(\cdot)$ is not smooth, we can generalize our notion of derivative.

A *subgradient* at $\boldsymbol{x}$ is a supporting hyperplanes:

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}_0) + \boldsymbol{u}^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}_0) \quad \leftrightarrow \quad \boldsymbol{u} \in \mathbb{R}^N \text{ is a subgradient at } \boldsymbol{x}_0$$

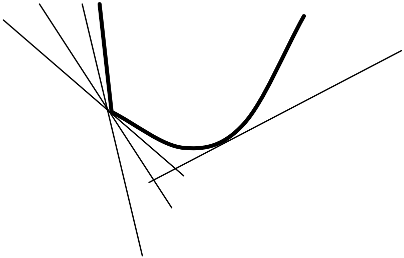
Subgradients

When $f(\cdot)$ is not smooth, we can generalize our notion of derivative.

A *subgradient* at $\boldsymbol{x}$ is a supporting hyperplanes:

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}_0) + \boldsymbol{u}^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}_0) \quad \leftrightarrow \quad \boldsymbol{u} \in \mathbb{R}^N \text{ is a subgradient at } \boldsymbol{x}_0$$

There can be many subgradients at a point:

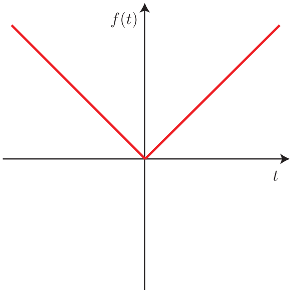


The set of subgradients is called the *subdifferential* at $\boldsymbol{x}_0$:

$$\partial f(\boldsymbol{x}_0) = \{\boldsymbol{u} \in \mathbb{R}^N \ : \ \boldsymbol{u} \text{ is a subgradient of } f \text{ at } \boldsymbol{x}_0\}$$

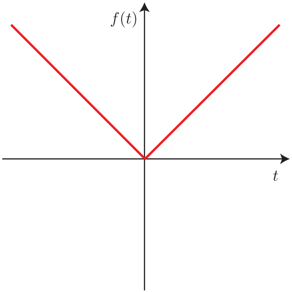
Subgradients: Example

What are the subgradients of $f(x) = |x|$?



Subgradients: Example

What are the subgradients of $f(x) = |x|$?

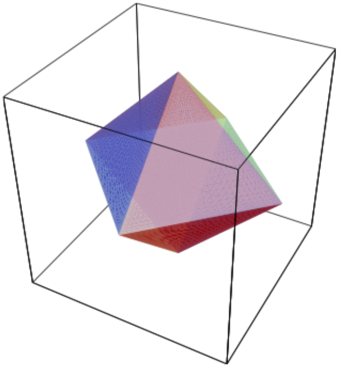


Answer:

$$\partial f(x) = \begin{cases} -1, & x < 0 \\ [-1, 1], & x = 0 \\ 1, & x > 0 \end{cases}$$

Subgradients: Example

What are the subgradients of $f(\boldsymbol{x}) = \|\boldsymbol{x}\|_1 = \sum_n |x[n]|$?



Subgradients: Example

What are the subgradients of $f(\boldsymbol{x}) = \|\boldsymbol{x}\|_1 = \sum_n |x[n]|$?
Answer: $\boldsymbol{u} \in \mathbb{R}^N$ is a subgrad

$$\partial f(\boldsymbol{x}) = \begin{cases} \mathrm{sign}(x[n]), & n : x[n] \neq 0 \\ [-1, 1], & n : x[n] = 0 \end{cases}$$

So $\boldsymbol{u} \in \mathbb{R}^N$ is a subgradient at $\boldsymbol{x}_0$ if

$$\begin{aligned} u[n] &= \mathrm{sign}(x_0[n]), \quad n \in \Gamma \\ |u[n] &\leq 1, \quad n \in \Gamma^c \end{aligned}$$

where $\Gamma =$ support of $\boldsymbol{x}_0$

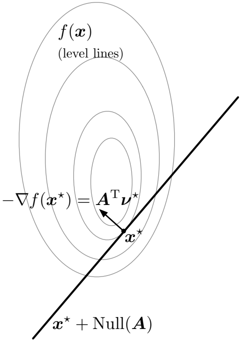
Constrained optimization, smooth

Linear constraints, $f(\boldsymbol{x})$ is differentiable

$$\underset{\boldsymbol{x}}{\text{minimize}} \;\; f(\boldsymbol{x}) \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$$

A necessary and sufficient condition for the solution:
there exists $\boldsymbol{\nu}^\star \in \mathbb{R}^M$ such that

$$\nabla f(\boldsymbol{x}^\star) + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\nu}^\star = \boldsymbol{0}$$



Constrained optimization, nonsmooth

Linear constraints, $f(\boldsymbol{x})$ is not differentiable

$$\underset{\boldsymbol{x}}{\text{minimize}} \;\; f(\boldsymbol{x}) \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$$

A necessary and sufficient condition for the solution:
there exists $\boldsymbol{u}^\star \in \partial f(\boldsymbol{x}^\star)$ and a $\boldsymbol{\nu}^\star$ such that

$$\boldsymbol{u}^\star + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\nu}^\star = \boldsymbol{0}$$

in other words, there exists $\boldsymbol{\nu}^\star$ such that

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\nu}^\star \in \partial f(\boldsymbol{x}^\star)$$