Adversarial Risk Analysis: An Overview

David Banks
Duke University

1. Introduction

Classical game theory has focused upon situations in which outcomes are known. When uncertainty is addressed, it makes unreasonable assumptions about common knowledge (cf. Harsanyi, 1967/68a,b). Also, game theory makes unreasonable assumptions about human decision-making (Camerer, 2003).

Classical risk analysis has focused upon situations in which the hazards arise at random. This is appropriate for accident and life insurance, but it does not apply when hazards result from the actions of an intelligent adversary.

Corporate competition, federal regulation, and counterterrorism all entail strategic problems with uncertain outcomes and partial information about the goals and actions of the opponents. This talk describes a Bayesian approach to Adversarial Risk Analysis (ARA). It extends the decision analysis proposed by Kadane and Larkey (1982) and Raiffa (1982) [and rejected by Harsanyi (1982)].

Myerson (1991, p. 114) points up the issues clearly:

> **"A fundamental difficulty may make the decision-analytic approach impossible to implement, however. To assess his subjective probability distribution over the other players' strategies, player i may feel that he should try to imagine himself in their situations. When he does so, he may realize that the other players cannot determine their optimal strategies until they have assessed their subjective probability distributions over i's possible strategies. Thus, player i may realize that he cannot predict his opponents' behavior until he understands what an intelligent person would rationally expect him to do, which is, of course, the problem that he started with. This difficulty would force i to abandon the decision analytic approach and instead undertake a game-theoretic approach, in which he tries to solve all players' decision problems simultaneously."**

However, instead of following Myerson in defaulting back to game theory, we use ARA. In some cases this may be viewed as a Bayesian version of Level-k thinking (Stahl and Wilson, 1995).

The ARA framework builds a model for the decision-making process of the opponents, and uses that to develop a subjective distribution on their actions. The model can be complex; e.g., it can be a mixture over several simpler models.

ARA conveniently partitions the uncertainty in the problem into

- aleatory uncertainty, which describes the randomness in the outcome conditional on the actions chosen;
- epistemic uncertainty, which describes Bayesian beliefs about the utilities, information and capabilities of an opponent; and
- concept uncertainty, which describes uncertainty about the solution concept that an opponent is using.

Parnell and Merrick (2011) compared Probabilistic Risk Analysis with various intelligent adversary methods, and preferred ARA, in large part because this division enables more transparent modeling.

We now explore ARA more formallhy.

2. ARA in General

In ARA one takes the side of one agent, using only her beliefs and knowledge, rather than assuming common knowledge and trying to solve all of the agents' problems simultaneously. The selected agent must have

- a subjective probability about the actions of each opponent,
- subjective conditional probabilities about the outcome for every set of possible choices, and
- perfect knowledge of her own utility function.

Thus Daphne believes Apollo has probability $\pi_D(a)$ of choosing action $a \in \mathcal{A}$. She has a subjective probability $p_D(s \,|\, d, a)$ for each possible outcome $s \in \mathcal{S}$ given every choice $(d, a) \in \mathcal{D} \times \mathcal{A}$. And she knows her own utility $u_D(d, a, s)$ for each combination of outcome and pair of choices.

Daphne maximizes her expected utility by choosing the action d^* such that

$$\begin{aligned} d^* &= \textbf{argmax}_{d \in \mathcal{D}} \, \mathbb{E}_{\pi_D, p_D}[u_D(d, A, S)] \\ &= \textbf{argmax}_{d \in \mathcal{D}} \int_{s \in \mathcal{S}} \int_{a \in \mathcal{A}} u_D(d, a, s) p_D(s \mid d, a) \pi_D(a) \, da \, ds \end{aligned}$$

where A is the random action chosen by Apollo and S is the random outcome that results from choosing A and d.

In practice, the most difficult quantity to obtain is $\pi_D(a)$. The $p_D(s \mid d, a)$ is found by conventional risk analysis and $u_D(d, a, s)$ is a personal utility.

Previously, we laid out ARA methods for obtaining $\pi_D(a)$, in the cases of the the non-strategic opponent, the Nash equilibrium seeking opponent, the opponent whose analysis mirrors that of the decision-maker, and the opponent who is a level-k thinker. Implementing these approaches imposes different cognitive loads upon the analyst.

The following shows how the cognitive load depends upon the kind of ARA. Each row corresponds to a different level of reasoning in level-k thinking.

The table displays the quantities that Daphne must assess in order to implement a level-k analysis. Row 0 corresponds to the utilities and beliefs of Daphne and Apollo, as perceived by themselves. Subsequently, row k contains the additional utilities and probabilities that Daphne would have to assess in order to perform a level-k analysis.

- The first column contains what Daphne believes are the utility functions that Apollo ascribes to her.
- The second column contains the probabilities of the outcome, conditional on both her action and Apollo's, that she believes Apollo ascribes to her.
- The third column contains her opinion of what Apollo thinks is her distribution for he will do.
- The fourth column contains the utility functions she ascribes to Apollo.
- The fifth column contains the conditional probabilities of the outcome, given her choice and Apollo's, that she ascribes to Apollo.
- The sixth column contains what she thinks is Apollo's distribution over her choice.

	1	**2**	**3**	**4**	**5**	**6**
0	u_D	$p_D(\cdot \mid d, a)$	$\pi_D(a)$	u_A	$p_A(\cdot \mid d, a)$	$\pi_A(d)$
1	U_D^1	$P_D^1(\cdot \mid d, a)$	$\Pi_D^1(a)$	U_A^1	$P_A^1(\cdot \mid d, a)$	$\Pi_A^1(d)$
2	U_D^2	$P_D^2(\cdot \mid d, a)$	$\Pi_D^2(a)$	U_A^2	$P_A^2(\cdot \mid d, a)$	$\Pi_A^2(d)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The upper case characters in rows 1 and higher indicate that these quantities are all random variables to Daphne.

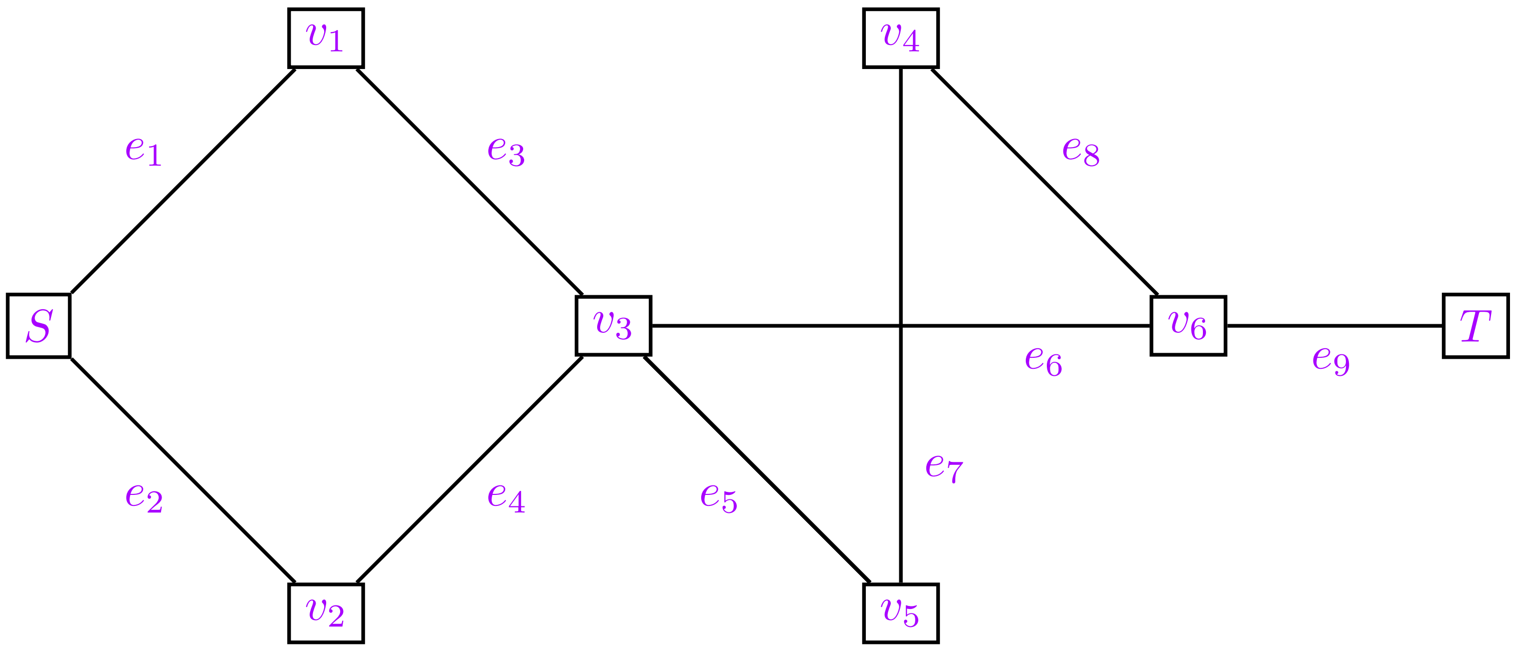
In terms of the table, different solution concepts require information in different cells:

- Traditional game theory requires cells (0,1), (0,2), (0,4), (0,5) and assumes that these are common knowledge.
- The non-strategic adversary analysis requires cells (0,1), (0,2) and (0,3), where the (0,3) cell is assessed from historical data and/or expert opinion.
- When the adversary seeks a Nash equilibrium solution, the analysis requires cells (0,1), (0,2) and (1,1), (1,2), (1,4) and (1,5). It uses these last four cells to infer cell (0,3).
- The level-k adversary approach requires cells (0,1), (0,2) and:
  - for a level-1 analysis, cells (1,4), (1,5) and (1,6) can produce (0,3);
  - for a level-2 analysis, cells (1,1), (1,2) and (1,3) produce (1,6), which, with (1,4), (1,5) can then produce (0,3);
  - and so forth for larger k.
- The mirror equilibrium approach requires cells (0,1), (0,2) and uses a consistency condition between (1,4), (1,5), (1,6) and (1,1), (1,2) and (1,3) to produce (0,3).

The main message is that all of these methods entail significant effort.

3. Routing Games

The most famous routing game is Nash. But a simpler game with practical importance is to route a convoy through a city street network when an adversary may place IEDs.



The Defender has imperfect information about the placement of the IEDs, and the Attacker's resources and utilities. Symmetrically, the Attacker has probabilistic knowledge about route choice, convoy value, and Defender utilities. But Harsanyi's common-knowledge analysis is untenable.

As with auctions, this leads to coupled probability equations. But here the stochastic payoffs for the Attacker and the Defender have additive structure.

$\boldsymbol{Y}$: the Defender's privately known loss matrix (which is unknown to the Attacker);

$\tilde{\boldsymbol{X}}$: the random variable which the Defender uses to model the Attacker's gain matrix—it has probability distribution F;

$\tilde{\boldsymbol{Y}}$: the random variable that the Defender uses to describe the Attacker's beliefs about the Defender's loss matrix—it has probability distribution G;

$\tilde{\boldsymbol{a}}$: the random vector that the Defender uses to model the Attacker's decision—it has distribution P with support $\mathscr{A}$;

$\tilde{\boldsymbol{r}}$: the random vector that the Defender uses to model the Attacker's belief about the Defender's decision—it has distribution Q with support $\mathscr{D}$.

ARA Algorithm: Assume $\tilde{\boldsymbol{X}} \sim F$, $\tilde{\boldsymbol{Y}} \sim G$, $\boldsymbol{a} \in \mathscr{A}$ and $\boldsymbol{r} \in \mathscr{D}$.

1. **Initialize.** The Defender starts with a pair of probability distributions (P_0, Q_0), where P_0 is a distribution for $\tilde{\boldsymbol{a}}$ and Q_0 is a distribution for $\tilde{\boldsymbol{r}}$.

2. **Iterate.** Given (P_k, Q_k), iterate to convergence.

   2.A Simulate many realizations of $\tilde{\boldsymbol{X}}$. For each, the Defender mimics the Attacker's analysis and solves $\boldsymbol{a}^* = \max_{\boldsymbol{a} \in \mathscr{A}} \boldsymbol{a}' \tilde{\boldsymbol{X}} \mathbb{E}_{Q_k}[\tilde{\boldsymbol{r}}]$. Since $\tilde{\boldsymbol{X}} \sim F$, the resulting maximizer is a random variable, and the distribution of the solutions is an estimate of P_{k+1}, the updated distribution of $\tilde{\boldsymbol{a}}$; i.e.,

$$\underset{\boldsymbol{a} \in \mathscr{A}}{\operatorname{argmax}}\, \boldsymbol{a}' \tilde{\boldsymbol{X}} \mathbb{E}_{Q_k}[\tilde{\boldsymbol{r}}] = \tilde{\boldsymbol{a}} \sim P_{k+1}.$$

   2.B Update Q_k of $\tilde{\boldsymbol{r}}$ using P_{k+1} by generating realizations of $\tilde{\boldsymbol{Y}}$ and solving

$$\underset{\boldsymbol{r} \in \mathscr{D}}{\operatorname{argmin}}\, \mathbb{E}_{P_{k+1}}[\tilde{\boldsymbol{a}}]' \tilde{\boldsymbol{Y}} \boldsymbol{r} = \tilde{\boldsymbol{r}} \sim Q_{k+1}.$$

3. **Terminate.** The Defender chooses $\boldsymbol{r}^* = \underset{\boldsymbol{r} \in \mathscr{D}}{\operatorname{argmin}}\, \mathbb{E}_{P^*}[\tilde{\boldsymbol{a}}]' \boldsymbol{Y} \boldsymbol{r}$. In this final step the Defender uses the true loss matrix $\boldsymbol{Y}$.

For this situation, we can obtain two theorems:

Theorem 1: A mirroring fixed-point for the system of equations exists.

Theorem 2: If there exists a total order $\succeq_{\mathscr{A}}$ on $\mathscr{A}$ and $\succeq_{\mathscr{D}}$ on $\mathscr{D}$ such that

1. $\tilde{V}_{a,r} := \boldsymbol{a}\tilde{\boldsymbol{X}}\boldsymbol{r}$ has increasing difference in $(\boldsymbol{a}, \boldsymbol{r})$,
2. $\tilde{W}_{a,r} := \boldsymbol{a}\tilde{\boldsymbol{Y}}\boldsymbol{r}$ has decreasing difference in $(\boldsymbol{a}, \boldsymbol{r})$,

then the ARA Algorithm converges to the mirroring fixed point.

The second theorem is technical, using submodularity, but is satisfied if most (some) of the losses or gains are of opposite sign. For example, it holds for zero-sum games.

In general, finding fixed-point solutions in game theory is hard. For the special structure of the routing game, we know that equilibria exist and can provide conditions under which a reasonable algorithm converges to the mirroring method solution.

The mirroring argument provides an explicit mechanism for modeling the reasoning of one's opponents. Previously, the decision-theoretic Bayesians who did game theory simply declared a distribution over the actions of their opponents.

Kadane (2009) points to a passage in Poe's The Purloined Letter that illustrates the naturalness of the ARA approach, in contrast to the minimax solution. Dupin recalls:

> **I knew one [school-boy] about eight years of age, whose success at guessing in the game of "even and odd" attracted universal admiration. This game is simple, and is played with marbles. One player holds in his hand a number of these toys and demands of another whether that number is even or odd. If the guess is right, the guesser wins one; if wrong, he loses one. The boy to whom I allude won all the marbles of the school. Of course he had some principle of guessing; and this lay in mere observation and admeasurement of the astuteness of his opponents.**

For example, an arrant simpleton is his opponent, and, holding up his closed hand, asks, "Are they even or odd?" Our school-boy replies, "Odd," and loses; but upon the second trial he wins, for he then says to himself: "The simpleton had them even upon the first trial, and his amount of cunning is just sufficient to make him have them odd upon the second; I will therefore guess odd"; he guesses odd, and wins. Now, with a simpleton a degree above the first, he would have reasoned thus: "This fellow finds that in the first instance I guessed odd, and, in the second, he will propose to himself, upon the first impulse, a simple variation from even to odd, as did the first simpleton; but then a second thought will suggest that this is too simple a variation, and finally he will decide upon putting it even as before. I will therefore guess even"; he guesses even, and wins. Now this mode of reasoning in the schoolboy, whom his fellows termed "lucky," what, in its last analysis, is it?

It is merely, I said, an identification of the reasoner's intellect with that of his opponent.

3. Le Relance: A Primitive Version of Poker

Pokeresque games have received considerable attention in the game theory literature. Early work by von Neumann and Morgenstern (1947) and Borel (1938) developed solutions under various simplifying assumptions. More recently, Ferguson and Ferguson (2008) provide approximate analyses pertinent to more complex games, such as Texas hold'em.

In the following, assume that Bart and Lisa play a game in which each privately and independently draws a $\mathcal{U}[0,1]$ random number. Each must ante an amount $a = 1$. First, Bart examines his number X and decides whether to bet b or fold. Then Lisa examines her Y and decides whether to bet b or fold. If both players bet, they compare their draws to determine who wins the pot. Otherwise, the first person to fold forfeits his or her ante.

Let V_x be the amount Bart wins. The table shows the four possible situations:

V_x	**Bart's Decision**	**Lisa's Decision**	**Outcome**
-1	**fold**		
1	**bet**	**fold**	
1+b	**bet**	**bet**	$X > Y$
-(1+b)	**bet**	**bet**	$X < Y$

From the table, the expected amount won by Bart, given his draw $X = x$, is:

$$\begin{aligned} \mathbb{E}[V_x] &= -\mathbb{P}[\textbf{ Bart folds }] + \mathbb{P}[\textbf{ Bart bets and Lisa folds }] \\ &\quad +(1+b)\mathbb{P}[\textbf{ Lisa bets and loses }] \\ &\quad -(1+b)\mathbb{P}[\textbf{ Lisa bets and wins }]. \end{aligned}$$

Bart must use mirroring to find a subjective distribution for the probabilities, based on the adversarial analysis he expects Lisa to perform.

Assume that Bart uses a "bluffing function" $g(x)$; given x, he bets with probability $g(x)$. Then

$$\begin{aligned} \mathbb{E}[V_x] &= -[1-g(x)] + g(x)\mathbb{P}[\textbf{ Lisa folds } \mid \textbf{ Bart bets }] \\ &\quad +(1+b)g(x)x\mathbb{P}[\textbf{ Lisa bets } \mid \textbf{ Bart bets }] \\ &\quad -(1+b)g(x)(1-x)\mathbb{P}[\textbf{ Lisa bets } \mid \textbf{ Bart bets }]. \end{aligned}$$

For optimal play, Bart needs to find $\mathbb{P}[$ Lisa bets $\mid$ Bart bets $]$.

So Bart must "mirror" the thinking that Lisa will perform in deciding whether to bet. He knows that Lisa's opinion about X is updated by the knowledge that Bart decided to bet. Further, suppose Bart has a subjective belief that Lisa thinks that his bluffing function is $\tilde{g}(x)$. In that case, Lisa should calculate the conditional density of X, given that Bart decided to bet, as

$$\tilde{f}(x) = \frac{\tilde{g}(x)}{\int \tilde{g}(z)\,dz}.$$

Note: If $\tilde{g}$ is a step function (i.e., Lisa believes that Bart does not bet if x is less than some value x_0, but always bets if it is greater), then the posterior distribution on X is truncated below the X value corresponding to x_0 and the weight is reallocated proportionally to values above x_0.

From this analysis, Bart believes that Lisa calculates her probability of winning as $\mathbb{P}[X \leq y| \text{ Bart bet }] = \tilde{F}(y)$, where $Y = y$ is unknown to Bart. And thus Bart believes that Lisa will bet if the expected value of her return V_y from betting b is greater than the loss of a that results from folding; i.e., Lisa would bet if

$$\mathbb{E}[V_y] = (1+b)\tilde{F}(y) - (1+b)[1-\tilde{F}(y)] \geq -1.$$

So Bart believes Lisa will bet if and only if $\tilde{F}(y) \geq b/2(1+b)$.

Set $\tilde{y} = \inf\{y : \tilde{F}(y) \geq b/2(1+b)\}$. The probability that Lisa has drawn $Y > \tilde{y}$ is $1 - \tilde{y}$ and this is the probability that she bets. So the expected value of the game for Bart, given $X = x$, is:

$$V_x = -[1-g(x)] + g(x)\tilde{y} + (1+b)g(x)[x-\tilde{y}]^+ - (1+b)g(x)(1-\tilde{y}-[x-\tilde{y}]^+).$$

Bart should choose $g(x)$ to maximize V_x.

Bart's expected value has the form $-1 + cg(x)$, where

$$c = 1 + \tilde{y} + (1+b)[x - \tilde{y}]^+ - (1+b)(1 - \tilde{y} - [x - \tilde{y}]^+).$$

To maximize the expectation, Bart should make $g(x)$ as small as possible when c is negative (i.e., $g(x) = 0$), but as large as possible when c is positive (i.e., $g(x) = 1$). Thus the optimal $g(x)$ is a step function. It implies that Bart should never bluff, no matter what he believes about the playing strategy used by Lisa.

When $x \le \tilde{y}$, Bart bets if $\tilde{y} > b/(b+2)$, he folds if $\tilde{y} < b/(b+2)$, and he may do as he pleases when $\tilde{y} = b/(b+2)$. When $x > \tilde{y}$, then Bart bets if $x > \tilde{x} = [b(1+\tilde{y})]/[2(1+b)]$, folds if $x < \tilde{x}$, and may do as he pleases when $x = \tilde{x}$.

As a sanity check, if $b = 0$ then Lisa should always bet. Here $\tilde{x} = 0$, properly implying that Bart also always bets.

The expected value of the game, to Bart, is $V = \int_0^1 V_x \, dx$. Its value depends on his belief about Lisa's play.

Case I: Bart Believes that Lisa Plays Minimax.

The traditional minimax solution has $\tilde{y} = b/(b+2)$. In that case it is known that Bart should bet if $x > \tilde{y}$, and he should bet with probability $2/(b+2)$ when $x \leq \tilde{y}$. The value of the game (to Bart) is $V = -b^2/(b+2)^2$; he is disadvantaged by the sequence of play.

In contrast, the ARA analysis finds that when Lisa uses the minimax threshold $\tilde{y} = b/(b+2)$, then Bart may bet or not, as he pleases, when $x \leq \tilde{x}$. This is slightly different from the minimax solution.

The difference arises because, if Lisa knows that Bart's bluffing function does not bet with probability $2/(b+2)$ when $x \leq b/(b+2)$, then she can improve her expected value for the game by changing the threshold at which she calls.

In the minimax game, Bart's bluff pins Lisa down, preventing her from using a more profitable rule. But for either game, the value for Bart is unchanged: $-\left(\frac{b}{b+2}\right)^2$.

Case II: Bart Believes that Lisa Is Rash.

Suppose that Bart's analysis leads him to think that Lisa is reckless, calling with $\tilde{y} < b/(b+2)$. Then the previous ARA shows that his bluffing function should be

$$g(x) = \begin{cases} 0 & \textbf{if } 0 \leq x \leq \max\{\tilde{y}, \tilde{x}\} \\ 1 & \textbf{if } \max\{\tilde{y}, \tilde{x}\} < x \leq 1 \end{cases}$$

where $\tilde{x} = [b(1+\tilde{y})]/[2(1+b)]$.

The value of this ARA game to Bart is

$$\begin{aligned} V &= -\int_0^{\tilde{x}} dx + \int_{\tilde{x}}^1 -1 + 2x + 2bx - b\tilde{y} - b\,dx \\ &= b\tilde{x} - b\tilde{y}(1-\tilde{x}) - (1+b)\tilde{x}^2. \end{aligned}$$

The value of this ARA game is strictly larger than the minimax value.

Case III: Bart Believes that Lisa Is Conservative.

Suppose Bart believes that Lisa is risk averse, calling with $\tilde{y} > b/(b+2)$. Then

$$V_x = -1 + g(x)\left[1 + \tilde{y} + (1+b)(1-\tilde{y})\frac{x-\tilde{y}}{1-\tilde{y}} - (1+b)(1-\tilde{y})\left(1 - \frac{x-\tilde{y}}{1-\tilde{y}}\right)\right].$$

When $x > \tilde{y}$, Bart's optimal play is to bet. On the other hand, when $x < \tilde{y}$, Bart's payoff is

$$V_x = -1 + g(x)\left[1 + \tilde{y} - (1+b)(1-\tilde{y})\right].$$

For $\tilde{y} > b/(b+2)$, the quantity in the square brackets is strictly positive. Thus, when $x < \tilde{y}$, Bart should bet.

The value V of this game is

$$V = \int_0^{\tilde{y}} \tilde{y} - (1+b)(1-\tilde{y}) + \int_{\tilde{y}}^1 \tilde{y} + (1+b)(x-\tilde{y}) - (1+b)(1-x).$$

Solving the integral shows $V = -b\tilde{y} + \tilde{y}^2(1+b)$. This value is increasing in $\tilde{y}$ for $\tilde{y} > b/(2+b)$ and it is equal to the minimax value at $\tilde{y} = b/(b+2)$. Thus the value of the ARA game when Lisa is conservative is strictly larger than the minimax value.

Note: This analysis of the Borel Game extends immediately to situations in which the two players draw independently from a continuous distribution W with density w. In that case, the conditional distribution that Bart imputes to Lisa is

$$\tilde{f}(x) = \frac{\tilde{g}(W(x))w(x)}{\int \tilde{g}(W(z))w(z)\,dz}$$

and Bart's bluffing function takes its step at

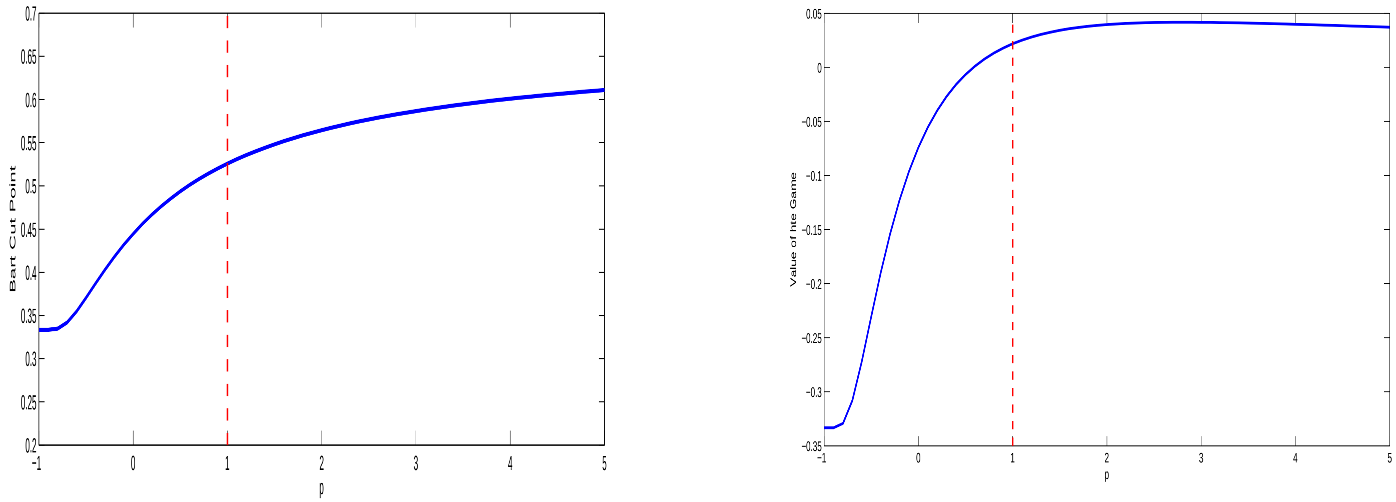
$$\tilde{x} = \frac{1}{2}\left[1 - \frac{1}{1+b}\frac{1+W(\tilde{y})}{1-W(\tilde{y}}\right].$$

If Bart and Lisa draw from a bivariate, possibly discrete distribution $W(x, y)$ (e.g., a deck of cards) then the analysis is trivial (in G. H. Hardy's sense): Bart's distribution for Y is the conditional $W(y|X = x)$, and he knows that Lisa's analysis is symmetric.

Note: Some may be uncomfortable with the specificity in requiring Bart to assume that Lisa thinks his bluffing function is $g(\tilde{x})$. They might argue that Bart could not guess that exactly—that it would be more reasonable to say that he has a subjective distribution over the set $\mathcal{G}$ of all possible bluffing functions. But when Bart integrates over that space with respect to his subjective distribution, he then obtains the $\tilde{g}$ that he needs for this analysis.

Example: The $\tilde{g}$ is a power function.

Suppose that Bart believes that Lisa thinks his bluffing function has the form $g(x) = x^p$ for some fixed value $p > -1$. Then $\tilde{y} = \sqrt[p+1]{\frac{1}{2}\frac{b}{1+b}}$. Large values of p imply that Lisa believes Bart tends to bet for large values of x, leading Lisa to fold more frequently and increasing Bart's expected payoff.



The left panel shows, for $b = 2$, the minimum value of x at which Bart should bet as a function of p. The right panel shows the game value, to Bart, as a function of p.

Continuous Bets

Consider a modification of the Borel Game, in which Bart is not constrained to bet any amount on some interval $(\epsilon, K]$.

Define the following notation:

ϵ, K: the lower and upper bounds of the bets Bart can choose, if he decides to bet; i.e. $[\epsilon, K]$ is Bart's betting strategy space, where $0 < \epsilon \ll K$ (usually ϵ is a very small positive number).

$g(x)$: the probability that Bart decides to bet after learning $X = x$.

$h(b|x)$: a probability density on $[\epsilon, K]$ that Bart will use to select his bet conditional on his decision to bet.

B_x: a random variable with value in $[\epsilon, K]$ representing Bart's bet after he learns $X = x$.

Let $\mathbb{P}_{h(\cdot|x)}[\cdot]$ and $\mathbb{E}_{h(\cdot|x)}[\cdot]$ denote the probability and expectation computed using the probability measure induced by the density $h(\cdot|x)$.

Bart must "mirror" Lisa's analysis given that she observes Bart's bet $B_x = b$. Define

$\tilde{g}(x)$**:** Bart's belief about Lisa's belief of the probability that he decides to bet with $X = x$.

$\tilde{h}(b|x)$**:** Bart's belief about Lisa's belief of the density on $[\epsilon, K]$ that Bart uses to bet.

$\tilde{f}(x|b)$**:** Bart's belief about Lisa's posterior density for X after she observes that he bets b:

$$\tilde{f}(x|b) = \frac{\tilde{h}(b|x)\tilde{g}(x)}{\int_0^1 \tilde{h}(b|z)\tilde{g}(z)\,dz}.$$

Given $g(x)$ and $h(\cdot|x)$, then $V_x = \mathbb{E}_{g(x),h(\cdot|x)}[V_B|X = x]$:

$$\begin{aligned} V_x \quad = \quad & \underbrace{-(1-g(x))}_{\textbf{Bart folds}} + g(x)\Big\{\mathbb{E}_{h(\cdot|x)}\Big[\mathbb{P}_{\tilde{f}(\cdot|B_x)}[\textbf{ Lisa folds } | \textbf{ Bart bets } B_x] \,|\, X = x\Big] \\ & + \mathbb{E}_{h(\cdot|x)}\Big[\mathbb{P}_{\tilde{f}(\cdot|B_x)}[\textbf{ Lisa loses } | \textbf{ Bart bets } B_x]\cdot(1+B_x)\,|\, X = x\Big] \\ & - \mathbb{E}_{h(\cdot|x)}\Big[\mathbb{P}_{\tilde{f}(\cdot|B_x)}[\textbf{ Lisa wins } | \textbf{ Bart bets } B_x]\cdot(1+B_x)\,|\, X = x\Big]\Big\}. \end{aligned}$$

Bart's first-order ARA solution is

$$\{g^*(x), h^*(\cdot|x)\} \in \underset{g(x),h(\cdot|x)}{\textbf{argmax}} \quad \mathbb{E}_{g(x),h(\cdot|x)}\left[V_B|X=x\right].$$

To solve for $\{g^*(x), h^*(\cdot|x)\}$, he studies Lisa's strategy and rolls back.

Bart believes Lisa will form the posterior assessment $\tilde{f}(\cdot|b)$ on his X, so for $Y = y$, Bart believes Lisa thinks her probability of winning is

$$\mathbb{P}_{\tilde{f}(\cdot|B_x)}[X \leq Y \mid B_x, Y = y] = \int_0^y \tilde{f}(z|B_x)\,dz.$$

So Bart believes that Lisa is, by calling, expecting a payoff of

$$\begin{aligned} V_y &= \mathbb{P}_{\tilde{f}(\cdot|B_x)}[\textbf{ Lisa wins } |B_x, Y=y, \textbf{ Lisa calls }] \cdot (1+B_x) \\ &\quad - \mathbb{P}_{\tilde{f}(\cdot|B_x)}[\textbf{ Lisa loses } \mid B_x, Y=y, \textbf{ Lisa calls }] \cdot (1+B_x) \\ &= 2(1+B_x)\int_0^y \tilde{f}(z|B_x)\,dz - (1+B_x). \end{aligned}$$

So Bart believes Lisa will call if and only if

$$-1 \le 2(1+B_x)\int_0^y \tilde{f}(z|B_x)\,dz - (1+B_x).$$

Since $\tilde{f}(z|B_x) \ge 0$, then for all $y \ge \tilde{y}^*(B_x)$ we must have

$$\int_0^y \tilde{f}(z|B_x)\,dz \ge \int_0^{\tilde{y}^*} (B_x)\tilde{f}(z|B_x)\,dz \ge \frac{B_x}{2(1+B_x)}.$$

Then Lisa will call if and only if

$$Y \ge \tilde{y}^*(B_x) \equiv \inf\left\{ y \in [0,1] : \int_0^y \tilde{f}(z|B_x)\,dz \ge \frac{B_x}{2(1+B_x)} \right\}.$$

Hence, Bart believes that the probability Lisa will call after he bets the amount B_x should be

$$\mathbb{P}_{\tilde{f}(\cdot|B_x)}[\ \textbf{Lisa calls}\ \mid\ \textbf{Bart bets}\ B_x] = \mathbb{P}[Y \ge \tilde{y}^*(B_x)\,|\,B_x] = 1 - \tilde{y}^*(B_x).$$

Now Bart is able to compute the following quantities:

$$\begin{aligned}
\mathbb{P}_{\tilde{f}(\cdot|B_x)}[\textbf{ Lisa folds } | \textbf{ Bart bets } B_x] &= \tilde{y}^*(B_x); \\
\mathbb{P}_{\tilde{f}(\cdot|B_x)}[\textbf{ Lisa loses } | \textbf{ Bart bets } B_x] &= \mathbb{P}[\tilde{y}^*(B_x) \le Y \le x|B_x] \\
&= [x - \tilde{y}^*(B_x)]^+; \\
\mathbb{P}_{\tilde{f}(\cdot|B_x)}[\textbf{ Lisa wins } | \textbf{ Bart bets } B_x] &= \mathbb{P}_{\tilde{f}(\cdot|B_x)}[\textbf{ Lisa calls } | \textbf{ Bart bets } B_x] \\
&\quad - \mathbb{P}_{\tilde{f}(\cdot|B_x)}[\textbf{ Lisa loses } | \textbf{ Bart bets } B_x] \\
&= 1 - \tilde{y}^*(B_x) - [x - \tilde{y}^*(B_x)]^+.
\end{aligned}$$

Combining these expressions shows:

$$\begin{aligned}
V_x &= -(1 - g(x)) + \\
&\quad g(x)\mathbb{E}_{h(\cdot|x)}\left[\tilde{y}^*(B_x) + 2[x - \tilde{y}^*(B_x)]^+(1 + B_x) - (1 - \tilde{y}^*(B_x))(1 + B_x)\right].
\end{aligned}$$

Theorem: For $x \in [0,1]$ and given $\tilde{f}(\cdot|b)$ positive and continuous in $b \in [\epsilon, K]$, let

$$\begin{aligned} b^*(x) &\in \operatorname*{\mathbf{argmax}}_{b\in[\epsilon,K]} \quad \tilde{y}^*(b) + 2(x - \tilde{y}^*(b))^+(1+b) - (1-\tilde{y}^*(b))(1+b), \\ \Delta^*(x) &\equiv \max_{b\in[\epsilon,K]} \quad \tilde{y}^*(b) + 2(x - \tilde{y}^*(b))^+(1+b) - (1-\tilde{y}^*(b))(1+b). \end{aligned}$$

Then, Bart's first-order ARA solution is

$$\begin{aligned} g^*(x) &= \begin{cases} 0 & \textbf{if} \;\; \Delta^*(x) < -1 \\ 1 & \textbf{if} \;\; \Delta^*(x) \geq -1; \end{cases} \\ h^*(b|x) &= \delta(b - b^*(x)), \end{aligned}$$

where $\delta(\cdot)$ is the Dirac delta function.

In other words, when he observes $X = x$, Bart will fold with probability 1 if $\Delta^*(x) < -1$ and bet $b^*(x)$ with probability 1 if $\Delta^*(x) \geq -1$. Of course, the regularity condition requiring that $\tilde{f}(\cdot|b)$ be positive and continuous in $b \in [\epsilon, K]$ is purely sufficient but not necessary.

Example: Lisa has a step-function posterior.

To illustrate the use of the theorem to find the ARA solution in a Borel game with continuous bets, suppose $\tilde{f}(\cdot|b)$ is of the following form:

$$\tilde{h}(x|b) = \begin{cases} \frac{1+K}{1+b} & \textbf{if } 0 \le x \le \frac{1+b}{1+K} \\ 0 & \textbf{otherwise}. \end{cases}$$

It is easy to see that $\tilde{y}^*(b) = \frac{b}{2(1+K)}$, and

$$\begin{aligned} &\tilde{y}^*(b) + 2(x - \tilde{y}^*(b))^+(1+b) - (1 - \tilde{y}^*(b))(1+b) \\ &= \begin{cases} -\frac{b^2}{2(1+K)} + (2x-1)(b+1) & \textbf{if } b \le 2(1+K)x \\ \frac{b^2}{2(1+K)} - \frac{K}{1+K}b - 1 & \textbf{if } b > 2(1+K)x. \end{cases} \end{aligned}$$

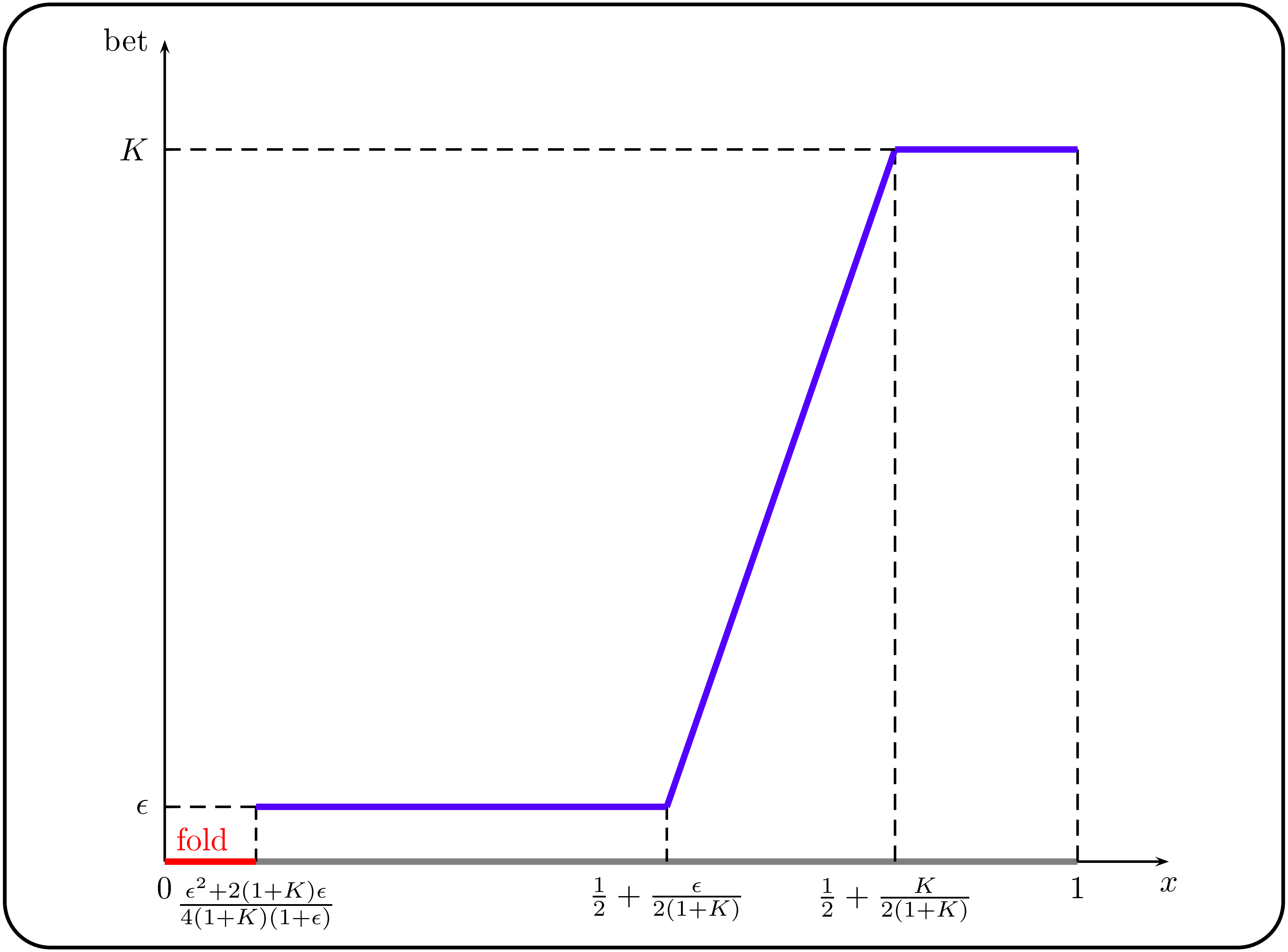
Assume that ϵ is small enough that $\frac{\epsilon^2+2(1+K)\epsilon}{4(1+K)(1+\epsilon)} < \frac{1}{2} + \frac{\epsilon}{2(1+K)}$. Consider the following cases:

1. For $x < \frac{\epsilon^2+2(1+K)\epsilon}{4(1+K)(1+\epsilon)}$, then $b^*(x) = \epsilon$ and $\Delta^*(x) = -\frac{\epsilon^2}{2(1+K)} + (2x-1)(\epsilon+1) < -1$. By the theorem, $g^*(x) = 1$; i.e., Bart will fold w.p. 1. There is no need to specify $h^*(\cdot|x)$.

2. For $\frac{\epsilon^2+2(1+K)\epsilon}{4(1+K)(1+\epsilon)} \leq x < \frac{1}{2} + \frac{\epsilon}{2(1+K)}$, then $b^*(x) = \epsilon$ and $\Delta^*(x) = -\frac{\epsilon^2}{2(1+K)} + (2x-1)(\epsilon+1) \geq -1$. By the theorem, $g^*(x) = 1$ and $h^*(b|x) = \delta(b-\epsilon)$, i.e. Bart will bet ϵ w.p. 1.

3. For $\frac{1}{2} + \frac{\epsilon}{2(1+K)} \leq x < \frac{1}{2} + \frac{K}{2(1+K)}$, then $b^*(x) = 2(1+K)x - (1+K)$ and $\Delta^*(x) = \frac{1+K}{2}(2x-1)^2 + (2x-1) \geq -1$. By the theorem, $g^*(x) = 1$ and $h^*(b|x) = \delta(b - (2(1+K)x - (1+K)))$; i.e., Bart will bet $2(1+K)x - (1+K)$ w.p. 1.

4. For $x \geq \frac{1}{2} + \frac{K}{2(1+K)}$, then $b^*(x) = K$ and $\Delta^*(x) = -\frac{K^2}{2(1+K)} + (2x-1)(K+1) \geq -1$. Then, by the Theorem, $g^*(x) = 1$ and $h^*(b|x) = \delta(b-K)$; i.e., Bart will bet K w.p. 1.



bet
K
ϵ
fold
0
$\frac{\epsilon^2+2(1+K)\epsilon}{4(1+K)(1+\epsilon)}$
$\frac{1}{2}+\frac{\epsilon}{2(1+K)}$
$\frac{1}{2}+\frac{K}{2(1+K)}$
1
x

5. Main Points

ARA allows the analyst to flexibly model the thought-process of the opponents. This fits naturally with a large body of modern work in behavioral game theory, and avoids awkward assumptions about rationality and common knowledge.

It also partitions the total uncertainty into usefully distinct parts (aleatory, epistemic, and concept uncertainty), which facilitates elicitation and calculation.

The talk described ARA perspectives in two game settings: convoy routing and *La Relance*. The examples find interestingly different results than one obtains under traditional solution concepts.

Finally, the ARA formulation leads to new research questions, as with the n-person version of *La Relance* in which one can model all the pairwise beliefs that bidders have about each other's valuations.