Augmented Probability Simulation based Stochastic Programming

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Two-Stage Stochastic Programming with Recourse

- Decision making under uncertainty (Powell, 2016): decision analysis, stochastic optimization, dynamic programming, multi-armed bandit problems, optimal stopping, sequential learning, Markov decision processes

$$\max_x E_\xi[Q(x,\xi)]$$

- Two-Stage Stochastic Programming with Recourse (Higle, 2005)
  - Postponing some decisions until the uncertainty disappears so that a corrective (recourse) action can be implemented
  - The first stage decision, x, is made before knowing the realization of the random variable, ξ and chosen such that it is feasible for all scenarios of ξ and has the minimum expected objective function value for both stages combined

$$x \longrightarrow \xi \longrightarrow y(x,\xi)$$

Formulation of the two stage problems with recourse (Birge and Louveaux (2011)

$$\begin{aligned}
\min_{x} \quad & cx - E[Q(x,\xi)] \\
\text{subject to} \quad & Ax \geqslant b \\
\text{where } Q(x,\xi) = \max_{y} \quad & q(\xi)y \\
\text{subject to} \quad & Tx + Wy \leqslant h, y \geqslant 0
\end{aligned} \tag{1}$$

- First stage:

$$\max_{x}(-cx + E_{\xi}[q(\xi)y(\xi)])$$

$$\mathbb{1}(Ax \geqslant b,\, Tx + Wy(\xi) \leqslant h, y(\xi) \geqslant 0, \forall \xi)$$

- Second stage for a given realization of ξ:

$$\max_{y}(qy)\mathbb{1}(Tx + Wy \leqslant h, y \geqslant 0)$$

Sample Average Approximation (SAA)

- Monte Carlo sampling-based methods: Homem-de-Mello and Bayraksan, 2015
- Estimation of the expectation and optimization

$$\begin{aligned}
\max \quad & -cx + \frac{\sum_{n=1}^{N} Q(x,\xi_n)}{N} \\
\text{subject to} \quad & Ax \geqslant b \\
\text{where } Q(x,\xi_n) = \max_y \quad & q(\xi_n)y_n \\
\text{subject to} \quad & Tx + Wy_n \leqslant h, y_n \geqslant 0
\end{aligned} \tag{2}$$

- Can be ineffective in high dimensions, high variance of draws
- Still needs to be solved by a proper deterministic optimization method
- Internal Sampling: estimation done within an optimization method such as L-shaped (Higle and Sen, 1991)

Augmented Probability Simulation (APS, Müller (1999), Bielza et al. (1999))

$$\max_{x} E_{\xi}[Q(x,\xi)]$$

where

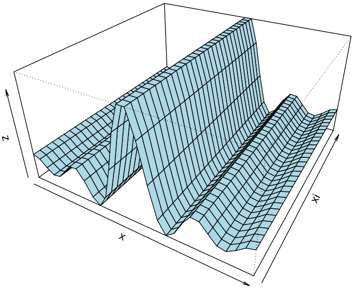
$$E_{\xi}[Q(x,\xi)] = Q(x) = \int Q(x,\xi)p(\xi)d\xi$$

- Requires computation of the expectation and optimization
- APS method treats x as random and constructs a joint distribution

$$\pi(x,\xi) \propto Q(x,\xi)p(x,\xi)$$

so that the marginal distribution of the decision variable x is

$$\pi(x) \propto \int Q(x,\xi)p(x,\xi)d\xi \Rightarrow \pi(x) \propto Q(x)$$



Finding the mode of the marginal distribution $\pi(x)$ is equivalent to finding the decision that provides the maximum expected utility

Augmented Probability Simulation

- Assumptions
  - $Q(x, \xi)$ is non-negative and the choice set of x is bounded
  - The decision variable, x, is treated as random for computational purposes
- Implementation issues
  - Flat expected utility, $Q(x)$: J copies idea, replacing $Q(x)$ with $Q^J(x)$ (Müller (1999), Müller et al (2004))
  - Sampling from constrained joint distribution, $\pi(x, \xi)$: MCMC and nested sampling based methods

Augmented Probability Simulation

- The augmented probability model is:

$$\pi_J(\xi_1, \xi_2, ..., \xi_J, x) \propto \prod_{j=1}^{J} Q(\xi_j, x) p(\xi_j, x)$$

  by using J *iid* draws from the distribution $p(\xi)$.
- $\pi_J(x) \propto (Q(x))^J = exp\{Jln[Q(x)]\}$
- The reciprocal $1/J$ is also known as the "annealing temperature" in the simulated annealing literature (Kirkpatrick et al. (1983))
- Computation of mode of $\pi_J(x)$: Explanatory graphical tools or the mean as an estimate for large J (Pincus (1968))

Simulation with MCMC

- MCMC techniques within a Gibbs conditional sampling framework
- The resulting full conditional distributions are:

$$\pi(x|\underline{\xi_J}, \underline{y_J^*}) \propto \prod_{j=1}^{J} (-cx + q(\xi_j)y_j^*)\mathbb{1}(Ax \geqslant b)$$

$$\pi(\xi_j|x, y_j^*) \propto (cx - q(\xi_j)y_j^*)p(\xi_j|x)$$

- As a result of this algorithm, the draws from the marginal density collapses on the optimal decision x^*

Simulation with Nested Sampling (Skilling, 2006)

Sampling from the joint prior distribution constrained to the minimum likelihood level (Polson and Scott, 2015)

1. Simulate draws for S live points $(x, \xi_j)_{(s)}$ from the joint prior $p(x, \xi_j)$ and evaluate utility levels $u_J((x, \boldsymbol{\xi}_J)_{(s)}) = \prod_{j=1}^{J} u((x, \xi_j)_{(s)})$ for $s = 1, ..., S$.
2. Find the live point that provides minimum utility and record it as the minimum utility level.
3. Replace that live point by sampling from the constrained prior so that new likelihood is larger than the current minimum
4. Repeat steps 2-3 for G iterations or until a stopping criterion is satisfied.

How to simulate from the constrained prior

- Sampling from the constrained prior distribution: $\pi(x, \xi_J | u(x, \xi_J) > U_{min})$
- Alternatives: **accept-reject**, MCMC, random walk MH, reversible jump based algorithms

1. Pick one of the (S-1) live points, excluding the one that provides the minimum utility, at random, as your initial point.
2. Simulate a candidate draw from the joint prior distribution, $\pi(x, \xi_J)$.
3. Accept the candidate draw if it results with an increased minimum utility.
4. Repeat Steps 1-3 for N iterations.

Convergence and solution quality

- APS: Bielza et al. (1999), Muller et al. (2004), Ekin et al. (2014)
- Convergence in the number of iterations, G: (Gamerman and Lopes (2006), Johannes and Polson (2004))
- Convergence in the augmentation parameter, J: Jacquier et al. (2007)
- Convergence of nested sampling: Robert and Chopin (2010) and Polson and Scott (2015)
- Assessment of the solution quality: Mak et al (1999), Solak et al. (2010)

Recourse problems with Decision Dependent Uncertainty

- This setup simply involves changing $p(\xi)$ to $p(\xi|x)$
  - $p(\xi|x)$ is a function of x: Ahmed (2000), Viswanath et al. (2004)
  - $p(\xi)$ is conditional on x by having different parameters for each x
- Operational planning of offshore gas development (Jonsbraten et al. (1998), Goel and Grossmann (2004)), aggregate workforce planning and scheduling (Fragnière et al. (2010), Morton and Popova (2004)), reliability (Galenko et al. (2006)), project portfolio management (Solak et al. (2010)), call center staffing.
- Mainly low dimensional problems solved due to computational complexity

Decision Dependent News-vendor Problem

- Decision: initial inventory level (the number) of newspapers to be purchased, x
- Impact of initial inventory on the uncertainty of demand, $d(\xi)$
- Demand is non-decreasing with a diminishing rate w.r.t. the initial inventory level (Balakrishnan et al. (2004), Ernst and Powell (1995))
- $p(\xi|x) \sim TN[\alpha x^{\beta}, \sigma_d]$

Decision Dependent News-vendor Model Formulation

$$\max_{x} \; -cx + E_{\xi}[Q(x,\xi)]$$

where the recourse function, $Q(x,\xi)$ is obtained from the linear program

$$Q(x,\xi) = \max_{y_1,y_2} \; sy_1 + ry_2$$

$$\text{subject to } \; y_1 \leq d(\xi) \;, \; y_1 + y_2 \leq x \;, y_1 \geq 0, y_2 \geq 0.$$

Optimal recourse: $y_1^*(x,\xi) = \min(d(\xi), x)$, $y_2^*(x,\xi) = \max(x - d(\xi), 0)$

- c $=$ 1, s $=$ 2, r $=$ 0.8, $\alpha = 5$, $\beta = 0.5$, $\sigma_d = 10$
- $x^* = 225.8$, $E[Q(x^*,\xi)] = 44.951$
- $p(x) \sim Unif(150, 300)$

Estimated optimality gaps for the same total sample size, 150, 000 draws

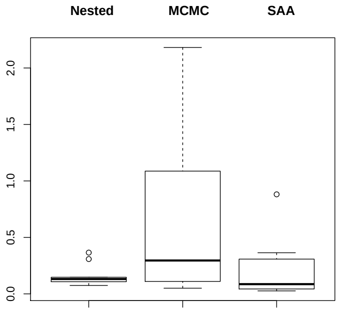


Figure: Estimated Optimality Gaps of the objective functions via MAPE for the same total sample size using nested based APS, MCMC based APS and SAA

Comparison for the same total sample size, 150, 000 draws

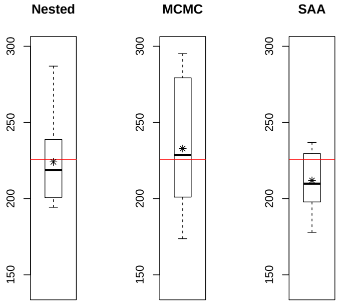


Figure: Box-plots of the candidate optimal decisions for the same total sample size using nested based APS, MCMC based APS and SAA

Choice of J

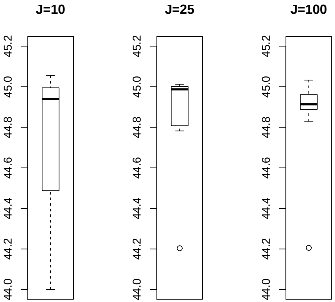


Figure: Box-plots of the optimal objective function values with J values of 10, 25 and 100

Trace plots

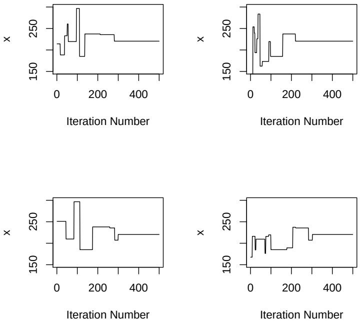


Figure: Trace plots of the decision variable for randomly selected live points for the nested sampling formulation

Decision Dependent News-stand Problem

- Decision: initial inventory levels of a number of newspapers to be purchased, $\boldsymbol{x}$
- Impact of initial inventory on the uncertainty of demand, $\boldsymbol{d}(\boldsymbol{\xi})$
- $p(\boldsymbol{\xi}|\boldsymbol{x}) \sim TN[\alpha \boldsymbol{x}^{\beta}, \boldsymbol{\sigma_d}]$.
- $M = 3$ identical items with $c = 10$, $s = 12$, $r = 1$, $p(x_m) \sim Unif(60, 110)$, Incapacitated
- $\alpha = 10$, $\beta = 0.5$, $\rho_{1,2} = 0.417$, $\rho_{1,3} = -0.078$, $\rho_{2,3} = -0.260$

Estimated optimality gaps of the dependent demand case

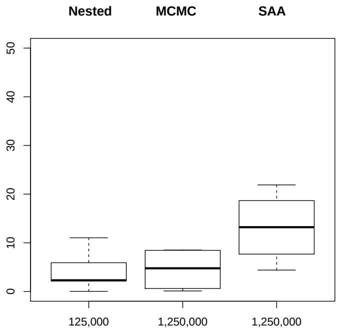


Figure: Estimated Optimality Gaps of the optimal objective functions via MAPE for the dependent demand case using nested based APS, MCMC based APS and SAA

Comparison of SAA vs Nested for dependent demand case

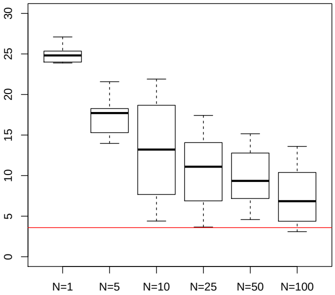


Figure: Estimated Optimality Gaps of the optimal objective functions via MAPE for the dependent demand case using SAA for different number of decision dependent draws, N

Performance for increasing dimensionality for the independent case

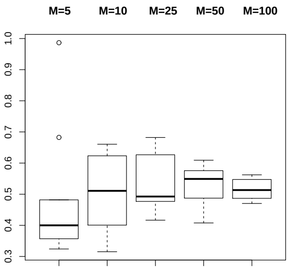


Figure: Box-plots of the optimal decisions for different sizes of multi-item problems using nested based APS

Conclusion

- Extends augmented probability simulation method of Müller (1999) and Bielza et al. (1999) to constrained stochastic optimization problems with decision dependent uncertainty
- Can be effective for problems with objective functions which are not analytically available
- Smarter sampling and without any gradient information
- Reduced variance for the samples, reduced sample size

Challenges and ongoing research

- Sampling from constrained domains
- Finding the mode for high dimensional decision variables: Multi-set sampler idea of Leman et al. (2009), cluster analysis
- Multi stage utilization of APS
- Two stage problems with randomness in both objective function and constraints
- Consideration of non-linear functions
- Call center staffing problem with uncertain arrival, service and abandonment rates
- Integrated maintenance and production planning problem where the yield is decision dependent

Thank you

Behavior of the objective function

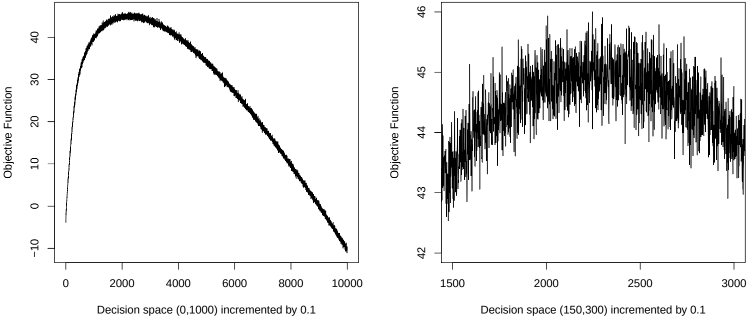


Figure: Objective function for the decision spaces (left:(0,1000) and right:(150,350))

Estimation of optimal decision

- For each decision point in the decision space of $(0, 1000)$ with increments of 0.1:
  1. Compute the objective function values for $1,000$ realizations of the random variable
  2. Estimate the expectation function by averaging these objective functions
- Record the decision that maximizes the objective function as the candidate optimal decision
- For each candidate optimal decision, $100,000$ realizations of the decision dependent random variables ξ are drawn and the optimal objective function estimate is computed.
- Repeat this 100 times
- Record the candidate optimal decision with the highest objective function value as the optimal decision, x^*.

Optimal decisions

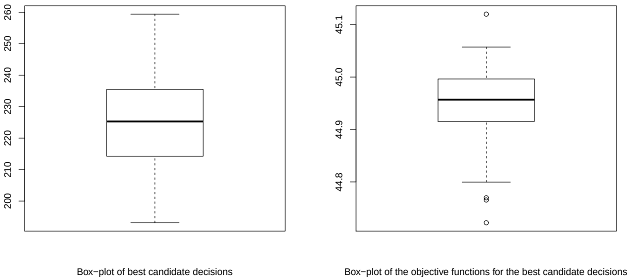


Figure: Box-plots of the candidate optimal decisions and candidate optimal objective function values

Choice of J

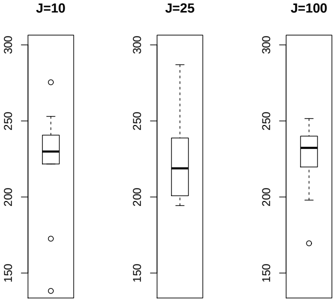


Figure: Box-plots of the optimal decisions with J values of 10, 25 and 100

Decisions for 5-item case with independent demand

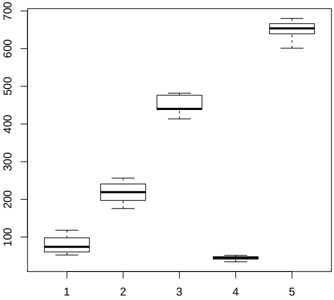


Figure: Box-plots of the optimal decisions for each of the five items using nested based APS

Recourse problems

Recourse problems with discrete random variables

- Explicit representation
- L-shaped method (van Slyke and Wets(1969)): Expectations computed to have upper bound approximations

Recourse problems with continuous random variables

- Bound approximation methods (Kall and Wallace(1994))
- Statistical approaches (Mitra(2008))
- Gradient-step based procedures (Prekopa (1988), Ermoliev (1988), Glynn(1989))
- Monte Carlo simulation based methods (Shapiro and de Mello(1998), Shapiro(2003))
    - External sampling
    - Internal sampling

Most of these methods need derivative information and may suffer from Monte Carlo errors

Internal sampling

- Sampling is done within an optimization method such as L-shaped
- Expectation functions estimated in order to have an upper bound approximation
- Types
  - Stochastic decomposition (Higle and Sen,1991 and 1996)
  - Importance sampling within L-shaped method (Infanger,1993)

Most of these methods need derivative information and may suffer from Monte Carlo errors

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