Distributed Topology using Harmonics

Harish Chintakunta
SAMSI Workshop on Topological Data Analysis

February 6, 2014

Why do we need Distributed Algorithms?

High Performance Computing

- Massively parallel architectures and supercomputers.

High Performance Computing

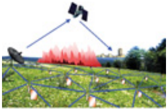
- Massively parallel architectures and supercomputers.
- Exploit the decoupled natured of processing requirements to speed up computations.

High Performance Computing

- Massively parallel architectures and supercomputers.
- Exploit the decoupled natured of processing requirements to speed up computations.
- Essential for coping with the large size of data.

Sensor Networks

- Will be ubiquitous in the future.
- Distributed algorithms ensure
  - In-network processing.
  - Low cost.
  - Robust to failures.
  - Fast response time



What are Harmonics?

- For a given complex K, and the boundary operators

$$\partial_{k+1} : C_{k+1} \to C_k, \partial_k : C_k \to C_{k-1}$$

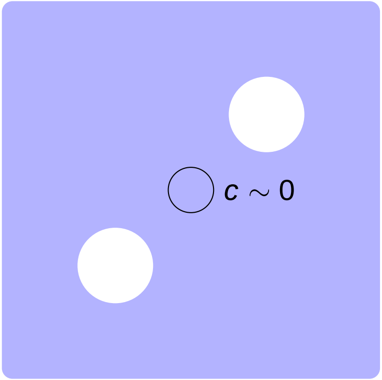
- Consider the k^{th} combinatorial Laplacian

$$L_k = \partial_{k+1}\partial_{k+1}^T + \partial_k^T \partial_k$$

- The elements in the null space of the k^{th} Laplacian, $ker(L_k)$, are denoted as k-harmonics.

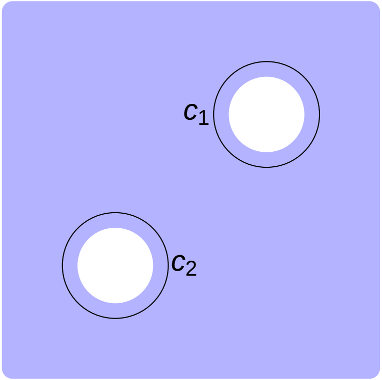
Harmonics

- Given a cycle c, is there a computationally efficient method to test for contractibility?



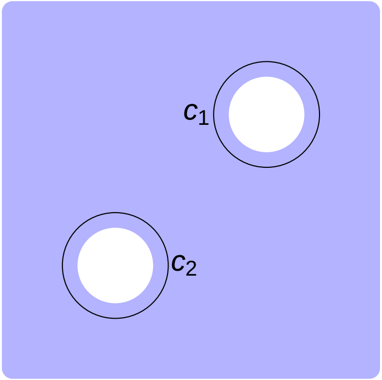
Harmonics

- Given a cycle c, is there a computationally efficient method to test for contractibility?



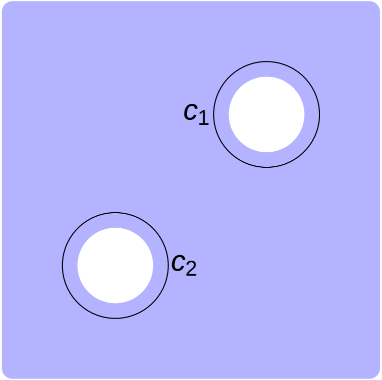
Harmonics

- Given a cycle c, is there a computationally efficient method to test for contractibility?
- The simplest method would be to construct the matrix $[\partial_{k+1} c]$ and reduce.



Harmonics

- Given a cycle c, is there a computationally efficient method to test for contractibility?
- The simplest method would be to construct the matrix $[\partial_{k+1} c]$ and reduce.
- Can we do better?



Let $Z_k \subseteq C_k$ denote the space of all k-cycles, and $B_k = Img(\partial_{k+1})$ denote the space of k-boundaries.

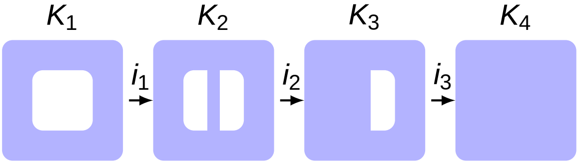
- $L_k = \partial_{k+1}\partial_{k+1}^T + \partial_k^T \partial_k$
- For any $y \in ker(L_k)$, $y \in Z_k$, and $y \perp B_k$, i.e., $\langle y, c \rangle = 0, \forall c \in B_k$.

Theorem

Let $c \in Z$ be a cycle in Z, and let y be a generic element in $ker(L_1)$. Then $c \in img(\partial_2) \iff \langle y, c \rangle = 0$ with probability 1.

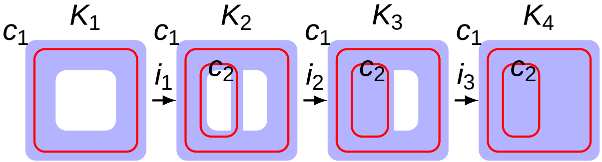
Harmonics

Persistence



Harmonics

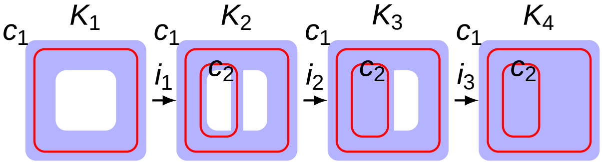
Persistence



- Let $[c] \in H_k(K_1)$. We want to check if $[c] \neq 0$, and if $i_*^2([c]) \neq 0$.

Harmonics

Persistence



- Let $[c] \in H_k(K_1)$. We want to check if $[c] \neq 0$, and if $i^2_*([c]) \neq 0$.
- Compute $y_1 \in ker(L_k(K_1))$, and $y_3 \in ker(L_k(K_3))$. If $\langle y_1, c\rangle \neq 0$, and $\langle y_3, c\rangle \neq 0$, then $[c]$ persists from K_1 through K_3.

Harmonics

Computation

$y \in ker(L_k)$ can be computed using the iteration

$$y^{k+1} = y^k - \delta L_k y^k$$

starting with a random vector y^0.

$y \in ker(L_k)$ can be computed using the iteration

$$y^{k+1} = y^k - \delta L_k y^k$$

starting with a random vector y^0.

Theorem

Let K be the matrix with column space equal to the null space of L_k. Then the iteration converges to $y^\infty = KK^T y^0$ if and only if δ satisfies the following inequality:

$$0 < \delta < 2/\lambda_i, \forall i$$

where $\lambda_1 > \lambda_2 > \cdots$ are the eigenvalues of L_k.

Consider the operation of

$$Y = LX$$

Consider the operation of

$$Y = LX$$

- the value at the i^{th} row is

$$y_i = \sum_j L_{ij} x_j = \sum_{j \in \mathcal{N}_i} x_j$$

where $\mathcal{N}_i$ is the set of neighbors of node i

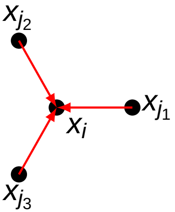
Consider the operation of

$$Y = LX$$

- the value at the i^{th} row is

$$y_i = \sum_j L_{ij} x_j = \sum_{j \in \mathcal{N}_i} x_j$$

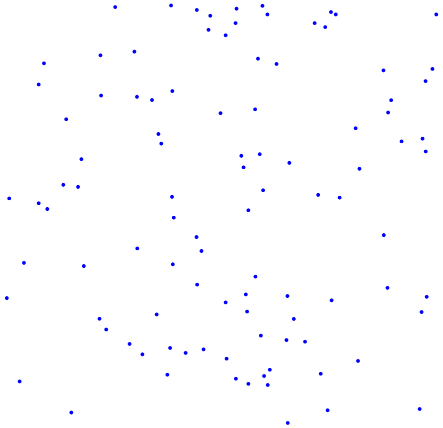
  where $\mathcal{N}_i$ is the set of neighbors of node i
- this can be accomplished by each node broadcasting its value to its neighbors



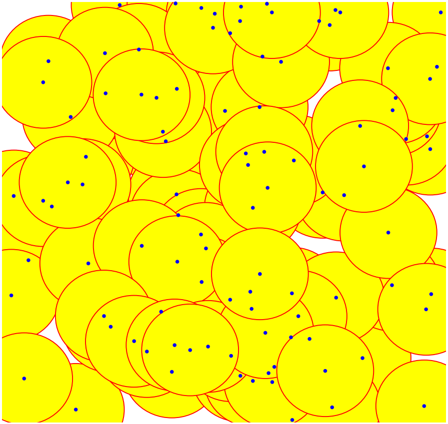
Application

Coverage in Sensor Networks

Coverage area

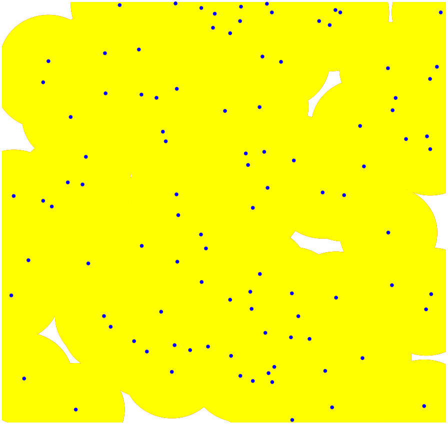


Coverage area



Balls of radius r_c

Coverage area



Coverage area

R_c

Coverage Problem in literature

Algebraic topological approaches

- R. Grhist and A. Muhammad. *Coverage and hole-detection in sensor networks via homology.* In Proceedings of the 4^{th} international symposium on Information processing in sensor networks. 2005.

Coverage Problem in literature

Algebraic topological approaches

- R. Grhist and A. Muhammad. *Coverage and hole-detection in sensor networks via homology.* In Proceedings of the 4^{th} international symposium on Information processing in sensor networks. 2005.
- V. de Silva and R. Ghrist. *Coordinate-free coverage in sensor networks with controlled boundaries via homology*. The International Journal of Robotics Research. 2006

Coverage Problem in literature

Algebraic topological approaches

- R. Grhist and A. Muhammad. *Coverage and hole-detection in sensor networks via homology.* In Proceedings of the 4^{th} international symposium on Information processing in sensor networks. 2005.
- V. de Silva and R. Ghrist. *Coordinate-free coverage in sensor networks with controlled boundaries via homology*. The International Journal of Robotics Research. 2006
- P. Dlotko, R. Ghrist, M. Juda, and M. Mrozek. *Distributed computation of coverage in sensor networks by homological methods*. Applicable Algebra in Engineering, Communication and Computing. 2012.

Coverage Problem in literature

Algebraic topological approaches

- R. Grhist and A. Muhammad. *Coverage and hole-detection in sensor networks via homology.* In Proceedings of the 4^{th} international symposium on Information processing in sensor networks. 2005.
- V. de Silva and R. Ghrist. *Coordinate-free coverage in sensor networks with controlled boundaries via homology*. The International Journal of Robotics Research. 2006
- P. Dlotko, R. Ghrist, M. Juda, and M. Mrozek. *Distributed computation of coverage in sensor networks by homological methods*. Applicable Algebra in Engineering, Communication and Computing. 2012.
- A. Tahbaz-Salehi and A. Jadbabaie. *Distributed coverage verification in sensor networks without location information.* IEEE transaction on Automatic Control. 2010.

Problem Statement

- Detection: is $H_1(R_c) \neq 0$

Problem Statement

- Detection: is $H_1(R_c) \neq 0$
- Localization: compute sparse generators for $H_1(R_c)$

Problem Statement

- Detection: is $H_1(R_c) \neq 0$
- Localization: compute sparse generators for $H_1(R_c)$
- Sparse generators also have many other applications †

† Busaryev, Oleksiy, et al. "Annotating simplices with a homology basis and its applications." Algorithm Theory-SWAT 2012.

A direct method: Čech complex

- let V be the set of points in $\mathbb{R}^2$ (the nodes)

A direct method: Čech complex

- let V be the set of points in $\mathbb{R}^2$ (the nodes)
- The Čech complex $\check{C}(V, r_c)$ has the homotopy type of the union of the balls $B(V, r_c)$

A direct method: Čech complex

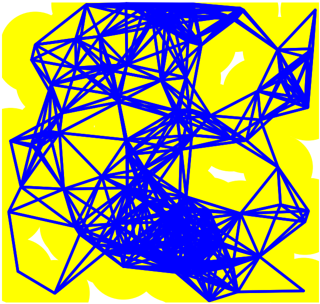
- let V be the set of points in $\mathbb{R}^2$ (the nodes)
- The Čech complex $\check{C}(V, r_c)$ has the homotopy type of the union of the balls $B(V, r_c)$
- The detection problem is then: is $H_1\left(\check{C}(V, r_c)\right) = 0$?

Rips Shadow

- Let $E = \{e = (v_i, v_j), \|v_i - v_j\| \leq 1\}$

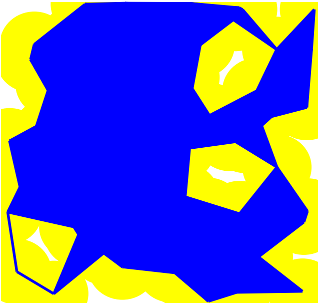
Rips Shadow

- Let $E = \{e = (v_i, v_j), \|v_i - v_j\| \leq 1\}$
- the graph $G_1 = (V, E)$ is called the Unit Disk Graph (UDG) and its flag complex is the Rips complex K_{G_1}



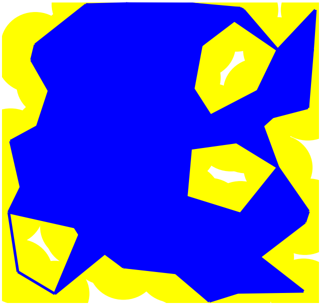
Rips Shadow

- Let $E = \{e = (v_i, v_j), \|v_i - v_j\| \leq 1\}$
- the graph $G_1 = (V, E)$ is called the Unit Disk Graph (UDG) and its flag complex is the Rips complex K_{G_1}
- for a simplex $\sigma \in K_{G_1}$, let $Conv(\sigma)$ denote the convex hull of its vertices in the plane. The Rips shadow R_s is then the union $\cup_\sigma Conv(\sigma)$.



Rips Shadow

- Let $E = \{e = (v_i, v_j), \|v_i - v_j\| \leq 1\}$
- the graph $G_1 = (V, E)$ is called the Unit Disk Graph (UDG) and its flag complex is the Rips complex K_{G_1}
- for a simplex $\sigma \in K_{G_1}$, let $Conv(\sigma)$ denote the convex hull of its vertices in the plane. The Rips shadow R_s is then the union $\cup_\sigma Conv(\sigma)$.
- When $r_c = 1/2$, Rips shadow is a good approximation to the coverage area.



Theorem (Chambers et al,2007)

For any set of points in $\mathbb{R}^2$, $\pi_1(p) : \pi_1(K_{G_1}) \to \pi_1(R_s)$ *is an isomorphism, where* p *is the projection map.*

Theorem (Chambers et al,2007)

For any set of points in $\mathbb{R}^2$, $\pi_1(p) : \pi_1(K_{G_1}) \to \pi_1(R_s)$ *is an isomorphism, where p is the projection map.*

. . . this justifies the use of Rips complex to analyze the coverage area.

Topological Noise

- The knowledge of G_1 requires perfect length information
- for $G_1 = (V, E)$,

$$(v_1, v_2) \in E \Leftrightarrow \|v_1 - v_2\| \leq 1$$

Topological Noise

- The knowledge of G_1 requires perfect length information
- for $G_1 = (V, E)$,

$$(v_1, v_2) \in E \Leftrightarrow \|v_1 - v_2\| \leq 1$$

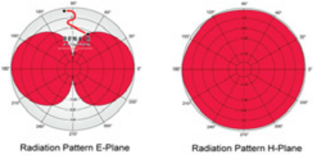
- In general, this is not true

Topological Noise

- The knowledge of G_1 requires perfect length information
- for $G_1 = (V, E)$,

$$(v_1, v_2) \in E \Leftrightarrow \|v_1 - v_2\| \leq 1$$

- In general, this is not true



Quasi unit disk graphs

- $G_1^{\epsilon} = (V, E)$

$$\|v_1 - v_2\| \leq 1 - \epsilon \Rightarrow (v_1, v_2) \in E$$

$$1 - \epsilon < \|v_1 - v_2\| \leq 1$$

$$\Rightarrow (v_1, v_2) \in E \text{ w.p } 0.5$$

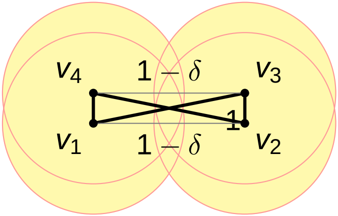
Quasi unit disk graphs

- $G_1^{\epsilon} = (V, E)$

  $$\|v_1 - v_2\| \leq 1 - \epsilon \Rightarrow (v_1, v_2) \in E$$

  $$1 - \epsilon < \|v_1 - v_2\| \leq 1$$
  $$\Rightarrow (v_1, v_2) \in E \text{ w.p } 0.5$$

- $K_{G_1^{\epsilon}}$ does not have the same homology as its shadow.



It gets worse!!

Theorem (Chambers et al, 2007)

Given any value ϵ, and any finitely presented group G, there exists a quasi-Rips complex $K_{G_1^\epsilon}$ with $\pi_1(K_{G_1^\epsilon}) \cong G * F$, where F is a free group.

It gets worse!!

Theorem (Chambers et al, 2007)

Given any value ϵ, and any finitely presented group G, there exists a quasi-Rips complex $K_{G_1^\epsilon}$ with $\pi_1(K_{G_1^\epsilon}) \cong G * F$, where F is a free group.

. . . good news is coming...

Persistence to the rescue

- We can get a good estimate of ϵ

Persistence to the rescue

- We can get a good estimate of ϵ
- consider the following filtration:

$$K_{G_1^\epsilon} \xrightarrow{i} K_{G_1} \xrightarrow{i} K_{G_{1+\epsilon}^\epsilon}$$

Persistence to the rescue

- We can get a good estimate of ϵ
- consider the following filtration:

$$K_{G_1^\epsilon} \xrightarrow{i} K_{G_1} \xrightarrow{i} K_{G^\epsilon_{1+\epsilon}}$$

- if $[c] \neq 0, [c] \in K_{G_1^\epsilon}$ and $i_*^2([c]) \neq 0$, then $i_*([c])$ is non-contractible in K_{G_1}

Detection

Given the complex $K_{G_1^{\epsilon}}$

1. Compute a basis Z for $ker(\partial_1)$

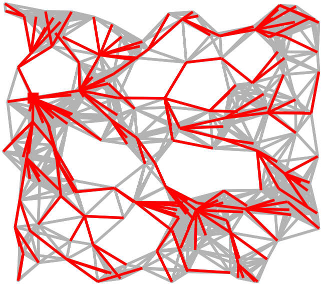
Detection

Given the complex $K_{G_1^\epsilon}$

1. Compute a basis Z for $ker(\partial_1)$
2. $H_1(K_{G_1^\epsilon}) \neq 0$ if and only if $\exists$ a cycle $c \in Z$ such that $[c] \neq 0$ $(c \notin B, B = img(\partial_2))$

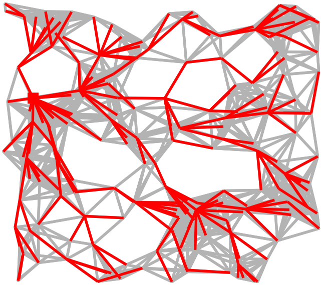
Computing cycle basis Z

1. compute a spanning tree T on G_1^{ϵ}



Computing cycle basis Z

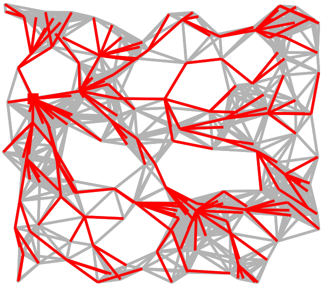
1. compute a spanning tree T on G_1^{ϵ}
2. let r be the root of T. Then for $e = (v_i, v_j)$, let $\gamma(T, e)$ be the algebraic representation of the cycle $(v_i - r - v_j - v_i)$



Computing cycle basis Z

1. compute a spanning tree T on G_1^ϵ
2. let r be the root of T. Then for $e = (v_i, v_j)$, let $\gamma(T, e)$ be the algebraic representation of the cycle $(v_i - r - v_j - v_i)$
3. the set $Z = \{\gamma(T, e), e \notin T\}$ forms a basis for $ker(\partial_1)$

†

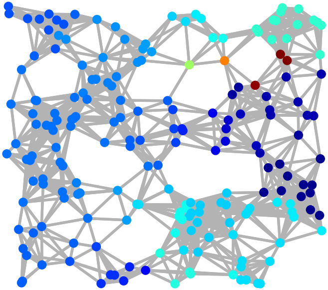


† Busaryev, Oleksiy, et al. "Annotating simplices with a homology basis and its applications." Algorithm Theory-SWAT 2012.

Distributed testing for contractibility

Computing dot products with harmonics

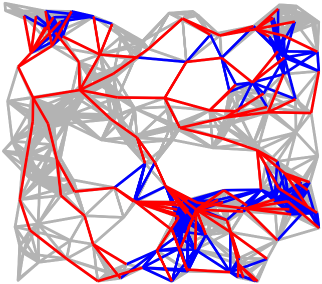
- Compute the integral function of a harmonic y on the tree T.



Distributed testing for contractibility

Computing dot products with harmonics

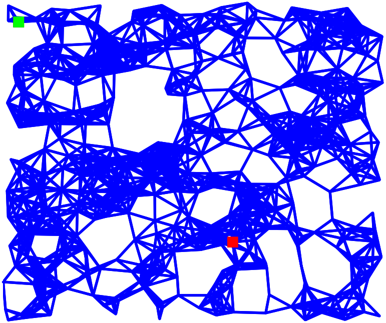
- Compute the integral function of a harmonic y on the tree T.
- Locally determine non-contractible cycles.



Localization

“Divide and Conquer”

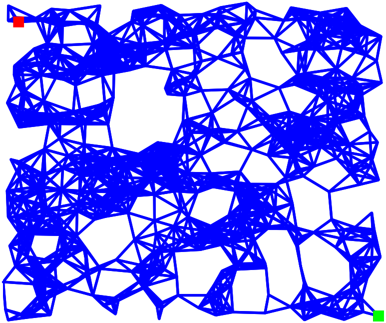
First, find a pair of nodes which are “far apart” from each other in the network.



Localization

"Divide and Conquer"

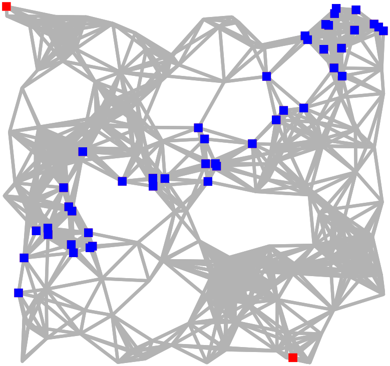
First, find a pair of nodes which are "far apart" from each other in the network.



Localization

"Divide and Conquer"

Find the set of nodes equidistant from the diameter nodes and connect the subgraph



Summary

Distributed

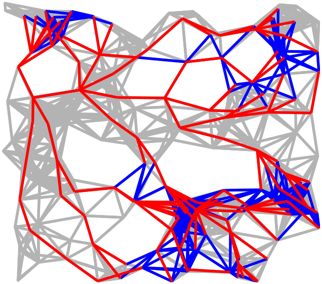
- computing the cycle basis Z,
- detecting non-contractible cycles in Z,
- localizing by dividing network, and
- checking for persistence for existence guarantees.

Computation of a sparse basis for Homology

Select representatives for homologous classes

Sparsifying the complex

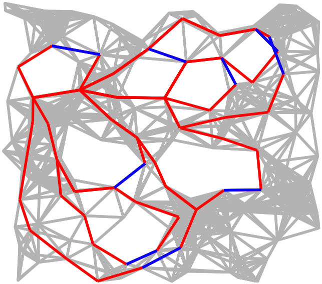
- Start with a non-contractible cycles in the basis Z.



Select representatives for homologous classes

Sparsifying the complex

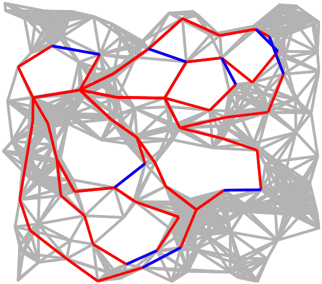
- Start with a non-contractible cycles in the basis Z.
- Select one cycle from each class of homologous cycles.



Select representatives for homologous classes

Sparsifying the complex

- Start with a non-contractible cycles in the basis Z.
- Select one cycle from each class of homologous cycles.
- Denote the above set of cycles by P.



Reduction using harmonics to obtain a basis set

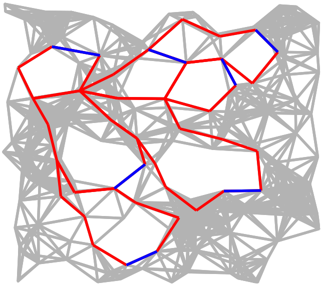
- Compute $|P|$ number of harmonics, Y.

Reduction using harmonics to obtain a basis set

- Compute $|P|$ number of harmonics, Y.
- Compute the matrix Φ where $\Phi_{ij} = \langle Y_i, P_j \rangle$.

Reduction using harmonics to obtain a basis set

- Compute $|P|$ number of harmonics, Y.
- Compute the matrix Φ where $\Phi_{ij} = \langle Y_i, P_j \rangle$.
- Column reduce Φ. The non-zero columns correspond to the basis elements.



Generalizations

- Generalization to higher dimensions: key issue is efficient computation of a cycle basis.
- Generalization to persistence.

Other techniques for reducing a complex

- Vidit Nanda and Konstantin Mischaikow. *Morse Theory for Filtrations and Efficient Computation of Persistent Homology.* Discrete and Computational Geometry. 2013.
- Adam Wilkerson, Harish Chintakunta, and Hamid Krim. *Computing persistent features in big data: A distributed dimension reduction approach.* To appear in International Conference on Acoustics, Speech, and Signal Processing (ICASSP), 2014.

References and Acknowledgements

- Harish Chintakunta and Hamid Krim. *Distributed localization of coverage holes using Topological Persistence.* To appear in IEEE Transactions on Signal Processing. 2014.
- Harish Chintakunta and Hamid Krim. *Distributed computation of homology using harmonics.* arXiv:1306.1158. 2013.

- sponsored by Defense Threat Reduction Agency (DTRA).
- Many thanks to all the members of the VISSTA group at NCSU for productive discussions.

Thank you for your attention!!