Adversarial Risk Analysis: Auctions

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1. One-Person Auctions

Suppose Daphne is bidding for a first edition of the Theory of Games and Economic Behavior. She is the only bidder, but the owner has set a secret reservation price v^* below which the book will not be sold. Daphne does not know v^*, and expresses her uncertainty as a subjective Bayesian distribution $F(v)$.

Daphne's utility function is linear in money and her personal valuation of the book is d_0. If money is infinitely divisible, her choice set is $\mathcal{D} = \mathbb{R}^+$. so her expected utility from a bid of d is $(d_0 - d)\mathbb{P}[d > V^*]$. Thus Daphne should maximize her expected utility by bidding

$$d^* = \mathbf{argmax}_{d \in \mathbb{R}^+} (d_0 - d)F(d).$$

This is a standard approach in Bayesian auction theory (cf. Raiffa, 2002).

2. Two-Person Auctions

Consider a two-person first-price independent private-value sealed-bid auction among risk-neutral opponents (hereafter, **auction**).

Specifically, suppose Daphne and Apollo are bidding for a first edition of the Theory of Games and Economic Behavior.

Aleatory uncertainty arises in this situation if the value of the book is a random variable. Perhaps it is damaged, or has marginalia by John Nash. So the profit or loss, conditional on the bids, is a random variable.

Epistemic uncertainty arises because neither opponent knows the value (or expected value) of the book to the other.

Concept uncertainty arises because Daphne does not know how Apollo will determine his bid. Will he be non-strategic? Will he seek a Bayes Nash equilibrium? Will he use level-k thinking?

To begin, we assume there is no aleatory uncertainty—both bidders know their personal value. We analyze the game from the perspective of Daphne, whose certain value for the book is x_0.

She may think that Apollo is non-strategic, and that he bids some random fraction P of his true value V. As a Bayesian, Daphne has a subjective distribution f_1 over V and a subjective distribution f_2 over P. In that case her belief about the distribution of Apollo's bid is

$$G(y) = \mathbb{P}[PV \leq y] = \int_0^\infty \int_0^{y/v} f_2(p) f_1(v)\, dp dv.$$

Then Daphne's optimal bid is

$$x^* = \mathbf{argmax}_{x \in \mathbb{R}^+} (x_0 - x) G(x)$$

since this maximizes her expected utility. If she wins, her profit is $(x_0 - x)$, and her subjective probability of winning is $G(x)$.

But Daphne may think Apollo is strategic. Perhaps he seeks a **Bayes Nash equilibrium** (BNE) solution.

The BNE formulation makes a strong common knowledge assumption: both Apollo and Daphne have distributions H_D and H_A for each other's valuation, and each knows both distributions and knows that the other knows them.

This leads to solving a system of first-order ODEs. For an asymmetric auction, when $H_A \neq H_D$, no solution algorithm exists, although it is known that if H_A and H_D are differentiable then a unique solution exists and is also differentiable (LeBrun, 1999).

Previous attempts at solutions are based on the **backshooting algorithm**. But Fibich and Gavish (2011) have recently shown that all such algorithms are inherently unstable. Kirkegaard (2009) established results on crossing conditions in the solutions, and Hubbard et al. (2012) used these to provide a visual test, but their work fails in examples. Tim Au (2014) has an algorithm that succeeds, based on the limit of discretized bids and points of indifference.

From an ARA perspective, the common knowledge assumption can be replaced by something more reasonable. Daphne has a subjective opinion about the distribution H_D that she thinks Apollo has for her value, and she has a subjective opinion about H_A, the distribution she believes he thinks is her distribution for his value. The H_A and H_D represent her epistemic uncertainty.

In that framework, Apollo solves the BNE equations:

$$\begin{aligned} \mathbf{argmax}_{d \in \mathbb{R}^+} (D^* - d) G(d) &\sim F \\ \mathbf{argmax}_{a \in \mathbb{R}^+} (A^* - a) F(a) &\sim G. \end{aligned}$$

where $D \sim H_D$ and $A \sim H_A$. The equilibrium solution gives G, her best guess, under the BNE solution concept, of the distribution for Apollo's bid.

Now Daphne should step outside the BNE framework and solve

$$x^* = \mathbf{argmax}_{x \in \mathbb{R}^+} (x_0 - x) G(x)$$

where x_0 is her true value. This is a **mirroring argument**.

Note: A nice feature of the ARA mirror equilibrium formulation is that it allows a new class of problems in n-person games. If Bob is also bidding for the book, then Daphne can have opinions about what Apollo thinks about Bob and what Apollo thinks Daphne thinks about Bob that are not expressible in the BNE common-knowledge framework.

As long as all of Daphne's opinions are coherent, then there is a solution that gives her best guess about the bidding distributions of each opponent, allowing her to find the solution that maximizes her expected utility.

Note: In terms of concept uncertainty, we first took Apollo to be non-strategic, and then assumed he used the BNE concept. In practice, Daphne might have probability p_1 that he is non-strategic, probability p_2 that he uses BNE, probability p_3 that he is a level-1 reasoner, and so forth. (There are many more possible solution concepts.)

She would then solve her decision theory problem under each scenario, and form the mixture distribution $G(x)$ with each solution component weighted by the p_i and then solve

$$x^* = \mathbf{argmax}_{x \in \mathbb{R}^+} (x_0 - x)G(x)$$

A fourth solution concept is **level-k thinking**. If Daphne is a level-0 thinker, she bids non-strategically. If she is a level-1 thinker, she believes Apollo is a level-0 thinker, and makes her best response given her subjective assessment of the probabilities. If she is a level-2 thinker, she believes Apollo is a level-1 thinker, and so forth.

The "I think that you think that I think ..." reasoning becomes intricate. (Recall Vizzini's analysis of the iocaine powder in The Princess Bride). An example will be more clear: Suppose Daphne is a level-2 thinker. She believes Apollo is a level-1 thinker who thus believes that she is non-strategic.

Specifically, assume her subjective belief is that Apollo thinks her value for the book has distribution $F_1(v)$ supported on [\$100, \$200] and that she bids bids a proportion of her value $F_2(p) = p^9$, $0 \leq p \leq 1$. Then

$$g(y) = \int_0^\infty f_1(v) f_2(y/v) \frac{1}{v} \, dv = \int_0^\infty g_1(v) 9 (y/v)^8 v^{-1} \, dv \propto y^8$$

so $G(y) = (y/200)^9$.

Apollo's best response is to bid x^* such that

$$x^* = \mathbf{argmax}_{x \in \mathbb{R}^+} (X_0 - x)G(x)$$

where X_0 is his true value (a random variable to Daphne). He should take the derivative, set it to 0, and solve:

$$0 = \frac{d}{dx}\left[(X_0 - x)G(x)\right] = 9\frac{x^8}{200^9}(X_0 - x) - \left(\frac{x}{200}\right)^9.$$

So Apollo's bid should be 90% of his true value X_0.

Daphne does not know Apollo's true value, but suppose she thinks it has the triangular distribution on [\$140, \$200] with peak at \$170. Since Apollo should bid 90% of his true value, Daphne believes that his bid will be a random variable with triangular distribution $F(x)$ that is supported on [\$126, \$180] with peak at \$153.

Finally, for y_0 her (known) true value for the book, Daphne solves

$$y^* = \mathbf{argmax}_{y \in \mathbb{R}^+} (y_0 - y)F(y).$$

3. More Than Two Bidders

An important advantage of ARA is that it enables a more nuanced treatment of many-player games. Specifically, the ARA formulation allows one to frame fresh problems in auction theory when there are more than two bidders, by permitting asymmetric models for how each opponent views the others.

If Bonnie is a level-1 thinker, then she assumes that Alvin and Clyde are non-strategic, and there is no novelty in the analysis. She has distributions over the non-strategic bids of each, and chooses her bid according to the maximum of those. Specifically, she has a subjective distribution F_A over Alvin's bid A and a subjective distribution F_C over Clyde's bid C, and she calculates the distribution F of $\max\{A, C\}$. Then she makes the bid

$$b^* = \text{argmax}_{b \in \mathbb{R}^+} (b_0 - b) F(b),$$

where b_0 is her true value for the book.

Now suppose Bonnie is a level-2 thinker. She thinks that Alvin has a belief about the distribution of her bid and also Clyde's bid; similarly, she thinks Clyde has a distribution for her bid and for Alvin's. Let $F_{IJ}(x)$ be what Bonnie thinks player I thinks is the distribution for player J's bid, and $G_{IJ}(x)$ be her belief about what player I thinks is the distribution for player J's value.

Since her level-2 analysis assumes both Alvin and Clyde are level-1 thinkers who believe their opponents are level-0 thinkers, then knowing F_{IJ} directly determines G_{IJ}.

The level-2 ARA formulation means that Bonnie thinks Alvin will make the bid $a^* = \max\{a_B^*, a_C^*\}$ for

$$\begin{aligned} a_B^* &= \mathbf{argmax}_{a \in \mathbb{R}^+} (a_0 - a) \mathbb{P}[B^* < a] \\ a_C^* &= \mathbf{argmax}_{a \in \mathbb{R}^+} (a_0 - a) \mathbb{P}[C^* < a], \end{aligned}$$

where a_0 is Alvin's true value, B^* is a random variable whose distribution is Alvin's opinion about Bonnie's bid, and C^* is a random variable whose distribution is Alvin's opinion about Clyde's bid.

Bonnie does not know a_0, and she does not know Alvin's distributions for the bids, but as a Bayesian, she has a subjective opinion about these. She regards a_0 as a random variable with distribution G_{BA}, and her best guess is that B^* and C^* have distributions F_{AB} and F_{AC}, respectively.

In order to find F_{AB}, Bonnie uses the fact that Alvin thinks she is a level-0 thinker. He views her as non-strategic, and thus thinks her bid follows some probability distribution, perhaps an unknown proportion of her unknown true value, where both the unknown proportion and the unknown true value can be modeled as random variables.

Thus, Bonnie's opinion about the distribution of Alvin's bid is found by solving

$$A_B^* = \mathbf{argmax}_{a \in \mathbb{R}^+} (A_0 - a) F_{AB}(a)$$
$$A_C^* = \mathbf{argmax}_{a \in \mathbb{R}^+} (A_0 - a) F_{AC}(a)$$

and then assuming that Alvin bids the larger of those two random variables. So his bid is $A^* = \max\{A_B^*, A_C^*\}$.

Similarly, Bonnie belief about Clyde's bid C^* is that it has the distribution of $\max\{C_A^*, C_B^*\}$, where

$$\begin{aligned} C_A^* &= \mathbf{argmax}_{c \in \mathbb{R}^+} (C_0 - c) F_{CA}(c) \\ C_B^* &= \mathbf{argmax}_{c \in \mathbb{R}^+} (C_0 - c) F_{CB}(c) \end{aligned}$$

and C_0 is Clyde's true value, with distribution G_{BC}, since it is unknown to Bonnie.

Just as before, Bonnie uses her beliefs about what Clyde thinks about Alvin's non-strategy and her non-strategy to identify F_{CA} and F_{CB}, respectively, and thus finds the distribution of C^*.

Bonnie has calculated her distribution for Alvin's bid A^* and Clyde's bid C^*. Now she should place the bid

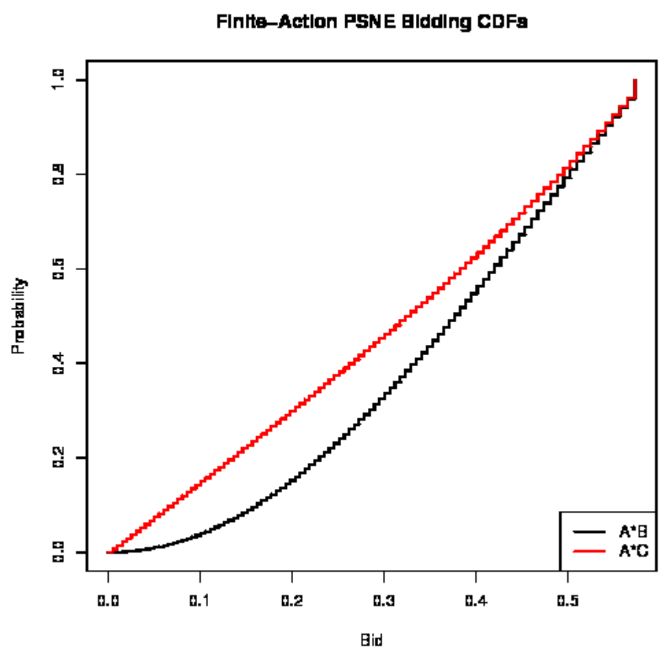
$$b^* = \mathbf{argmax}_{b \in \mathbb{R}^+} (b_0 - b) \mathbb{P}[\max\{A^*, C^*\} < b].$$

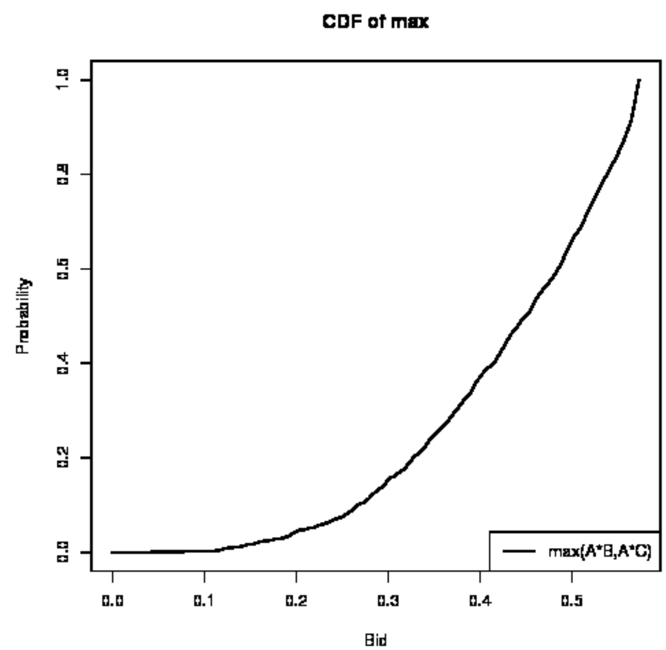
For example, suppose Bonnie believes that Alvin thinks her value for the first edition is Beta(1,1), and that Clyde's value is Beta(2, 1).

Similarly, she believes that Clyde thinks her value for the first edition is Beta(4, 1), and she thinks Clyde thinks Alvin's value is Beta(3,1).

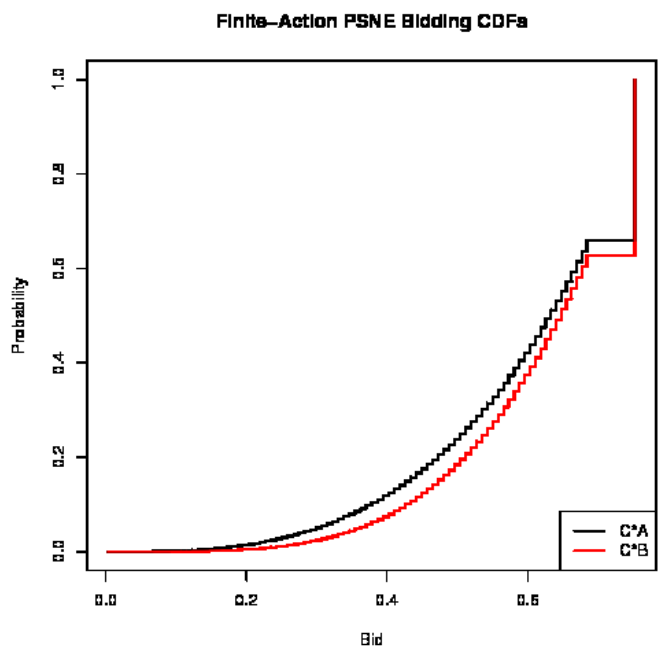
One can now use the BID algorithm to solve this three-person game. In this application, of course, we are supporting Bonnie.

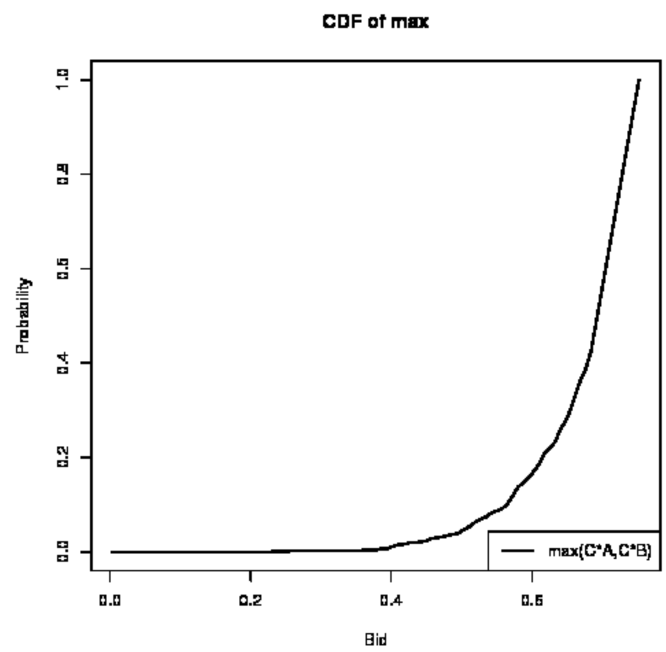
Alvin's bid is $A^* = \max\{A_B^*, A_C^*\}$. The left panel shows the cdfs of A_B^* and A_C^*, and the right shows the cdf of A^*.



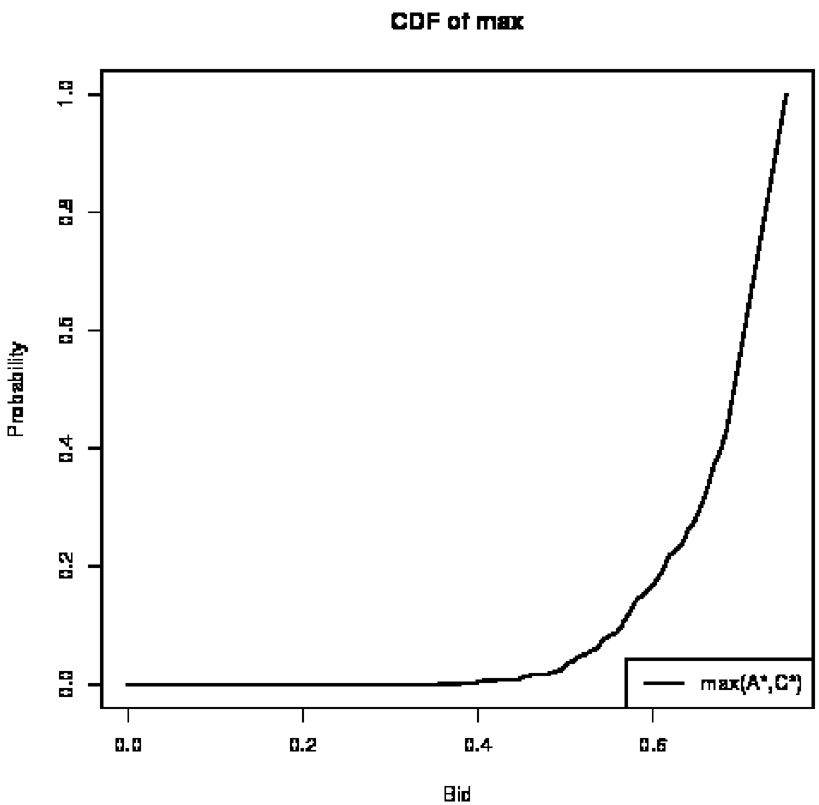


Clyde's bid is $C^* = \max\{C_A^*, C_B^*\}$. The left panel shows the cdfs of C_A^* and C_B^*, and the right shows the cdf of C^*.





This figure shows the distribution of



Under these assumptions about the beliefs of Alvin and Clyde, if Bonnie's true value for the book is 0.95, then her optimal bid is 0.7523.

Now consider the use of the mirror equilibrium solution concept when there are three bidders. This assumes that all bidders are solving the problem in the same way, but with possibly different subjective distributions over all unknown quantities.

The two-person system extends so that the basic problem is to solve

$$
\begin{aligned}
A^* &= \mathbf{argmax}_{a \in \mathbb{R}^+} (A_0 - a) F_A^*(a) \\
B^* &= \mathbf{argmax}_{b \in \mathbb{R}^+} (B_0 - b) F_B^*(b) \\
C^* &= \mathbf{argmax}_{c \in \mathbb{R}^+} (C_0 - c) F_C^*(c)
\end{aligned} \tag{1}
$$

from the perspective of each of the players, where $F_I^*(x)$ is what bidder I thinks is the chance that a bid of x will win. Bonnie does not know F_I^*, but she can use ARA to find F_I, which is her belief about what each opponent thinks is the chance that a given bid is successful.

The figure shows the notation that describes what Bonnie thinks each person believes about the distributions for each of the other bidders' true values. The G_{IJ} is what Bonnie thinks bidder I believes is distribution of the true value for bidder J, and G_{IJK} is the distribution that Bonnie thinks bidder I thinks bidder J has for the true value of the book to bidder K.

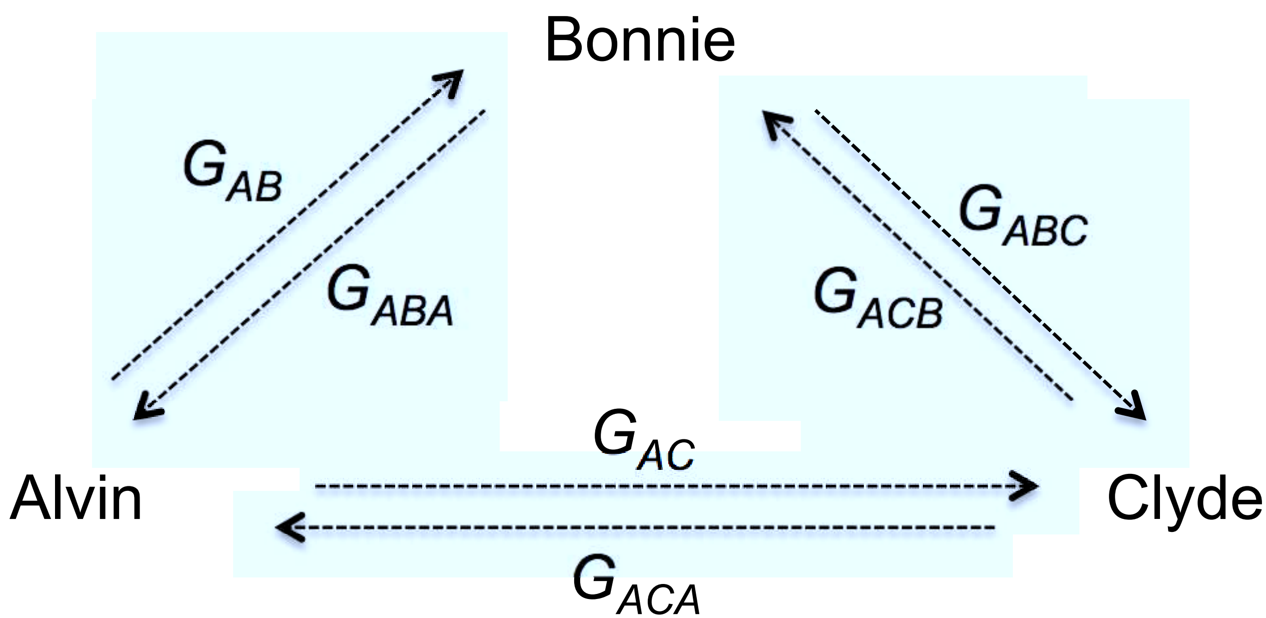


Figure 1: A representation of what Bonnie believes about the opinions held by each of the bidders regarding the value of the book to each the other bidders.

First, she models Alvin's logic. Bonnie thinks he obtains his distribution for her bid by solving (1) with $A_0 \sim G_{ABA}$, $B_0 \sim G_{AB}$, and $C_0 \sim G_{ABC}$. Since he, like Bonnie, does not know the true F_I^*, he must develop his own beliefs about them.

Here, his F_A is the distribution of the maximum of B^* and C^*, F_B is the distribution of the maximum of A^* and C^*, and F_C is the distribution of the maximum of B^* and C^*. After numerical computation to find the equilibrium solution, he obtains F_{AB}, his belief about the distribution of Bonnie's bid.

Next, Alvin considers Clyde. Bonnie thinks he solves (1) with $A_0 \sim G_{ACA}$, $B_0 \sim G_{ACB}$, and $C_0 \sim G_{AC}$. He proceeds as before, and obtains F_{AC}, his belief about the distribution of Clyde's bid. From this, Bonnie thinks his distribution for the probability of winning with a bid of a is F_A, where F_A is the distribution of the maximum of $B \sim F_{AB}$ and $C \sim F_{AC}$.

Bonnie's analysis for Clyde is analogous. To find Clyde's distribution for Bonnie's bid, she thinks he solves (1) with $A_0 \sim G_{CBA}$, $B_0 \sim G_{CB}$, and $C_0 \sim G_{CBC}$ to obtain F_{CB}.

Similarly, to find Clyde's distribution for Alvin's bid, he uses $A_0 \sim G_{CA}$, $B_0 \sim G_{CAB}$, and $C_0 \sim G_{CAC}$ to obtain F_{CA}. Putting these together, Bonnie thinks that Clyde thinks the probability that a bid of c will win is $F_C(c)$, which is the distribution of the maximum of $A \sim F_{CA}$ and $B \sim F_{CB}$.

Based on this reasoning, Bonnie thinks that Alvin's bid will be

$$A^* = \text{argmax}_{a \in \mathbb{R}^+}(A_0 - a)F_A(a) \sim F_{BA},$$

where $A_0 \sim G_{BA}$.

Bonnie thinks Clyde's bid will be

$$C^* = \text{argmax}_{c \in \mathbb{R}^+} (C_0 - c) F_C(c) \sim F_{BC},$$

where $C_0 \sim G_{BC}$. From this, the chance that a bid of b will win is $F_B(b)$, where F_B is the distribution of the maximum of $A^* \sim F_{BA}$ and $C^* \sim F_{BC}$. Now Bonnie uses her known value b_0 and solves

$$b^* = \mathbf{argmax}_{b \in \mathbb{R}^+} (b_0 - b) F_B(b)$$

to obtain her best bid under the mirror equilibrium solution concept.

Lebrun (1999, 2006) shows that an equilibrium solution always exists, and that, under a mild log concavity condition, the equilibrium is unique.

5. Conclusions

The auction game illustrates four kinds of thinking: the non-strategic player, the Bayes Nash equilibrium seeking player, the mirror equilibrium, and level-k thinking.

The examples have shown how each model can be analyzed. In principle, it is straightforward to incorporate model uncertainty, by mixing over the bidding distributions obtained under different solutions concepts.

One of the most exciting outcomes is the novel treatment of many-player games. Numerical solution of these requires use of the BID algorithm.

It isn't easy, but neither is standard game theory.