The X-Ray and Attenuated X-Ray Transforms

Research Training Group:

Inverse Problems and Partial Differential Equations

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Preface

(Last updated: Thursday 23rd June, 2011.) These notes introduce the X-ray transform and the attenuated X-ray transform, and present inversion formulas for each of them. The results are by no means the strongest possible; in particular we will generally assume functions to be infinitely differentiable and rapidly decaying at infinity. This simplifies issues of convergence of integrals, interchanging orders of integration, or passing limits into integrals, for example. Such an assumption allows us to focus on the overall structure rather than the technical details.

Acknowledgments: In preparing these notes, I have made use of some notes prepared by Peter Kutchment (2010) [4], the text "The Mathematics of Computerized Tomography" by Frank Natterer (2001) [6], and a survey article "The attenuated X-ray transform: recent developments" by David Finch (2003) [2].

1 Introduction

A thin beam of X-rays is transmitted into the body at a known initial intensity I_0, and a detector measures the intensity of the beam exiting the other side of the body. It is assumed that the beam travels in a straight line. The Beer-Lambert law states that X-rays of intensity I traveling a small distance Δx at position x experience a relative drop in intensity given by

$$\frac{\Delta I}{I} = -a(x)\Delta x$$

where $a(x)$ is the **attenuation coefficient** (often called the *absorption coefficient*). We consider $a(x)$ to be material specific, and so tells us something about the medium we are trying to image. It is this function $a(x)$ that we are trying to determine.

Infinitesimally this leads to the ODE

$$\frac{dI}{dx} = -a(x)I$$

and so the intensity I_1 recorded at a detector will be

$$I_1 = I_0 \exp\Big\{-\int_L a(x)\,ds\Big\}, \qquad \text{or, equivalently,} \qquad \int_L a(x)\,ds = \log\frac{I_0}{I_1}.$$

The integrals above are line-integrals along the line L describing the path of the X-ray beam, with respect to arc-length. Assuming we are able to change the source-detector positions without limit, scanning the body with X-ray beams results in knowing the line-integrals of $a(x)$ along every line through the support of a. Of course, in practice, one can never scan more than finitely many such lines, but for the moment we shall assume we are not limited in this way.

Our ultimate question is: from such information can we (uniquely) determine the function $a(x)$?

DEFINITION 1 (X-ray Transform). The X-ray transform maps a function $a(x)$ (defined in $\mathbb{R}^2$) into the set of all line-integrals of a:

$$a(x) \mapsto Ra(L) := \int_L a(x)\, ds.$$

The 2D X-ray transform is also known as the 2D Radon transform, which has a generalization to higher dimensions. For example, the 3D Radon transform maps a function (defined in $\mathbb{R}^3$) to integrals of a over *planes*. Hence the choice of notation to use "R" to denote the X-ray transform.

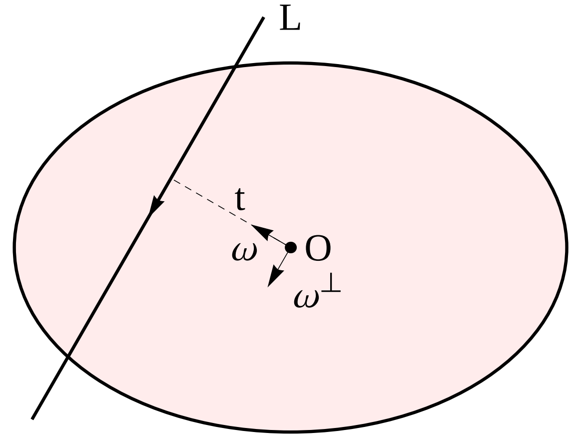
Parameterization of the X-ray transform:

We parameterize each element L in the set of lines by a unit vector $\omega \in \mathbb{S}$ (which is perpendicular to the line) and a distance t of the line from the origin. The line is thus the solution set to $x \cdot \omega = t$, and can be parameterized (by arc-length) as

$$t\omega + s\omega^{\perp}, \quad -\infty < s < \infty,$$

where for $\omega = (\omega_1, \omega_2)$, $\omega^{\perp} = (-\omega_2, \omega_1)$. Consequently,

$$\boxed{(Ra)(\omega, t) = \int_{x\cdot\omega = t} a(x)\, dx = \int_{-\infty}^{\infty} a(t\omega + s\omega^{\perp})\, ds.}$$



A natural question is: for what functions f is Rf well defined? For the present we will assume that functions are in the Schwartz space $\mathscr{S}(\mathbb{R}^2)$:

DEFINITION 2. The Schwartz space, $\mathscr{S}(\mathbb{R}^2)$ is the linear space of C^∞ functions f for which

$$\sup_{x\in\mathbb{R}^2} |x^\alpha \partial^\beta f(x)| < \infty$$

for all multi-indices $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbb{N}^2$; we use here the notation $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}$ and $\partial^\beta = \left(\frac{\partial}{\partial x_1}\right)^{\beta_1}\left(\frac{\partial}{\partial x_2}\right)^{\beta_2}$. Similarly, $\mathscr{S}(\mathbb{S}\times\mathbb{R})$ is defined to be the space of $C^\infty(\mathbb{S}\times\mathbb{R})$ functions g for which

$$\sup_{t\in\mathbb{R}} |t^n \partial_\omega^{\alpha_1} \partial_t^{\alpha_2} g(\omega, t)| < \infty$$

for all $n \in \mathbb{N}$ and multi-indices α.

EXERCISE 3. If $f \in \mathscr{S}(\mathbb{R}^2)$ then $Rf \in \mathscr{S}(\mathbb{S}\times\mathbb{R})$.

Note: it is helpful to use the fact that, for example, $f \in \mathscr{S}(\mathbb{R}^2) \Rightarrow |\partial^\beta f(x)| < \dfrac{C_{\alpha,\beta}}{(1+|x|)^{|\alpha|}}$ where $C_{\alpha,\beta}$ depends on α and β but is independent of x.

Note that if f is compactly supported, $f \in C_0(\mathbb{R}^2)$, then Rf is also compactly supported (there is M such that $Rf(\omega, t) = 0$ for all $t > M$).

The X-ray transform is even: One immediate property of Rf is that it is an even function on the cylinder $\mathbb{S} \times \mathbb{R}$, i.e. $Rf(-\omega, -t) = Rf(\omega, t)$.

2 The Fourier transform

Our goal, ultimately, is to try to invert the X-ray transform. What facilitates this is the way in which the X-ray and Fourier transforms intertwine.

DEFINITION 4. If $f \in \mathscr{S}(\mathbb{R}^2)$, define its Fourier transform to be

$$\mathcal{F}[f](\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x)\, dx.$$

We also define the partial Fourier transform below for functions defined on $\mathbb{S} \times \mathbb{R}$, with respect to the second variable, and use "tilde" notation to distinguish it from the full Fourier transform defined above:

DEFINITION 5. Let $g \in \mathscr{S}(\mathbb{S} \times \mathbb{R})$. Denote by $\widetilde{g}(\omega, \rho)$ the Fourier transform of g with respect to the $\mathbb{R}$ variable:

$$\mathcal{F}_t[g](\omega, \rho) = \widetilde{g}(\omega, \rho) := \int_{\mathbb{R}} e^{-it\rho} g(\omega, t)\, dt.$$

The Fourier transform of $f(x)$ is well defined provided f decays sufficiently fast; this is certainly the case when $f \in \mathscr{S}(\mathbb{R}^n)$, is also easily seen to be the case when $f \in L^1(\mathbb{R}^n)$ and in fact the Fourier transform has a unique extension to $L^2(\mathbb{R}^n)$ (see [3]). We present some important and useful properties:

THEOREM 6. *If $f \in C^k(\mathbb{R}^n)$, $\partial^\alpha f \in L^1(\mathbb{R}^n)$ for $|\alpha| \leq k$, and $\partial^\alpha f \in C_0(\mathbb{R}^n)$ for $|\alpha| \leq k-1$, then*

$$\widehat{\partial^\alpha f}(\xi) = (i\xi)^\alpha \widehat{f}(\xi).$$

In particular, the Fourier transform maps $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}(\mathbb{R}^n)$.

LEMMA 7. *If $f, g \in L^1(\mathbb{R}^n)$ then* $\displaystyle\int_{\mathbb{R}^n} f\widehat{g} = \int_{\mathbb{R}^n} \widehat{f} g.$

Perhaps most importantly, we have the **Fourier inversion formula**:

THEOREM 8. *If* $f \in L^1(\mathbb{R}^n)$ *and* $\widehat{f} \in L^1(\mathbb{R}^n)$*, then*

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \widehat{f}(\xi)\, d\xi \quad a.e. \tag{1}$$

In particular, (1) *holds for* $f \in \mathscr{S}(\mathbb{R}^n)$.

We introduce the notation

$$\mathcal{F}^{-1}[f](x) = \overset{\vee}{f}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\, d\xi$$

and so (when defined) $(\widehat{f})^\vee = \mathcal{F}[\overset{\vee}{f}] = \mathcal{F}^{-1}[\widehat{f}] = (\overset{\vee}{f})\widehat{} = f \; a.e.$

As promised, here is how the Fourier transform of the X-ray transform relates to the Fourier transform of the original function:

PROPOSITION 9. *If* $f \in \mathscr{S}(\mathbb{R}^2)$ *then* $\widetilde{Rf}(\omega, \rho) = \widehat{f}(\rho\omega)$.

Proof. We compute

$$\widetilde{Rf}(\omega, \rho) = \int_{\mathbb{R}_t} e^{-it\rho}(Rf)(\omega, t)\, dt = \int_{\mathbb{R}_t} e^{-it\rho} \int_{-\infty}^{\infty} f(t\omega + s\omega^\perp)\, ds\, dt.$$

If we set $x = t\omega + s\omega^\perp$, then $\left|\frac{\partial(x_1, x_2)}{\partial(s, t)}\right| = 1$ and $x \cdot \omega = t$ so

$$\widetilde{Rf}(\omega, \rho) = \int_{\mathbb{R}^2} e^{-i\rho x\cdot\omega} f(x)\, dx = \widehat{f}(\rho\omega).$$

□

3 Convolution

In the next section we will compute the adjoint operator $R^\#$ to R and investigate the composition $R^\# R$. To understand this, and for use in implementing inversion algorithms, we need to define *convolution* and understand its relationship with the Fourier transform.

DEFINITION 10. The convolution of functions f and g on $\mathbb{R}^n$ is the function $f * g$ defined by

$$f * g(x) = \int_{\mathbb{R}^n} f(x - y) g(y)\, dy.$$

An immediate property of convolution is $f * g = g * f$. Notice also that

$$\operatorname{supp}(f * g) \subset \operatorname{supp} f + \operatorname{supp} g := \{x + y \;:\; x \in \operatorname{supp} f,\; y \in \operatorname{supp} g\}.$$

A consequence of this is that if f is compactly supported, but g is not, then $f * g$ is also not compactly supported. Another straight-forward computation shows that

$$\partial^\alpha (f * g) = (\partial^\alpha f) * g = f * (\partial^\alpha g)$$

and so, roughly speaking, $f * g$ is at least as smooth as *either* f or g. For this reason we view convolution is a *smoothing* operation. From the point of view of reconstructing images, for example, convolution is considered as a *blurring* process since sharp, discontinuous edges get smoothed away. In particular, if $f, g \in \mathscr{S}$ then $f * g \in \mathscr{S}$.

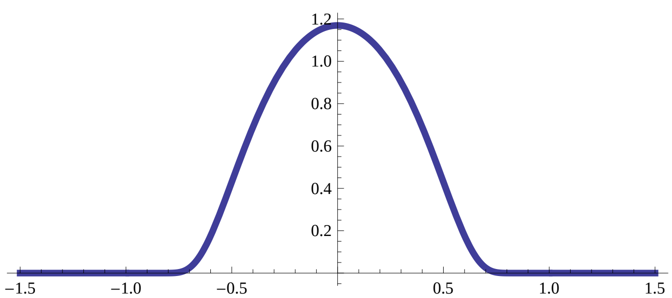
An important application of convolution is in constructing "approximations to the identity," a concept we now make precise. Given a function φ on $\mathbb{R}^n$ and $t > 0$ define

$$\varphi_t(x) = \frac{1}{t^n}\varphi\Big(\frac{x}{t}\Big).$$

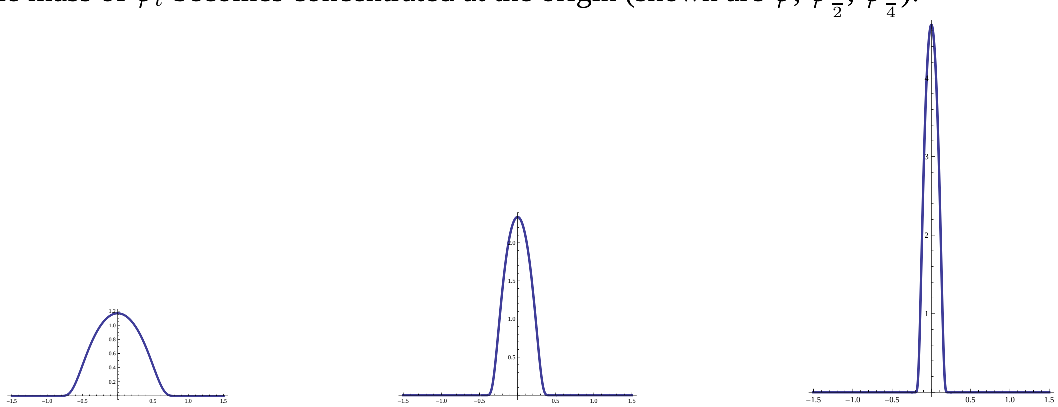
If φ is integrable, then substituting $y = x/t$, $dx = t^n\, dy$, and so

$$\int \varphi_t = \int \frac{1}{t^n}\varphi\Big(\frac{x}{t}\Big)\, dx = \int \varphi(y)\, dy = \int \varphi$$

independent of t. Of particular interest is when φ is a smooth "bump" function centered at 0. For example, we might take $\varphi \in C^\infty(\mathbb{R}^n)$, $\operatorname{supp}\varphi \subset B_1(0)$, $\varphi \geq 0$, and $\displaystyle\int \varphi = 1$. An illustration in $\mathbb{R}$ is given (the function is $\varphi(x) = ce^{-\tan(\pi x/2)^2}\chi_{[-1,1]} \in C^\infty(\mathbb{R})$ where c is such that $\int \varphi = 1$):



As $t \to 0$, the mass of φ_t becomes concentrated at the origin (shown are φ, $\varphi_{\frac{1}{2}}$, $\varphi_{\frac{1}{4}}$):



THEOREM 11. *Let* $\varphi \in L^1(\mathbb{R}^n)$ *with* $\int \varphi = 1$. *Then*

1. *If* $f \in L^1(\mathbb{R}^n)$, *then* $f * \varphi_t \to f$ *in* $L^1(\mathbb{R}^n)$ *as* $t \to 0$. *I.e.* $\int_{\mathbb{R}^n} |\varphi_t - f| \, dx \to 0$ *as* $t \to 0$.
2. *If* f *is bounded and uniformly continuous then* $f * \varphi_t \to f$ *uniformly as* $t \to 0$.

If φ is as in the example above (compactly supported, for example) then $f * \varphi_t(x) \to f(x)$ a.e., and for every x at which f is continuous.

Convolution interacts with the Fourier transform in an especially pleasing way:

EXERCISE 12. Show that $\widehat{f * g} = \widehat{f}\,\widehat{g}$ and, similarly, $\widehat{fg} = (2\pi)^n \widehat{f} * \widehat{g}$.

While the bump function shown above is qualitatively very nice, we cannot compute its Fourier transform analytically. Another bump-type function (on $\mathbb{R}$) is the Gaussian $G(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$. One has $\int G = 1$, but G is no longer compactly supported. (It does, however, have sufficiently fast decay that the remark following Theorem 11 still holds for G_t.)

EXERCISE 13. Show that:

1. $\frac{dG}{dx} + 2xG = 0$,
2. which implies $\frac{d\widehat{G}}{d\xi} + \frac{1}{2}\xi\widehat{G} = 0$,
3. and hence $\widehat{G}(\xi) = e^{-\xi^2/4}$.
4. If $H(x) = \frac{a}{\sqrt{\pi}} e^{-a^2x^2}$ then it still holds that $\int H = 1$; can you find a so that $\int \widehat{H} = 1$ also?

Filters:

If φ is such that $\widehat{\varphi}$ is smooth and very quickly decaying, then consider $\varphi * f$. We have $\widehat{\varphi * f} = \widehat{\varphi}\,\widehat{f}$ is then very quickly decaying, which implies $\varphi * f$ is very smooth. Intuitively, when the Fourier transform of a function is rapidly decaying, the function has little to no "high oscillation" – more precisely, Theorem 6 shows that decay in the Fourier transform translates to smoothness in the original function. One says that multiplication of $\widehat{f}$ by $\widehat{\varphi}$ "filters out" the high frequency components of $\widehat{f}$; the function $\widehat{\varphi}$ is called a *filter*, or *window function*; the function φ is called a *mollifier*.

While smoothing f to some degree might be desirable, it is also desirable not to have "lost" f too much. As we saw above, choosing φ in such a way that (a) φ_t concentrates well to 0 as $t \to 0$ and (b) $\widehat{\varphi}_t$ is quickly decaying, we have $\varphi_t * f$ mollifies f and $\varphi_t * f \to f$ as $t \to 0$. Of course, you can't have your cake and eat it too, as $\widehat{\varphi}_t(\xi) = \widehat{\varphi}(t\xi)$, so the decay of $\widehat{\varphi}_t(\xi)$ worsens as $t \to 0$.

4 The adjoint operator

We wish to determine the adjoint operator $R^\# : \mathscr{S}(\mathbb{S} \times \mathbb{R}) \to \mathscr{S}(\mathbb{R}^2)$ which has the property that

$$\int_{\mathbb{R}^2} (R^\# g)(x) f(x)\, dx = \langle R^\# g, f \rangle = \langle g, Rf \rangle = \int_{\mathbb{R}} \int_{\mathbb{S}} g(\omega, t)(Rf)(\omega, t)\, d\omega\, dt.$$

Starting with the right hand side, we use the change of variables $x = t\omega + s\omega^\perp$ and then apply Fubini to change the order of integration:

$$\begin{aligned} \langle g, Rf \rangle &= \int_{\mathbb{S}} \int_{\mathbb{R}} g(\omega, t) \int_{\mathbb{R}} f(t\omega + s\omega^\perp)\, ds\, dt\, d\omega \\ &= \int_{\mathbb{S}} \int_{\mathbb{R}^2} g(\omega, x \cdot \omega) f(x)\, dx\, d\omega = \int_{\mathbb{R}^2} f(x)(R^\# g)(x)\, dx \end{aligned}$$

where

$$\boxed{(R^\# g)(x) = \int_{\mathbb{S}} g(\omega, x \cdot \omega)\, d\omega.} \tag{2}$$

EXERCISE 14. Show that $R^\# : \mathscr{S}(\mathbb{S} \times \mathbb{R}) \to \mathscr{S}(\mathbb{R}^2)$

Unlike the X-ray transform itself, the adjoint operator does not map compactly supported functions to compactly supported functions: if $f(\omega, t) = 0$ for all $t > M$, it does not hold that $R^\# f \in C_0(\mathbb{R}^n)$.

Geometrically: if G is measured data, then $g(\omega, x \cdot \omega)$ is an integral of (some) f along a line; the line is one which goes through x. As ω varies over $\mathbb{S}$, we obtain all the lines through x. Thus $R^\# g(x)$ is the "average" of the measured data g, where the averaging is over all points in $\mathbb{S} \times \mathbb{R}$ which correspond to lines passing through x. The operator $R^\#$ is often referred to as **back-projection**.

It is worth investigating the composition of back-projection with the X-ray transform:

$$R^\# Rf(x) = \int_{\mathbb{S}} (Rf)(\omega, x \cdot \omega)\, d\omega = \int_{\mathbb{S}} \int_{\mathbb{R}} f\big((x \cdot \omega)\omega + s\omega^\perp\big)\, ds\, d\omega.$$

It is clear that, for fixed ω, we are thus integrating f over the line through x which is perpendicular to ω, and then integrating over ω. This is essentially integrating with respect to polar variables based at x, except that the lines are integrated in *both* directions, not just "out" from the origin x, and there is no change of volume radial factor. We proceed more carefully as follows: for fixed ω, first split the the inner integral into two pieces and then make the substitution $t = s - x \cdot \omega^\perp$:

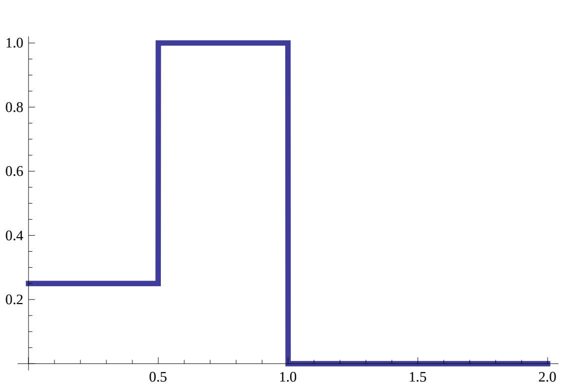
$$\begin{aligned}\int_{\mathbb{R}} f\big((x\cdot\omega)\omega + s\omega^\perp\big)\,ds &= \int_{-\infty}^{x\cdot\omega^\perp} + \int_{x\cdot\omega^\perp}^{\infty} f\big((x\cdot\omega)\omega + s\omega^\perp\big)\,ds \\ &= \int_{-\infty}^{0} + \int_{0}^{\infty} f\big((x\cdot\omega)\omega + (t + x\cdot\omega^\perp)\omega^\perp\big)\,dt \\ &= \int_{-\infty}^{0} + \int_{0}^{\infty} f(x + t\omega^\perp)\,dt.\end{aligned}$$

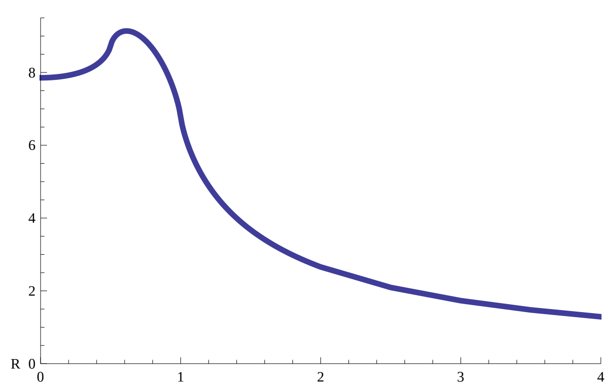
Now when we integrate this over $\mathbb{S}$, we make the substitution $y = x + t\omega^\perp$. Then $dt\,d\omega = \dfrac{dy}{|y-x|}$ and we obtain

$$R^\# Rf(x) = 2\int_{\mathbb{R}^2} \frac{f(y)}{|y-x|}\,dy = \frac{2}{|x|} * f(x).$$

Thus, by back-projecting the data Rf, we obtain a blurred version of the original function f. We hope to be able to do better than this!

Example: If $f(x) = f(|x|)$ is a radial function, then it is easy to see that $2\int_{\mathbb{R}^2} \frac{f(y)}{|y-x|}\,dy$ is again a radial function. Consider the function f shown below and the result of $2\int_{\mathbb{R}^2} \frac{f(y)}{|y-x|}\,dy$:





5 Inversion of the X-ray transform

We begin by expressing f as the inverse transform of its Fourier transform and then use the above connection between the Fourier and X-ray transforms. Indeed, using polar coordinates,

$$\begin{aligned} f(x) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{ix\cdot\xi}\widehat{f}(\xi)\,d\xi = \frac{1}{(2\pi)^2}\int_{\mathbb{S}}\int_0^\infty e^{i\rho x\cdot\omega}\widehat{f}(\rho\omega)\rho\,d\rho\,d\omega \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{S}}\frac{1}{2}\Big(\int_0^\infty + \int_0^\infty\Big)e^{i\rho x\cdot\omega}\widehat{f}(\rho\omega)\rho\,d\rho\,d\omega \end{aligned}$$

In one of the integrals we make the substitution $\omega \mapsto -\omega$, $\rho \mapsto -\rho$ to obtain:

$$f(x) = \frac{1}{2}\frac{1}{(2\pi)^2}\int_{\mathbb{S}}\int_{\mathbb{R}} e^{i\rho x\cdot\omega}\widehat{f}(\rho\omega)|\rho|\,d\rho\,d\omega$$

where, on the half-line $\rho < 0$ we have made the substitution $\omega \mapsto -\omega$. Now that we have the Fourier transform of f in polar coordinates, we may apply Proposition 9 to obtain

$$\begin{aligned} f(x) &= \frac{1}{2}\frac{1}{(2\pi)^2}\int_{\mathbb{S}}\int_{\mathbb{R}} e^{i\rho x\cdot\omega}\widetilde{Rf}(\omega,\rho)|\rho|\,d\rho\,d\omega \\ &= \frac{1}{4\pi}\int_{\mathbb{S}}\Big(\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\rho t}\widetilde{Rf}(\omega,\rho)|\rho|\,d\rho\Big)\Big|_{t=x\cdot\omega}\,d\omega. \end{aligned}$$

The inner integral is the inverse one-dimensional Fourier transform applied to the function $\widetilde{Rf}(\omega,\rho)|\rho|$ (in the variable ρ). Were it not for the presence of $|\rho|$, the result would be Rf and we would have expressed f in terms of the data Rf. Note further, that if instead of $|\rho|$ it were $i\rho$, then the result would be $\frac{d}{dt}Rf$, which would again be satisfactory. Instead, we write $|\rho| = -i\,\mathrm{sgn}\,\rho\cdot i\rho$ where

$$\mathrm{sgn}\,\rho = \begin{cases} 1 & \text{if } \rho > 0, \\ 0 & \text{if } \rho = 0, \\ -1 & \text{if } \rho < 0. \end{cases}$$

To give name to the result in the above, we introduce the **Hilbert transform** H of a function of one variable: (expressed in terms of the Fourier transform of u)

$$Hu(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\rho t}\widehat{u}(\rho)(-i\,\mathrm{sgn}\,\rho)\,d\rho.$$

Expressed in terms of the function u itself,

$$Hu(t) = \frac{1}{\pi}\,\mathrm{p.\,v.}\int_{\mathbb{R}}\frac{u(s)}{t-s}\,ds$$

where the integral is a principal value integral:

$$Hu(t) = \frac{1}{\pi}\lim_{\varepsilon\searrow 0}\Big(\int_{-\infty}^{t-\varepsilon} + \int_{t+\varepsilon}^{\infty}\Big)\frac{u(s)}{t-s}\,ds = -\frac{1}{\pi}\lim_{\varepsilon\searrow 0}\int_{\varepsilon}^{\infty}\frac{u(t+s)-u(t-s)}{s}\,ds.$$

(The Hilbert transform can be shown to be well defined on L^p, $1 \leq p < \infty$; it is a bounded operator $L^p \to L^p$ for $1 < p < \infty$; when $p = 1$, the limit exists $a.e.$ but the resulting function need not be in L^1.) Using the expression for H in terms of the Fourier transform, and using $i\rho\widetilde{Rf}(\omega,\rho) = \widetilde{\frac{d}{dt}Rf}(\omega,\rho)$ we see that

$$H\frac{d}{dt}Rf(\omega,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\rho t}\widetilde{\frac{d}{dt}Rf}(\omega,\rho)(-i\,\mathrm{sgn}\,\rho)\,d\rho = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\rho t}\widetilde{Rf}(\omega,\rho)i\rho(-i\,\mathrm{sgn}\,\rho)\,d\rho$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\rho t}\widetilde{Rf}(\omega,\rho)|\rho|\,d\rho.$$

Thus, returning to the expression for f above, we have the **inversion formula** for the X-ray transform:

$$\boxed{f(x) = \frac{1}{4\pi}\int_{\mathbb{S}} H\frac{d}{dt}Rf(\omega, x\cdot\omega)\,d\omega = \frac{1}{4\pi}R^{\#}H\frac{d}{dt}Rf(x).} \tag{3}$$

Note that implementation of $H\frac{d}{dt}Rf$ is equivalent to computing $\mathcal{F}_\rho^{-1}(|\rho|\widetilde{Rf})$, where $\mathcal{F}_\rho^{-1}(\cdot)$ denotes the inverse Fourier transform in the ρ variable. Suppose that, in some sense, $\widetilde{\Phi}(\rho) \approx |\rho|$; then

$$H\frac{d}{dt}Rf = \mathcal{F}_\rho^{-1}(|\rho|\widetilde{Rf}) \approx \mathcal{F}_\rho^{-1}(\widetilde{\Phi}\widetilde{Rf}) = \mathcal{F}_\rho^{-1}(F_\rho(\Phi *_t Rf) = \Phi *_t Rf.$$

Here, $*_t$ indicates convolution in the t variable. Now in exactly what sense do we need $\widetilde{\Phi}$ to approximate $|\rho|$? We need the approximate equality in the above expression to hold, and we need $\Phi *_t Rf$ to approximate f. Achieving this leads to the *filtered back-projection* algorithm, which is the subject of the next section.

6 The Filtered Back-projection Algorithm

We begin with the following observation about how the adjoint operator intertwines with convolution.

THEOREM 15. *Let $f \in \mathscr{S}(\mathbb{R}^2)$ and $\varphi \in \mathscr{S}(\mathbb{S}\times\mathbb{R})$. Then*

$$(R^{\#}\varphi) * f = R^{\#}(\varphi *_t Rf) \tag{4}$$

where $*_t$ denotes convolution in the $\mathbb{R}$ variable (in $\mathbb{S}\times\mathbb{R}$).

Proof. We simply compute

$$\begin{aligned}(R^{\#}\varphi) * f(x) &= \int_{\mathbb{R}^2} R^{\#}\varphi(x-y)f(y)\,dy \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{S}} \varphi(\omega,(x-y)\cdot\omega)\,d\omega f(y)\,dy && \text{by (2)} \\ &= \int_{\mathbb{S}}\int_{\mathbb{R}^2} \varphi(\omega,(x-y)\cdot\omega)f(y)\,dy\,d\omega.\end{aligned}$$

Now make the substitution $y = s\omega + t\omega^{\perp}$ (as we did in Proposition 9) and obtain

$$\begin{aligned}(R^{\#}\varphi) * f(x) &= \int_{\mathbb{S}}\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(\omega, x\cdot\omega - s)f(s\omega + t\omega^{\perp})\,dt\,ds\,d\omega \\ &= \int_{\mathbb{S}}\int_{\mathbb{R}} \varphi(\omega, x\cdot\omega - s)Rf(\omega,s)\,ds\,d\omega \\ &= \int_{\mathbb{S}} (\varphi *_t Rf)(\omega, x\cdot\omega)\,d\omega \\ &= R^{\#}(\varphi *_t Rf)(x) && \text{by (2) again.}\end{aligned}$$

□

The approach now is to choose (as φ) a sequence of functions φ_n in such a way that $R^{\#}\varphi_n(x) \to \delta_0(x)$ as $n \to \infty$. If this is so, then the left hand side of (4) converges to $f(x)$ (or is an approximation of $f(x)$ for large n), while the right hand side of (4) is computable from the data Rf.

That is, we wish to choose φ_n so that $R^{\#}\varphi_n(x)$ is an approximate identity, and we do so by way of a "low pass filter" with cut-off frequency n in the Fourier domain:

$$\widehat{R^{\#}\varphi_n}(\xi) = \widehat{\Phi}\Big(\frac{|\xi|}{n}\Big) \tag{5}$$

where $0 \le \widehat{\Phi} \le 1$ and $\widehat{\Phi}(\sigma) = 0$ for $\sigma \ge 1$. As $n \to \infty$, we want $\widehat{\Phi}\big(\sigma/n\big) \to 1$ and so $R^{\#}\varphi_n \to \delta$. To be more precise consider the following exercise:

EXERCISE 16. Let $f, \hat{f} \in L^1(\mathbb{R}^2)$; suppose that $\widehat{\Phi} \in L^1(\mathbb{R})$ is such that $\widehat{\Phi}(\sigma) = 0$ for $|\sigma| \ge 1$, $\widehat{\Phi}(0) = 1$, $\widehat{\Phi}$ is continuous at 0, and $0 \le \widehat{\Phi}(\sigma) \le 1$. If (5) holds, prove that $\Big|\int R^{\#}\varphi_n(x)f(x)\,dx - f(0)\Big| \to 0$ as $n \to \infty$.

Of course, this begs the question as to how one might achieve (5). To answer this we compute the interaction of the adjoint operator with the Fourier transform.

PROPOSITION 17. *For $\varphi \in \mathscr{S}(\mathbb{S}\times\mathbb{R})$,*

$$\widehat{R^{\#}\varphi}(\xi) = \frac{2\pi}{|\xi|}\Big[\widetilde{\varphi}\Big(\frac{\xi}{|\xi|}, |\xi|\Big) + \widetilde{\varphi}\Big(-\frac{\xi}{|\xi|}, -|\xi|\Big)\Big]. \tag{6}$$

Proof. For any $f \in \mathscr{S}(\mathbb{R}^2)$, since $R^\#$ is the adjoint of R, and by Lemma 7,

$$\int_{\mathbb{R}^2} \widehat{R^\# \varphi}(\xi) f(\xi)\, d\xi = \int_{\mathbb{R}^2} R^\# \varphi(x) \widehat{f}(x)\, dx = \int_{\mathbb{S}} \int_{\mathbb{R}} \varphi(\omega, s) R\widehat{f}(\omega, s)\, ds\, d\omega$$
$$= \int_{\mathbb{S}} \int_{\mathbb{R}} \widetilde{\varphi}(\omega, \rho) \mathcal{F}_s^{-1}[R\widehat{f}](\omega, \rho)\, d\rho\, d\omega$$

by Lemma 7 again. Applying Proposition 9 (it is useful here to note that $\mathcal{F}_s^{-1}[h](\sigma) = \frac{1}{2\pi}\widetilde{h}(-\sigma)$ and $\frac{1}{(2\pi)^2}\mathcal{F}[\widehat{f}](-\xi) = f(\xi)$),

$$\begin{aligned}
\int_{\mathbb{R}^2} \widehat{R^\# \varphi}(\xi) f(\xi)\, d\xi &= \frac{1}{2\pi} \int_{\mathbb{S}} \int_{\mathbb{R}} \widetilde{\varphi}(\omega, \rho) \widetilde{R\widehat{f}}(\omega, -\rho) \\
&= \frac{1}{2\pi} \int_{\mathbb{S}} \int_{\mathbb{R}} \widetilde{\varphi}(\omega, \rho) \mathcal{F}[\widehat{f}](-\rho\omega)\, d\rho\, d\omega \\
&= \frac{(2\pi)^2}{2\pi} \int_{\mathbb{S}} \int_{\mathbb{R}} \widetilde{\varphi}(\omega, \rho) f(\rho\omega)\, d\rho\, d\omega \\
&= 2\pi \int_{\mathbb{S}} \int_{\mathbb{R}} \big(\widetilde{\varphi}(\omega, \rho) + \widetilde{\varphi}(-\omega, -\rho)\big) f(\rho\omega)\, d\rho\, d\omega
\end{aligned}$$

where we have split the integral $d\rho$ into $\rho > 0$ and $\rho < 0$; on the integral for $\rho < 0$ we substitute $\rho \mapsto -\rho$ and $\omega \mapsto -\omega$. Finally, set $\xi = \rho\omega$, so $\rho = |\xi|$ and $\omega = \xi/|\xi|$, $d\xi = \rho\, d\rho\, d\omega = |\xi|\, d\rho\, d\omega$, to obtain

$$\int_{\mathbb{R}^2} \widehat{R^\# \varphi}(\xi) f(\xi)\, d\xi = 2\pi \int_{\mathbb{R}^2} \frac{1}{|\xi|} \Big[\widetilde{\varphi}\Big(\frac{\xi}{|\xi|}, |\xi|\Big) + \widetilde{\varphi}\Big(-\frac{\xi}{|\xi|}, -|\xi|\Big)\Big] f(\xi)\, d\xi$$

□

Combining (6) with (5) we have

$$\widehat{\Phi}\Big(\frac{|\xi|}{n}\Big) = \frac{2\pi}{|\xi|} \Big[\widetilde{\varphi}\Big(\frac{\xi}{|\xi|}, |\xi|\Big) + \widetilde{\varphi}\Big(-\frac{\xi}{|\xi|}, -|\xi|\Big)\Big];$$

the LHS is radial, so $\widetilde{\varphi}$ will be radial and we can drop the dependence on $\xi/|\xi|$. If we choose to make $\widetilde{\varphi}$ to be even, we then obtain

$$\widehat{\Phi}\Big(\frac{|\xi|}{n}\Big) = \frac{2\pi}{|\xi|} 2\widetilde{\varphi}_n(|\xi|), \qquad \text{or, simply,} \qquad \widetilde{\varphi}_n(\rho) = \frac{|\rho|}{4\pi} \widehat{\Phi}\Big(\frac{|\rho|}{n}\Big).$$

Evaluation of the right hand side of (4) involves convolving the data with φ_n, which we may view as a filtering step, for each direction ω, and then applying the back-projection operator $R^\#$. Hence the name "filtered back-projection algorithm."

7 The attenuated X-ray transform

The attenuated X-ray transform is like the regular X-ray transform in that it is based on integrals along lines through the body, but this time the integrals are against an exponential weight. One situation where such a transform arises is in single photon emission computed tomography (SPECT). Here a radio-pharmaceutical is injected into the body; it is assumed to accumulate at regions of interest and to then emit photons. The distribution f of this source of emission is what is sought. Any intensity of photons emitted from a point x undergoes attenuation as it travels toward the detector. Let $a(x)$ again be the attenuation coefficient of the body and assume the Beer-Lambert law. The intensity I recorded "at ∞" at the end of the half-line L^+ from x (with direction ω) due to the emission $f(x)$ from x is then

$$I(\infty) = I_0 e^{-\int_0^\infty a(x+s\omega)\,ds} = f(x)e^{-\int_0^\infty a(x+s\omega)\,ds}.$$

But what the detector actually receives is this contribution from $f(x)$ at *all* points x on the line with direction ω. The half-line integral of a is called the "divergent beam X-ray transform" of a, so we introduce the notation

$$Da(x,\omega) = \int_0^\infty a(x+s\omega)\,ds.$$

Adding up all these contributions, the intensity measured by the detector is thus

$$I = \int_{L^+} f(x)e^{-Da(x,\omega)}\,dx.$$

Using the same conventions as for the regular X-ray transform, we parameterize the lines by normal vectors ω and distance from the origin. Then the attenuated X-ray transform of a function f is

$$R_a f(\omega,t) = \int_{x\cdot\omega=t} f(x)e^{-Da(x,\omega^\perp)}\,dx$$

Unlike for the regular X-ray transform, we consider the attenuation a to be known and seek to determine the source f from $R_a f$.

Such integral transforms also arise in inverse stationary transport theory.

8 Preliminaries

Let L be a simple closed curve in $\mathbb{C}$, oriented positively. Let $g(t)$ be a Hölder continuous function of order $\alpha > 0$ defined on L:

$$|g(t) - g(t_0)| \le A|t - t_0|^\alpha.$$

Fix a point t_0 on L. We will use t and t_0 for points (in $\mathbb{C}$) on L. On $\mathbb{C} \setminus L$, define the function $G(z)$, analytic on $\mathbb{C} \setminus L$,

$$G(z) = \frac{1}{2\pi i} \int_L \frac{g(t)}{t - z} \, dt = \frac{1}{2\pi i} \int_L \frac{g(t) - g(t_0)}{t - z} \, dt + \frac{g(t_0)}{2\pi i} \int_L \frac{1}{t - z} \, dt$$
$$= \begin{cases} \Psi(z) + g(t_0), & \text{for } z \text{ in the interior of } L, \\ \Psi(z), & \text{for } z \text{ in the exterior of } L. \end{cases}$$

where we have defined $\Psi(z)$ and appealed to Cauchy's integral formula. We wish to investigate the non-tangential limit of $\Psi(z)$ as $z \longrightarrow t_0$. We claim:

PROPOSITION 18. *With g satisfying the above, $\Psi(z)$ tends uniformly (with respect to the point t_0 on L) to the limit*

$$\Psi(t_0) = \frac{1}{2\pi i} \int_L \frac{g(t) - g(t_0)}{t - t_0} \, dt.$$

We assume that z approaches t_0 such that the acute angle between the line-segment $\overline{z, t_0}$ and the tangent to L at t_0 is bounded away from zero, uniformly with respect to t_0.

Note: the integral in the proposition is non-singular due to the Hölder continuity of g.

Proof. It is clearly sufficient to prove the result for the integral over a small piece l of L close to t_0; set

$$\psi(t_0) = \frac{1}{2\pi i} \int_l \frac{g(t) - g(t_0)}{t - t_0} \, dt, \qquad \psi(z) = \frac{1}{2\pi i} \int_l \frac{g(t) - g(t_0)}{t - z} \, dt.$$

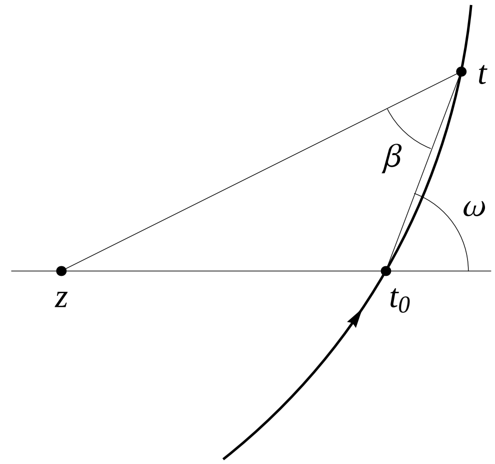
Then

$$\psi(z) - \psi(t_0) = \frac{1}{2\pi i} \int_l (g(t) - g(t_0)) \Big(\frac{1}{t - z} - \frac{1}{1 - t_0} \Big) \, dt = \frac{z - t_0}{2\pi i} \int_l \frac{g(t) - g(t_0)}{(t - z)(t - t_0)} \, dt.$$

Let $\rho > 0$ be small and consider the part $l_1$of l which lies inside the disk of radius ρ centered at t_0 and the part $l_2 = l \setminus l_1$. Write $\psi(z) - \psi(t_0) = I_1 + I_2$, the integrals over l_1 and l_2 respectively.

By the Holder continuity of g,

$$|I_1| \le \frac{|z-t_0|}{2\pi} \int_{l_1} \frac{A|t-t_0|^{\alpha-1}}{|t-z|}\, dt.$$



If ω is the acute angle between the line-segments $\overline{z, t_0}$ and $\overline{t, t_0}$ then, within l, we have

$$\frac{|t-z|}{\sin(\pi-\omega)} = \frac{|t-z|}{\sin\omega} = \frac{|z-t_0|}{\sin\beta} \ge |z-t_0|$$

(where β is the angle between the line-segments $\overline{z, t}$ and $\overline{t, t_0}$). Now within the disk of radius ρ we certainly have $\omega \ge \omega_0 > 0$ for some ω_0, so

$$|t-z| \ge |z-t_0| \sin\omega_0.$$

Thus

$$|I_1| \le \frac{A}{2\pi \sin\omega_0} \int_{l_1} |t-t_0|^{\alpha-1}\, dt.$$

The integral is not singular at t_0 since the power is greater than -1; thus the integral goes to zero as the length of l_1 goes to zero. So, given $\varepsilon > 0$, $|I_1| < \varepsilon$ for sufficiently small ρ, independent of t_0 and z.

We now consider I_2. For $|z-t_0| < \rho/2$, we have $|z-t| \ge \rho/2$ for all $t \in l_2$. We also clearly have $|t-t_0| \ge \rho$ for $t \in l_2$. Thus

$$|I_2| \le \frac{|z-t_0|}{2\pi} \int_{l_2} \frac{|g(t)-g(t_0)|}{(\rho/2)\rho}\, dt \le \frac{C|z-t_0|}{\rho^2}$$

for a constant C independent of t_0 and z. Thus $|I_2| < \varepsilon$ for sufficiently small $|z-t_0|$. $\square$

If we denote by $G_+(t_0)$ the (uniformly non-tangential) limit of $G(z)$ as z approaches t_0 from the left (with respect to the orientation of the curve L), and by $G_-(t_0)$ the corresponding limit from the right, we thus have

$$G_+(t_0) = g(t_0) + \frac{1}{2\pi i} \int_L \frac{g(t)-g(t_0)}{t-t_0}\, dt,$$
$$G_-(t_0) = \frac{1}{2\pi i} \int_L \frac{g(t)-g(t_0)}{t-t_0}\, dt.$$

If we consider the principal value integral then we have (exercise)

$$\frac{1}{2\pi i} \int_L \frac{1}{t-t_0}\, dt = \frac{1}{2}$$

and so we may re-write the above formulae as

$$G_+(t_0) = \frac{1}{2}g(t_0) + \frac{1}{2\pi i}\int_L \frac{g(t)}{t-t_0}\,dt,$$
$$G_-(t_0) = -\frac{1}{2}g(t_0) + \frac{1}{2\pi i}\int_L \frac{g(t)}{t-t_0}\,dt.$$

REMARK. In what follows, we denote elements of $\mathbb{S}$ in several ways; a vector $\omega \in \mathbb{S}$ has a Cartesian expression $(\cos\sigma, \sin\sigma)$, say, and has a $\mathbb{C}$ expression $z = e^{i\sigma}$.

COROLLARY 19. *If g is Holder continuous of order $\alpha > 0$ on the unit circle $\mathbb{S}$ and L is $\mathbb{S}$ oriented counter-clockwise, then for $\omega \in \mathbb{S}$,*

$$G_+(-\omega^\perp) - G_+(\omega^\perp) = \frac{1}{2}\big(g(-\omega^\perp) - g(\omega^\perp)\big) + \frac{1}{2\pi i}\,\mathrm{p.\,v.}\int_{\mathbb{S}} \frac{1}{\omega\cdot\theta} g(\theta)\,d\theta.$$

Proof. With $\omega = (\cos\sigma, \sin\sigma)$, set $z = e^{i\sigma}$. Then $\omega^\perp = iz$ and so, from the above,

$$G_+(-iz) - G_+(iz) = \frac{1}{2}\big(g(-iz) - g(iz)\big) + \frac{1}{2\pi i}\,\mathrm{p.\,v.}\int_{\mathbb{S}}\Big(\frac{1}{w+iz} - \frac{1}{w-iz}\Big)g(w)\,dw.$$

Now

$$\mathrm{p.\,v.}\int_{\mathbb{S}}\Big(\frac{1}{w+iz} - \frac{1}{w-iz}\Big)g(w)\,dw = \mathrm{p.\,v.}\int_{\mathbb{S}} \frac{-2iz}{w^2+z^2} g(w)\,dw$$

Parameterizing $\mathbb{S}$ by $w = e^{i\psi}$ the integral becomes

$$\mathrm{p.\,v.}\int_0^{2\pi} \frac{-2iz}{w^2+z^2} g(w) iw\,d\psi = \mathrm{p.\,v.}\int_0^{2\pi} \frac{2e^{i\sigma}e^{i\psi}}{e^{2i\sigma}+e^{2i\psi}} g(e^{i\psi})\,d\psi.$$

But

$$\frac{e^{2i\sigma}+e^{2i\psi}}{2e^{i\sigma}e^{i\psi}} = \frac{e^{i(\sigma-\psi)}+e^{-i(\sigma-\psi)}}{2} = \cos(\sigma-\psi) = \omega\cdot\theta$$

where we let θ be the $\mathbb{S}$ representation of $w = e^{i\psi}$. Thus we have

$$\mathrm{p.\,v.}\int_0^{2\pi} \frac{1}{\omega\cdot\theta} g(e^{i\psi})\,d\psi = \mathrm{p.\,v.}\int_{\mathbb{S}} \frac{1}{\omega\cdot\theta} g(\theta)\,d\theta.$$

□

9 Inverting the attenuated X-ray transform

We present Natterer's inversion formula [5] with some modification to the proof as presented in [2], which is similar to what was found by Boman and Strömberg [1].

Define

$$h(\theta,t) = \frac{1}{2}(I+iH)Ra(\theta,s) \tag{7}$$

where I is the identity operator, R is the X-ray transform, and the Hilbert transform H is applied to Ra in the second variable.

LEMMA 20. *The coefficients of the Fourier series expansion in the angular variable of the function $h(\theta,\theta\cdot x) - Da(x,\theta^\perp)$ are zero for negative or even index.*

Proof. Since, using the substitution $s = x\cdot\theta - t$,

$$\begin{aligned} Ra(\theta, x\cdot\theta) &= \int_{y\cdot\theta = x\cdot\theta} a(y)\,dy = \int_{\mathbb{R}} a\big((x\cdot\theta)\theta + t\theta^\perp\big)\,dt = \int_{\mathbb{R}} a(x - s\theta^\perp)\,ds \\ &= \Big(\int_{-\infty}^{0} + \int_0^\infty\Big) a(x - s\theta^\perp)\,ds \end{aligned}$$

we see that $Ra(\theta, x\cdot\theta) = Da(x,-\theta^\perp) + Da(x,\theta^\perp)$ by changing $s\mapsto -s$ in the first integral. Thus

$$h(\theta,\theta\cdot x) - Da(x,\theta^\perp) = \frac{1}{2}\big(Da(x,-\theta^\perp) - Da(x,\theta^\perp)\big) + \frac{i}{2} HRa(\theta, x\cdot\theta).$$

We compute $HRa(\theta, x\cdot\theta)$: First

$$Ra(\theta,t) = \int_{y\cdot\theta = t} a(y)\,dy = \int_{\mathbb{R}} a(t\theta + s\theta^\perp)\,ds$$

so

$$\begin{aligned} HRa(\theta, x\cdot\theta) &= \frac{1}{\pi}\,\mathrm{p.\,v.}\int \frac{Ra(\theta,t)}{x\cdot\theta - t}\,dt = \frac{1}{\pi}\,\mathrm{p.\,v.}\int\int \frac{a(t\theta + s\theta^\perp)}{x\cdot\theta - t}\,ds\,dt \\ &= \frac{1}{\pi}\,\mathrm{p.\,v.}\int_{\mathbb{R}^2} \frac{a(y)}{(x-y)\cdot\theta}\,dy. \end{aligned}$$

Setting setting $y = x + r\omega$, $dy = r\,dr\,d\omega$,

$$HRa(\theta, x\cdot\theta) = \frac{1}{\pi}\,\mathrm{p.\,v.}\int_{\mathbb{S}}\int_0^\infty \frac{a(x+r\omega)}{-r\omega\cdot\theta} r\,dr\,d\omega = -\frac{1}{\pi}\,\mathrm{p.\,v.}\int_{\mathbb{S}} \frac{Da(x,\omega)}{\theta\cdot\omega}\,d\omega.$$

So we now have

$$\begin{aligned} h(\theta,\theta\cdot x) - Da(x,\theta^\perp) &= \frac{1}{2}\big(Da(x,-\theta^\perp) - Da(x,\theta^\perp)\big) + \frac{1}{2\pi i}\,\mathrm{p.\,v.}\int_{\mathbb{S}} \frac{Da(x,\omega)}{\theta\cdot\omega}\,d\omega \\ &= \tilde{G}_+(-\theta^\perp) - \tilde{G}_+(\theta^\perp) \end{aligned}$$

where $\tilde{G}(z) = \dfrac{1}{2\pi i}\displaystyle\int_{\mathbb{S}} \frac{Da(x,t)}{t-z}\,dt$. Thus $h(\theta,\theta\cdot x) - Da(x,\theta^\perp)$ is the boundary value of a function analytic on the unit disk, and from Cauchy's integral theorem, its negative Fourier coefficients are all zero. Furthermore, since it is an odd function, the Fourier coefficients of even index are also zero. ☐

LEMMA 21. *Let* $\theta = (\cos\varphi, \sin\varphi)$. *Then*

$$\mathrm{p.\,v.} \int_0^{2\pi} \frac{\theta}{x\cdot\theta} e^{il\varphi}\, d\varphi = \begin{cases} 0, & l \text{ odd}, \\ \frac{2\pi}{|x|}\frac{x}{|x|}, & l = 0, \\ -\frac{2\pi i}{|x|}\left(\frac{x^\perp}{|x|}\right)^{l+1}, & l > 0 \text{ even.} \end{cases}$$

(In writing $\left(\frac{x^\perp}{|x|}\right)^{l+1}$*, we are identifying* $\mathbb{S}$ *with the unit circle in* $\mathbb{C}$*.)*

Proof. We apply Corollary 19 to the function $g(\theta) = \theta^{l+1}$, identifying θ with $e^{i\varphi}$. Thus, with $\omega = x/|x|$, $\omega^\perp = ix/|x|$,

$$\begin{aligned} \frac{1}{2\pi i}\,\mathrm{p.\,v.} \int_0^{2\pi} \frac{\theta}{x\cdot\theta} e^{il\varphi}\, d\varphi &= \frac{1}{2\pi i|x|}\,\mathrm{p.\,v.} \int_{\mathbb{S}} \frac{\theta^{l+1}}{\omega\cdot\theta}\, d\theta \\ &= \frac{1}{|x|}\big[G_+(-\omega^\perp) - G_+(\omega^\perp) - \frac{1}{2}\big(g(-\omega^\perp) - g(\omega^\perp)\big)\big]. \end{aligned}$$

Now extending g to the interior of $\mathbb{S} \subset \mathbb{C}$, $g(z) = z^{l+1}$, we have

$$G(z_0) = \int_{\mathbb{S}} \frac{z^{l+1}}{z - z_0}\, dz = z_0^{l+1} \qquad \forall\ |z_0| < 1$$

so $G_+(\pm\omega^\perp) = (\pm\omega^\perp)^{l+1} = g(\pm\omega^\perp)$. Thus

$$\begin{aligned} \mathrm{p.\,v.} \int_0^{2\pi} \frac{\theta}{x\cdot\theta} e^{il\varphi}\, d\varphi = \frac{\pi i}{|x|}\big(g(-\omega^\perp) - g(\omega^\perp)\big) &= \begin{cases} 0, & l \text{ odd}, \\ -\frac{2\pi i}{|x|}(\omega^\perp)^{l+1}, & l \text{ even} \end{cases} \\ &= \begin{cases} 0, & l \text{ odd}, \\ \frac{2\pi}{|x|}\frac{x}{|x|}, & l = 0, \\ -\frac{2\pi i}{|x|}\left(\frac{x^\perp}{|x|}\right)^{l+1}, & l > 0 \text{ even} \end{cases} \end{aligned}$$

□

With these preparations we can present the inversion formula of Natterer.

THEOREM 22. *If* $f \in \mathscr{S}(\mathbb{R}^2)$ *then, with* h *as in* (7)*,*

$$f(x) = \frac{1}{4\pi}\,\mathrm{Re}\ \mathrm{div} \int_{\mathbb{S}} \theta e^{Da(x,\theta^\perp)} e^{-h(\theta, x\cdot\theta)} H(e^h R_a f)(\theta, x\cdot\theta)\, d\theta.$$

Here, the Hilbert transform H *is applied to* $e^h R_a f$ *in the second variable.*

Proof. We start with

$$
\begin{aligned}
H(e^h R_a f)(\theta, s) &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{e^{h(\theta,t)}}{s-t} \int_{y\cdot\theta=t} e^{-Da(y,\theta^\perp)} f(y)\, dy\, dt \\
&= \frac{1}{\pi} \int_{\mathbb{R}} \frac{e^{h(\theta,t)}}{s-t} \int_{\mathbb{R}} e^{-Da(t\theta+r\theta^\perp,\theta^\perp)} f(t\theta + r\theta^\perp)\, dr\, dt \\
&= \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{f(y)}{s - y\cdot\theta} e^{h(\theta,y\cdot\theta)-Da(y,\theta^\perp)}\, dy.
\end{aligned}
$$

Let $u(y,\theta) = h(\theta, y\cdot\theta) - Da(y,\theta^\perp)$ be the function which appears in Lemma 20. Then

$$
\theta e^{-u(x,\theta)} H(e^h R_a f)(\theta, x\cdot\theta) = \frac{1}{\pi} \int_{\mathbb{R}^2} f(y) \frac{\theta}{(x-y)\cdot\theta} e^{u(y,\theta)-u(x,\theta)}\, dy.
$$

We claim that

$$
\operatorname{Re} \int_{\mathbb{S}} \frac{\theta}{(x-y)\cdot\theta} e^{u(y,\theta)-u(x,\theta)} = 2\pi \frac{x-y}{|x-y|}, \tag{8}
$$

and further, for any $g \in \mathscr{S}(\mathbb{R}^2)$,

$$
\int_{\mathbb{R}^2} g(y) \operatorname{div} \frac{x-y}{|x-y|^2}\, dy = 2\pi g(x). \tag{9}
$$

We leave the proofs of these to Lemmas 23 and 24 below. Given these claims,

$$
\begin{aligned}
&\frac{1}{4\pi} \operatorname{Re}\ \operatorname{div} \int_{\mathbb{S}} \theta e^{Da(x,\theta^\perp)} e^{-h(\theta,x\cdot\theta)} H(e^h R_a f)(\theta, x\cdot\theta)\, d\theta \\
&\quad = \frac{1}{4\pi^2} \operatorname{Re}\ \operatorname{div} \int_{\mathbb{S}} \int_{\mathbb{R}^2} f(y) \frac{\theta}{(x-y)\cdot\theta} e^{u(y,\theta)-u(x,\theta)}\, dy\, d\theta \\
&\quad = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} f(y) \operatorname{div} \operatorname{Re} \int_{\mathbb{S}} \frac{\theta}{(x-y)\cdot\theta} e^{u(y,\theta)-u(x,\theta)}\, d\theta\, dy \\
&\quad = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(y) \operatorname{div} \frac{x-y}{|x-y|^2}\, dy \\
&\quad = f(x).
\end{aligned}
$$

□

LEMMA 23. *With* $u(x,\theta) = h(\theta, x\cdot\theta) - Da(x,\theta^\perp)$*, the function which appears in Lemma 20, it holds that*

$$
\operatorname{Re} \int_{\mathbb{S}} \frac{\theta}{(x-y)\cdot\theta} e^{u(y,\theta)-u(x,\theta)}\, d\theta = 2\pi \frac{x-y}{|x-y|}.
$$

Proof. Setting $\theta = (\cos\varphi, \sin\varphi)$, from Lemma 20,

$$
u(y,\theta) - u(x,\theta) = \sum_{l>0 \text{ odd}} (u_l(y) - u_l(x)) e^{il\varphi}
$$

for functions $u_l(x)$. To understand $\exp(u(y,\theta)-u(x,\theta))$ we use the facts that $e^\alpha = \cosh\alpha + \sinh\alpha$. Now if $\alpha = \alpha_1 e^{i\varphi} + \alpha_3 e^{3i\varphi} + \cdots$ then

$$\begin{aligned}
\cosh\alpha &= \frac{1}{2}(e^\alpha + e^{-\alpha}) \\
&= \frac{1}{2} + \frac{1}{2}\Big(\sum_{l>0 \text{ odd}} \alpha_l e^{il\varphi}\Big) + \frac{1}{4}\Big(\sum_{l>0 \text{ odd}} \alpha_l e^{il\varphi}\Big)^2 + \cdots \\
&\qquad\qquad + \frac{1}{2} - \frac{1}{2}\Big(\sum_{l>0 \text{ odd}} \alpha_l e^{il\varphi}\Big) + \frac{1}{4}\Big(\sum_{l>0 \text{ odd}} \alpha_l e^{il\varphi}\Big)^2 - \cdots \\
&= 1 + \sum_{j>0 \text{ even}} \frac{1}{j!}\Big(\sum_{l>0 \text{ odd}} \alpha_l e^{il\varphi}\Big)^j = 1 + \sum_{l>0 \text{ even}} \beta_l e^{il\varphi}
\end{aligned}$$

for some coefficients β_l. Applying this to $u(y,\theta)-u(x,\theta)$,

$$\cosh\big(u(y,\theta)-u(x,\theta)\big) = 1 + \sum_{l>0 \text{ even}} u_l(x,y)e^{il\varphi},$$

for some functions $u_l(x,y)$, l even. Similarly,

$$\sinh\big(u(y,\theta)-u(x,\theta)\big) = \sum_{l>0 \text{ odd}} u_l(x,y)e^{il\varphi},$$

for some functions $u_l(x,y)$, l odd. Applying Lemma 21 with $\omega = (\cos\psi, \sin\psi) = \dfrac{(x-y)^\perp}{|x-y|}$ we thus obtain

$$\begin{aligned}
\text{p. v.} \int_0^{2\pi} &\frac{\theta}{(x-y)\cdot\theta}\cosh\big(u(y,\theta)-u(x,\theta)\big)\, d\varphi \\
&= 2\pi\frac{x-y}{|x-y|^2} - 2\pi i \sum_{l>0 \text{ even}} u_l(x,y)e^{il\psi}\frac{(x-y)^\perp}{|x-y|^2} \\
&= 2\pi\frac{x-y}{|x-y|^2} - 2\pi i\big[\cosh\big(u(y,\omega)-u(x,\omega)\big) - 1\big]\frac{(x-y)^\perp}{|x-y|^2}, \\
\text{p. v.} \int_0^{2\pi} &\frac{\theta}{(x-y)\cdot\theta}\sinh\big(u(y,\theta)-u(x,\theta)\big)\, d\varphi = 0.
\end{aligned}$$

Next, since $(x-y)\cdot\omega = (x-y)\cdot\big((x-y)/|x-y|\big)^\perp = 0$, $x\cdot\omega = y\cdot\omega$, and so

$$\begin{aligned}
u(y,\omega)-u(x,\omega) &= h(\omega, y\cdot\omega) - Da(y,\omega^\perp) - h(\omega, x\cdot\omega) + Da(x,\omega^\perp) \\
&= Da(x,\omega^\perp) - Da(y,\omega^\perp) = -\int_y^x a\, ds
\end{aligned}$$

where the integral is along the line joining y to x. This shows that $u(y,\omega)-u(x,\omega)$ is *real*, and so from the above,

$$\begin{aligned}
\operatorname{Re}\int_0^{2\pi} \frac{\theta}{(x-y)\cdot\theta} e^{u(y,\theta)-u(x,\theta)}\, d\varphi &= \operatorname{Re}\int_0^{2\pi} \frac{\theta}{(x-y)\cdot\theta}\cosh\big(u(y,\theta)-u(x,\theta)\big)\, d\varphi \\
&= 2\pi\frac{x-y}{|x-y|}.
\end{aligned}$$

□

LEMMA 24. *The fundamental solution of the divergence operator on* $\mathbb{R}^2$ *is* $\dfrac{x}{2\pi|x|^2}$.

Proof. Formally, we have

$$\frac{x}{|x|^2} = \operatorname{grad}(\log|x|) \quad \Rightarrow \quad \operatorname{div}\frac{x}{|x|^2} = \operatorname{div}\operatorname{grad}(\log|x|) = \Delta(\log|x|) = 2\pi\delta(x).$$

□

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