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Abstract

To ascertain the viability of a project, undertake resource allocation, take part in bidding processes and similar project related decisions, modern project management requires forecasting techniques for costs, duration and performance of a project, not only under normal circumstances, but also under external events that might abruptly change the *status quo*. We provide a Bayesian framework for such problem, in which we predict the probability and the impact of various disruptive events and, consequently, a global forecast of project performance. We focus on project costs to introduce the methodology, but the ideas apply equally to project duration or performance forecasting.

KEYWORDS: Expert opinion, Project costing, Project risk, Decision analysis, Risk management, Bayesian analysis, Copula functions.

1 Introduction

The globalization of the economy, more competitive markets and an increasingly hazardous environment have forced organizations to optimize all operations in projects, often adopting riskier alternatives, e.g. ordering from cheaper vendors that might be less reliable for meeting deadlines. As a consequence, risk analysis and management, see Bedford and Cooke (2001) or Henley and Kumamoto (1992) for reviews, have become crucial in project management. This issue has been recently acknowledged in the literature, as in Dillon et al. (2003), who propose a decision-support system to allocate a budget among projects, considering technical and managerial risk trade-offs, and assessing the acceptability of such risks; or Gasparini et al. (2004), who focus on risk prevention during the project development stage.

Traditional project cost, duration or performance forecasting under normal circumstances, see e.g. Palomo et al. (2004b), is no longer sufficient, and an approach to account for disruptive events that might affect the project is needed. Furthermore, a formal approach is required since the common practice of increasing the variance while estimating the performance of the tasks involved in a project does not account for the potentially very adverse effects of these events, if they occur. As an example, when undertaking a large dam construction, an engineering company should consider, say, the possibility of a sudden shortage of materials or unexpected strikes that could lead to a longer and/or more expensive project. Indeed, this information would enable project managers to understand, for example, why the construction of the Opera House in Sydney, which was planned to take 6 years and 7.2 million Australian dollars, was finished after 16 years with an actual cost of 102 million Australian dollars. More recently, the Spanish railway project Madrid-Lleida will cost 38% more than what it was planned three years before. Such forecasts may be extremely important while preparing bids in auctions, see Palomo et al. (2004a); for reserving insurance costs, see Garvey (2000); and for many other project-related decisions.

We, therefore, provide a general Bayesian framework to model unplanned low probability-high impact events in projects, French and Rios Insua (2000), Berger (1985), Robert (2001) and Casella and Berger (2001) provide multiple references. It will allow us to consider both expert's opinions and past company performance as valuable information when responding, in a coherent way, to engineering common interests such as:

1. assessing the project global performance,
2. computing probabilities of overrunning certain threshold costs,
3. identifying events that may entail very poor project performance,
4. identifying potential corrective actions,

5. etc.

To do so, we focus both on inference about events' probabilities and on their impacts on the consequences on project performance, which we call *gravities*. For example, we could focus on project cost and a delay in material supply could entail an increase in cost; we are therefore interested in the probability of such delay happening and the probability of the increase in project's cost. The former provides insights about the probability of changes in the initial project performance assessment, whereas the latter quantifies such changes. We propose, first, assessing the likelihood of various disruptive events, including an event *nothing happens*, and, then, assess the gravities of the potential events on the project performance. Finally, we combine both assessments to provide insights about to the engineering interests, including forecasting the project total gravity, that is the total impact on the consequences of interest for the project, say cost, duration, performance,... Note that for decision making purposes, we shall evaluate these consequences in (expected) utility terms, as shown in Section 5 or in Palomo et al. (2004a) where the problem of bidding in procurement contract auctions is addressed. For general references in decision analysis, see Clemen (1997) and French and Rios Insua (2000).

1.1 Problem description and notation

We assume that, based on past experience or some reasoning process like brainstorming, etc., we have identified in a project k potentially disruptive events $E_1, E_2, \ldots,$ E_k. We call E_0 the event '*nothing happens*' or '*normal circumstances*'. Clearly, $E_0 = \cap_{l=1}^{k} E_l^c$. Events $\{E_l\}_{l=1}^{k}$ are not necessarily exclusive, nor they are necessarily independent. Events $\{E_l\}_{l=0}^{k}$ are exhaustive. Let $q_l = P(E_l)$, $l = 0, 1, \ldots, k$, be the probability that event E_l happens, and q_l^* be the probability that E_l happens alone. Let $q_{i_1 \cdots i_r} = P(\cap_{j=1}^{r} E_{i_j})$ be the probability of simultaneous occurrence of events $E_{i_1}, \ldots, E_{i_r}$, and $q_{i_1 \cdots i_r}^*$ that of simultaneous occurrence of events $E_{i_1}, \ldots, E_{i_r}$, the rest of the events not happening. Section 2 describes an inference process about event probabilities.

To study the impact of, say, events E_i and E_j happening, we focus on the decrement of the project performance, g_{ij}, that they would entail. As an example, assuming we are focusing on costs, g_{ij} would be the increase in project cost, assuming that, only, E_i and E_j happen. Following the standard term by engineers and project managers, we call these increases *gravities*. We shall assume that we only observe total gravities, not the individual ones associated with events E_i and E_j. Neither gravities will necessarily be additive. However, we shall assume that we know the aggregation rule, e.g. addition (typically when the focus is on the cost), maximum (typically when the focus is on project duration),... Our second objective will be, therefore, to learn about gravities $g_{i_1 \dots i_r}$, for all relevant combinations of events $E_{i_1}, \dots, E_{i_r}$, in order to estimate the project gravity. For example, we may be interested in g_3, if event E_3 happens alone, and in g_{57}, if events E_5 and E_7 may happen simultaneously, and the rest do not happen. Section 3 describes an inference process about gravities when the aggregation rule is the sum, whereas, in Section 4, we assume such rule is the maximum.

Finally, once we have learned about event probabilities and gravities, the key interest will be on the project total gravity g, defined as the mixture

$$g = \sum_{i=1}^{k} q_i^* \cdot g_i + \sum_{i \neq j} q_{ij}^* \cdot g_{ij} + \sum_{i \neq r \neq j} q_{ijr}^* \cdot g_{ijr} + \cdots + q_{1\dots k}^* \cdot g_{1\dots k}$$

In Section 5, we show how to obtain such distribution through an example, and how to use it for several risk management tasks, such as computing the probability of overrunning a threshold level $\bar{g}$,

$$P(g \geq \bar{g}) = \sum_{l=1}^{k} q_i^* \cdot P(g_i \geq \bar{g}) + \sum_{i \neq j} q_{ij}^* \cdot P(g_{ij} \geq \bar{g}) + \cdots + q_{1\dots k}^* \cdot P(g_{1\dots k} \geq \bar{g})$$

or forecasting the expected total cost of the project, which will be

$$E[c] = \iint (c + g) \cdot \pi(c) \cdot \pi(g) \; dc \; dg$$

where $\pi(c)$ and $\pi(g)$ are, assuming the gravity is a cost, the distribution of the project cost, under normal circumstances, and project gravity respectively.

We assume that we have access to gravity data from n past projects in terms of a matrix such as Table 1, in which 1 (0) means that the event happened (did not). For example, in project #1, only event E_1 happened with gravity g_1; in project #2, nothing happened (E_0), with gravity 0; in project #3, only events E_1, E_2 and E_k happened, with (joint) gravity g_{12k}, and so on. We shall summarize the information available in such table as follows:

Project\Event	E_0	E_1	E_2	$\cdots$	E_k	Gravity
#1	0	1	0	$\cdots$	0	g_1
#2	1	0	0	$\cdots$	0	0
#3	0	1	1	$0 \cdots 0$	1	g_{12k}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
#n	0	0	1	$\cdots$	0	g_2

Table 1: Example of events and gravities registered from n past projects.

- In n_i cases, event E_i happened alone, with gravities $g_i^1, \ldots, g_i^{n_i}$, $i = 0, \ldots, k$. Clearly, $g_0^1 = \cdots = g_0^{n_0} = 0$,
- In n_{ij} cases, only events E_i and E_j happened, with gravities $g_{ij}^1, \ldots, g_{ij}^{n_{ij}}$,

and so forth. Clearly, $n = n_0 + \sum_{i=1}^k n_i + \sum_{j \neq i} n_{ij} + \cdots + n_{1 \ldots k}$. We shall also say that E_i, $i = 1, \ldots, k$, has happened $m_i = n_i + \sum_{j \neq i} n_{ij} + \sum_{i \neq r \neq j} n_{ijr} + \cdots + n_{1 \ldots k}$ times. We shall use $\mathbf{D}$ to refer to the data available in each case.

`Introduce it better!!`

We shall illustrate these concepts with a real case study. Due to proprietary reasons we cannot include here the actual data.

A company has a new product and plans to build a factory to produce it. Project managers would like to evaluate project's risks so as to decide whether or not to outsource the construction of the new plant. From previous experience, the company considers that there are four relevant disruptive events that could affect such project: strikes, accidents, power supply failure and materials delays due to suppliers. We shall assume that they are mutually independent, their effects are also independent, and they affect the project just in terms of cost. Data from past similar projects are structured as in Table 2. The engineering department of the company has developed a design, and several external companies have submitted offers to construct it, the lowest being equal to $970,000$ euros.

The company also has access to an expert who is asked to provide both the probability that each of these events arises, as well as their potential impact in the project.

Project	Nothing happens	Labor strike	Accident	Power supply failure	Material delays	Gravity
#1	0	0	0	1	1	$g^1_{34} = 207$
#2	0	1	1	0	0	$g^2_{12} = 315$
#3	1	0	0	0	0	0
#4	0	0	0	1	0	$g^4_3 = 143$
#5	0	1	0	1	1	$g^5_{134} = 387$

Table 2: Data from five projects with four possible disruptive events. They can be summarized by $m_0 = 1$, $m_1 = 2$, $m_2 = 1$, $m_3 = 3$, $m_4 = 2$, $n = 5$, $k = 4$, $n_3 = 1$, $n_{34} = 1$, $n_{12} = 1$, $n_{134} = 1$, $n_0 = 1$. Gravities are provided in thousands of euros.

2 Inference about event probabilities

We face now the problem of estimating the probability of various disruptive events happening, and affecting the original project performance.

2.1 The independent case

We start with the simplest case in which the k disruptive events are mutually independent. In such case, we have that for $i, j = 1, \dots, k$,

$$
\begin{aligned}
P(E_i \cap E_j) &= q_i \cdot q_j \qquad\qquad i \neq j \\
P(E_i \cap E_j^c) &= q_i \cdot (1 - q_j) \\
&\vdots \\
P(E_0) = P(\cap_{i=1}^k E_i^c) &= \prod_{i=1}^k (1 - q_i)
\end{aligned}
$$

With mutually independent events, whatever the choice of q_i ($0 \leq q_i \leq 1$), $i = 1, \dots, k$, we have that $0 \leq P(\cup_{i=1}^k E_i) \leq 1$. Hence, we may elicit the q_i's independently and obtain a coherent assessment. Therefore, our approach will be based on the standard beta prior assumptions

$$q_i \sim Beta(\alpha_i, \beta_i), \quad i = 1, \dots, k$$

together with their independence. We may, then, easily prove that as in standard beta-binomial models, see e.g. Berger (1985), a posteriori,

$$q_i|\mathbf{D} \sim Beta(\alpha_i + m_i, \beta_i + n - m_i), \quad i = 1, \ldots, k$$

The posterior moments of the event *nothing happens* are

$$E[q_0|\mathbf{D}] = \prod_{i=1}^{k} \left[\frac{\beta_i + n - m_i}{\alpha_i + \beta_i + n} \right]$$

$$Var[q_0|\mathbf{D}] = \prod_{i=1}^{k} \left[\frac{(\alpha_i + m_i)(\beta_i + n - m_i)}{(\alpha_i + \beta_i + n)^2(\alpha_i + \beta_i + n + 1)} + \frac{(\beta_i + n - m_i)^2}{(\alpha_i + \beta_i + n)^2} \right] - \prod_{i=1}^{k} E[q_i|\mathbf{D}]^2$$

2.2 The dependent case

In some projects, the assumption of disruptive event independence may be untenable. We shall consider now four approaches to tackle such a situation, while preserving the assumption of beta marginals for the basic disruptive event probabilities. Each approach differs in the assumptions about event interactions and on the cognitive demands on the expert.

We first partition the event space and use a standard Dirichlet-multinomial model. Since the number of events may be very large, and some of them will be very unlikely, we propose a second approach to avoid overloading the expert eliciting events with almost zero probability: we consider only two event interactions, ignoring higher order ones. This idea is frequently found in the experimental design literature, see, for example, Myers and Montgomery (1995), in which only main effects and second order interactions are considered. A third solution is based on assessing the marginal probabilities of the basic events, and then inducing dependence through the deterministic constraints required to ensure coherence. Finally, we consider copulas as the basis for modeling dependence among disruptive events. These functions will be used to construct the multivariate distribution function for the events, given their beta marginals.

Partitioning the event space

Let A be the set of all possible intersections of the events and their complements, forming the corresponding partition. The cardinal of A is $2^k - 1$. Without loss of generality, we assume that all elements in A are nonempty. We denote $A_i = E_i \cap_{j \neq i} E_j^c$, $A_{ij} = E_i \cap E_j \cap_{i \neq r \neq j} E_r^c$, $i, j, r = 1, \ldots, k$, and so on. Let $A_0 = E_0$ and let q_i^* be the probability of A_i.

We use a standard Dirichlet-multinomial model, see e.g. French and Rios Insua (2000), to learn about the q_i^*'s. The learning goes as follows:

- Prior:
$$q_0^*, q_1^*, \ldots, q_{1\ldots k}^* \sim Dir(a_0, a_1, \ldots, a_{1\ldots k})$$

- Likelihood:
$$\mathbf{D}|q_0^*, q_1^* \ldots, q_{1\ldots k}^* \sim Mult(n, q_0^*, q_1^*, \ldots, q_{1\ldots k}^*)$$

- Posterior:
$$q_0^*, q_1^*, \ldots, q_{1\ldots k}^*|\mathbf{D} \sim Dir(a_0 + n_0, a_1 + n_1, \ldots, a_{1\ldots k} + n_{1\ldots k})$$

Once we have estimated the posterior probabilities of each partition, we may easily recover the posterior distributions for each event, applying the properties of the Dirichlet distribution, see Ferguson (1973),

$$q_i|\mathbf{D} \sim Beta\left(\alpha_i^{'} + m_i, \beta_i^{'} + n - m_i\right), \qquad i = 0, 1, \ldots, k$$

where $\alpha_i^{'} = \sum_{j \in \hat{\imath}} a_j$ and $\beta_i^{'} = \sum_{j \notin \hat{\imath}} a_j$, $\hat{\imath} = \{i : E_i \cap A_j \neq \emptyset, j = 0, 1, \ldots, \{1, \ldots, k\}\}$.

To elicit prior probabilities of each partition, we may use the following procedure. First, we ask for the probability q_0^* of A_0 (nothing happens), and the probability z_1 that only one event occurs ($0 \leq z_1 \leq 1 - q_0^*$), along with some confidence measure on this belief. We spread the probability z_1 uniformly among the k single events, $q_i^* = \frac{z_1}{k}$, $i = 1, \ldots, k$, and assume that each one has the same confidence. Similarly, the probability z_2 that whatever two events occur only (and not the others) is assessed ($0 \leq z_2 \leq 1 - q_0^* - z_1$). Then, giving equal probability of occurrence to each of the $\binom{k}{2}$ possible combinations of two events, we have $q_{ij}^* = \frac{2z_2}{k(k-1)}$, and so forth. Other possibilities, not uniform, could also be considered but are more demanding to the expert.

Limiting the interactions

Although we have suggested an approach to prior assessment that aims at mitigating expert's effort, in general, the size of the partition space and the rarity of many of such events may render almost impossible the elicitation for all combinations of events. As a consequence, we propose to ignore high order interactions, considering, a priori, virtually impossible that more than two events happen simultaneously. This leads to adopting the Dirichlet-multinomial model with prior

$$q_0^*, q_1^*, \ldots, q_{1\ldots k}^* \;\sim\; Dir(a_0, a_1, \ldots, a_k, a_{12}, \ldots, a_{(k-1)k}, 0, \ldots, 0)$$

leading to the posterior distributions

$$\begin{aligned} q_i|\mathbf{D} &\sim Beta(\alpha_i^{'} + m_i, \beta_i^{'} + n - m_i), \qquad && i = 0, 1, \ldots, k \\ q_{ij}|\mathbf{D} &\sim Beta(\alpha_{ij}^{'} + m_{ij}, \beta_{ij}^{'} + n - m_{ij}), && i \neq j \end{aligned}$$

Setting marginals and deterministic constraints

Our third approach assumes, again, beta prior marginals for the single disruptive events. However, we do not ask for further information from the expert, but we add the deterministic constraints required to ensure coherence which, in turn, induce correlation. In general, for k events, we obtain the constraints $\mathbf{B}$ and $\mathbf{B}^*$:

$$\mathbf{B} = \left\{ \max_{i=1,\ldots,k} q_i \leq 1 - q_0 \leq \sum_{i=1}^{k} q_i \right\} \qquad \mathbf{B}^* = \left\{ \max_{i=1,\ldots,k} q_i^* \leq \sum_{i=1}^{k} q_i^* \leq 1 - q_0 \right\} \quad (1)$$

Therefore, the priors are

$$\left\{ \begin{array}{rl} q_i^* & \sim Beta(\alpha_i^{'}, \beta_i^{'}) \quad i = 0, 1, \ldots, k \\ (q_0^*, q_1^*, \ldots, q_k^*) & \in \mathbf{B}^* \end{array} \right.$$

which, following the procedure introduced before, will be updated to the posteriors

$$\left\{ \begin{array}{rl} q_i|\mathbf{D} & \sim Beta(\alpha_i^{'} + m_i, \beta_i^{'} + n - m_i) \quad i = 0, 1, \ldots, k \\ (q_0, q_1, \ldots, q_k) & \in \mathbf{B} \end{array} \right.$$

which may be sampled by rejection.

A copula-based joint density

The fourth approach assumes also beta marginals for the single event probabilities, but induces correlation through a copula function. A copula parameterizes the dependence structure of the random variables q_i, thereby capturing their joint behavior, and leaves the location and scale parameters of each variable to be parameterized through the marginals $\pi(q_i)$, see Nelsen (1998) for a full description. The importance of copulas in risk management derives from Sklar theorem which proves that any joint distribution can be written in copula form.

The are several copula families available. We shall use the multivariate normal copula, as it encodes dependence among variables using only pairwise correlation among variables, but may do so with beta marginals. A common criticism to this family is that the expert should specify an entire correlation matrix. However, this strong condition, in our case, may be easily met if the number of relevant single disruptive events is not too large. Also, an adequate combination of correlation assessment methods, such as direct elicitation of correlations, concordance probabilities or conditional fractile estimates, could help in obtaining such a matrix. Clemen and Reilly (1999) provide details about these methods.

Typically, correlation assessments are more difficult than probability elicitations, specially for experts with little statistical training. Therefore, we propose a correlation assessment method based on concordance probabilities, such as rank correlation. We consider Spearman's ρ^S as dependence measure, because it does not depend on the marginal distributions, as the linear correlation coefficient does. Kendall's τ^K and Gini's γ^G indices also verify these properties, see Nelsen (1998) for more details. ρ^S requires pairwise comparisons that can be asked to an expert with questions like: *having selected randomly two past projects A and B from a company, and given that in project A the probability of occurrence of event E_i is greater than in project B, what is the probability that event E_j in project A is also more likely than in project B?* By eliciting these pairwise measures of dependence, ρ^S_{ij}, for all disruptive events and transforming them by

$$r_{ij} = \begin{cases} 2\sin\left(\frac{\pi\cdot\rho^S_{ij}}{6}\right), & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$$

we obtain the linear correlation coefficients, r_{ij}, required by the multivariate normal distribution, and, so, the correlation matrix $V(r_{ij})$.

The joint prior density, with the multivariate normal copula density as dependence function, can be formulated as

$$\pi(q_1,\ldots,q_k|V) = \frac{1}{|V|^{\frac{1}{2}}} \cdot \exp\left\{-\frac{1}{2}\cdot y' \cdot (V^{-1} - I) \cdot y\right\} \cdot \prod_{i=1}^{k} \pi_i(q_i)$$

where $y = (y_1,\ldots,y_k)'$, $y_i = \Phi^{-1}[\Pi_i(q_i)]$; Φ denotes the univariate standard normal cdf; $\Pi_i(q_i)$, $i = 1,\ldots,k$, are the specified (beta) marginals and I is the identity matrix. As the adopted dependence measures are invariant under monotonic transformations, the q_i's have the assessed rank-order correlations in the matrix V.

With such prior, and taking into account the multinomial model, we have that the posterior is

$$\pi(q_1,\ldots,q_k|V,\mathbf{D}) \propto \exp\left\{-\frac{1}{2}\cdot y' \cdot (V^{-1} - I) \cdot y\right\} \times \prod_{i=1}^{k} q_i^{\alpha_i'+m_i-1}(1-q_i)^{\beta_i'+n-m_i-1}$$

from which we may sample using a Metropolis-Hastings algorithm, see Appendix A.1.

3 Forecasting the effects of events: the additive case

We now address the issue of forecasting the gravities that the disruptive events might entail, under the assumptions that they are independent and additive. This will be the case, e.g. when they affect the project only in terms of costs. Recall that we are observing the joint effect and not the individual events' impacts.

We shall assume that the gravity g_i due to event E_i, $i = 1,\ldots,k$, is a random variable with unknown upper bound θ_i. We also assume that the values in the interval $[0,\theta_i]$ are equally likely, reflecting the uncertainty about the effectiveness of corrective action plans in the actual gravity. Companies, while developing a project, apply corrective actions, if disruptive events occur, described on their contingency plans in order to lessen the impacts. For example, if event E_i occurs alone, an actual gravity $g_i = 0$ would correspond to full effectiveness or total correction, i.e. E_i would have no impact; whereas, $g_i = \theta_i$ would correspond to a totally useless action. As prior distribution for θ_i, we shall use a *Gamma* distribution. Summing up, we have

$$\begin{aligned} g_i|\theta_i &\sim U(0,\theta_i), \\ \theta_i &\sim Gamma(\mu_i,\phi_i) \qquad i=1,\dots,k, \text{ independent.} \end{aligned}$$

In this case, we may easily conclude that the (prior) predictive for the gravity g_i is

$$\pi(g_i) = \int \pi(g_i|\theta_i)\pi(\theta_i)\; d\theta_i = \frac{\phi_i\Gamma(\mu_i-1)}{\Gamma(\mu_i)}\left(1 - F_{Gamma(\mu_i-1,\phi_i)}(g_i)\right)$$

with $\mu_i - 1 > 0$.

Our key interest will be to sample from the relevant posterior predictive distributions of the gravities required to assess future project risks. Suppose we are interested in, say, sampling from the predictive distribution for g_i, $g_{ij} = g_i + g_j$, $g_{ijr} = g_i + g_j + g_r$. Then, assuming we have procedures to sample from the posteriors of θ_i, θ_j and θ_r, we could use the following algorithm

`Sample` $\theta_i^*, \theta_j^*, \theta_r^* \sim \pi(\theta_i,\theta_j,\theta_r|\mathbf{D})$
`Sample` $g_i^* \sim \pi(g_i|\theta_i^*)$
`Sample` $g_j^* \sim \pi(g_j|\theta_j^*)$
`Sample` $g_r^* \sim \pi(g_r|\theta_r^*)$
`Output` $g_i = g_i^*$, $g_{ij} = g_i^* + g_j^*$, $g_{ijr} = g_i^* + g_j^* + g_r^*$.

From these samples, we may use standard density estimators. Alternatively, we may use the following approximations:

- The predictive gravity density for g_i

$$\pi(g_i|\mathbf{D}) = \int \pi(g_i|\theta_i)\pi(\theta_i|\mathbf{D})\; d\theta_i \simeq \frac{1}{N}\sum_{l=1}^{N}\frac{1}{\theta_i^l} I_{[0,\theta_i^l]}(g_i) \tag{2}$$

- The predictive gravity density for g_{ij}

$$\begin{aligned} \pi(g_{ij}|\mathbf{D}) &= \iint \pi(g_{ij}|\theta_i,\theta_j)\pi(\theta_i,\theta_j|\mathbf{D})\; d\theta_i\; d\theta_j \simeq \\ &\simeq \frac{1}{N^2}\sum_{l=1}^{N}\sum_{m=1}^{N}\frac{1}{\theta_i^l\theta_j^m}\{\min(g_{ij},\theta_i^l) - \max(0, g_{ij}-\theta_j^m)\} \end{aligned}$$

- The predictive gravity density for g_{ijr}

$$\pi(g_{ijr}|\mathbf{D}) = \iiint \pi(g_{ijr}|\theta_i,\theta_j,\theta_r)\pi(\theta_i,\theta_j,\theta_r|\mathbf{D})d\theta_i\ d\theta_j\ d\theta_r \simeq$$
$$\simeq \frac{1}{N^3}\sum_{l=1}^{N}\sum_{m=1}^{N}\sum_{s=1}^{N}\frac{1}{\theta_i^l\theta_j^m\theta_r^s}\int_{\max(0,g_{ijr}-\theta_r^s)}^{\min(g_{ijr},\theta_i^l+\theta_j^m)}\{\min(g_{ij},\theta_i^l)-\max(0,g_{ij}-\theta_j^m)\}\ dg_{ij}$$

where $\{\theta_i^l\}_{l=1}^N$, $\{\theta_j^m\}_{m=1}^N$, $\{\theta_r^s\}_{s=1}^N$ are samples from the following posteriors, depending on what is observed,

a) when only event E_i occurs n_i times, the likelihood is

$$\pi(\mathbf{D}|\theta_1,\ldots,\theta_k) = \prod_{v=1}^{n_i}\frac{1}{\theta_i}I_{[\max_v(g_i^v),\infty)}(\theta_i) = \frac{1}{\theta_i^{n_i}}I_{[\max(\mathbf{D}),\infty)}(\theta_i)$$

Therefore, the posterior density $\pi(\theta_i|\mathbf{D})$ is a $Gamma(\mu_i - n_i, \phi_i)$ truncated at $I_{[\max(\mathbf{D}),\infty)}(\theta_i)$, which could be sampled by rejection with the following algorithm (assuming $\mu_i - n_i > 0$)

`Until` $\theta_i \geq \max(g_i^1,\ldots,g_i^{n_i})$

`Sample` $\theta_i \sim Gamma(\mu_i - n_i, \phi_i)$

`Output` θ_i

The posterior densities for the other θ_j's would coincide with their priors, as no data about them is observed.

b) when only simultaneous observations of events E_i and E_j are received n_{ij} times, the gravity is $g_{ij} = g_i + g_j$. The likelihood is

$$\pi(\mathbf{D}|\theta_1,\ldots,\theta_k) = \frac{1}{\theta_i^{n_{ij}}\theta_j^{n_{ij}}}\cdot\prod_{h=1}^{n_{ij}}\left\{\min(g_{ij}^h,\theta_i)-\max(0,g_{ij}^h-\theta_j)\right\}$$

Therefore, the posterior is

$$\pi(\theta_i,\theta_j|\mathbf{D}) \propto \left[e^{-\phi_i\theta_i}\theta_i^{\mu_i-n_{ij}-1}\right]\left[e^{-\phi_j\theta_j}\theta_j^{\mu_j-n_{ij}-1}\right]\times$$
$$\times\prod_{h=1}^{n_{ij}}\left\{\min(g_{ij}^h,\theta_i)-\max(0,g_{ij}^h-\theta_j)\right\}$$

To sample from it, we may use a Metropolis-Hastings algorithm, as in Appendix A.2.

c) when event E_i happened alone n_i times; E_j happened alone n_j times; both E_i and E_j happened n_{ij} times; and nothing happened n_0 times, the likelihood is

$$\begin{aligned}\pi(\mathbf{D}|\theta_i,\theta_j) =& \frac{1}{\theta_i^{n_i+n_{ij}}} I_{[\max_v(g_i^v),\infty)}(\theta_i) \frac{1}{\theta_j^{n_j+n_{ij}}} I_{[\max_p(g_j^p),\infty)}(\theta_j) \times \\ &\times \prod_{h=1}^{n_{ij}} \{\min(g_{ij}^h,\theta_i) - \max(0, g_{ij}^h - \theta_j)\}\end{aligned}$$

where $v = 1,\ldots,n_i$ and $p = 1,\ldots,n_j$. Therefore, the posterior is

$$\begin{aligned}\pi(\theta_i,\theta_j|\mathbf{D}) \propto & \left[e^{-\phi_i\theta_i}\theta_i^{\mu_i-n_i-n_{ij}-1}\right] I_{[\max_v(g_i^v),\infty)}(\theta_i) \left[e^{-\phi_j\theta_j}\theta_j^{\mu_j-n_j-n_{ij}-1}\right] \times \\ &\times I_{[\max_p(g_j^p),\infty)}(\theta_j) \times \prod_{h=1}^{n_{ij}} \left\{\min(g_{ij}^h,\theta_i) - \max(0, g_{ij}^h - \theta_j)\right\}\end{aligned}$$

To sample from it, we may use a Metropolis-Hastings algorithm, as in Appendix A.3.

d) when only simultaneous observations of events E_i, E_j and E_r are received n_{ijr} times, the gravity is $g_{ijr} = g_i + g_j + g_r$. We may easily see that the posterior is

$$\begin{aligned}\pi(\theta_i,\theta_j,\theta_r|\mathbf{D}) \propto & \left[e^{-\phi_i\theta_i}\theta_i^{\mu_i-n_{ijr}-1}\right] \left[e^{-\phi_j\theta_j}\theta_j^{\mu_j-n_{ijr}-1}\right] \left[e^{-\phi_r\theta_r}\theta_r^{\mu_r-n_{ijr}-1}\right] \times \\ &\times \prod_{c=1}^{n_{ijr}} \left[\int_{\max(0,g_{ijr}^c-\theta_r)}^{\min(g_{ijr}^c,\theta_i+\theta_j)} \{\min(g_{ij},\theta_i) - \max(0, g_{ij} - \theta_j)\}\, dg_{ij}\right] \times \\ &\times I_{[\max(\mathbf{D}),\infty)}(\theta_i+\theta_j+\theta_r)\end{aligned}$$

We could sample from it using the Metropolis-Hastings algorithm in Appendix A.4.

Other possible cases are dealt with in a similar fashion.

4 Forecasting the effects of events: the maximum case

In the previous section, we assumed that gravities were additive. This approach may be appropriate when the events' impacts are costs. However, when gravities refer to duration of activities scheduled in parallel, as in a PERT network, they may not be additive. We consider now the case in which the aggregation rule is $\max_i(g_i)$, where, as before, g_i are the individual gravities. We again assume

$$\begin{aligned} g_i|\theta_i &\sim U(0,\theta_i), \\ \theta_i &\sim Gamma(\mu_i,\phi_i) \qquad i=1,\dots,k \text{ independent.} \end{aligned}$$

As before, the goal is to sample from the relevant posterior predictive gravity distributions and make inferences on the θ_i's. Suppose we are interested in, say, sampling from the predictive distribution for g_{ij}. Then, we could use the following algorithm

`Sample` $\theta_i^*, \theta_j^* \sim \pi(\theta_i,\theta_j|\mathbf{D})$
`Sample` $g_{ij}^* \sim \pi(g_{ij}|\theta_i^*,\theta_j^*)$
`Output` $g_{ij} = g_{ij}^*$.

From them, we may use standard density estimators. Alternatively, we may use the following approximations:

- The predictive gravity density approximation for g_i remains as in (2).
- The predictive gravity density approximation for g_{ij}

$$\begin{aligned} \pi(g_{ij}|\mathbf{D}) \simeq \frac{1}{N^2}\sum_{l=1}^{N}\sum_{m=1}^{N}\frac{1}{\theta_i^l\theta_j^m}\big\{2g_{ij}\cdot I_{[0,\min(\theta_i^l,\theta_j^m)]}(g_{ij}) + \min(\theta_i^l,\theta_j^m)\times \\ \times I_{[\min(\theta_i^l,\theta_j^m),\max(\theta_i^l,\theta_j^m)]}(g_{ij})\big\} \end{aligned}$$

- The predictive gravity density approximation for g_{ijr}

$$\begin{aligned} \pi(g_{ijr}|\mathbf{D}) \simeq \frac{1}{N^3}\sum_{l=1}^{N}\sum_{m=1}^{N}\sum_{s=1}^{N}\frac{1}{\theta_i^l\theta_j^m\theta_r^s}\Bigg\{3g_{ijr}^2\cdot I_{[0,\min]}(g_{ijr}) + 2g_{ijr}\min\times \\ \times I_{\left[\min,\frac{\theta_i^l\theta_j^m\theta_r^s}{\min\cdot\max}\right]}(g_{ijr}) + \frac{\theta_i^l\theta_j^m\theta_r^s}{\max}\cdot I_{\left[\frac{\theta_i^l\theta_j^m\theta_r^s}{\min\cdot\max},\max\right]}(g_{ijr})\Bigg\} \end{aligned}$$

where $\min = \min(\theta_i,\theta_j,\theta_r)$, $\max = \max(\theta_i,\theta_j,\theta_r)$ and $\{\theta_i^l\}_{l=1}^N$, $\{\theta_j^m\}_{m=1}^N$, $\{\theta_r^s\}_{s=1}^N$ are samples from the following posteriors, depending on what is observed,

a) when only event E_i occurs, the posteriors remain as in the additive case.

b) when $\mathbf{D} = \{g_{ij}^1, \ldots, g_{ij}^{n_{ij}}\}$ is observed, the gravity is $g_{ij} = \max(g_i, g_j)$. The posterior is the joint prior density times the likelihood as follows

$$\pi(\theta_i, \theta_j|\mathbf{D}) \propto \left[e^{-\phi_i\theta_i}\theta_i^{\mu_i-n_{ij}-1}\right]\left[e^{-\phi_j\theta_j}\theta_j^{\mu_j-n_{ij}-1}\right] \times$$
$$\times \prod_{h=1}^{n_{ij}} \left\{2g_{ij}^h \cdot I_{[0,\min(\theta_i,\theta_j)]}(g_{ij}^h) + \min(\theta_i,\theta_j) \cdot I_{[\min(\theta_i,\theta_j),\max(\theta_i,\theta_j)]}(g_{ij}^h)\right\}$$

with $\mu_i - n_{ij} - 1 > 0$ and $\mu_j - n_{ij} - 1 > 0$.

We could sample from it using the Metropolis-Hastings algorithm of Appendix A.2.

c) when $\mathbf{D} = \{g_i^v, g_j^p, g_{ij}^h\}$, $v = 1, \ldots, n_i$, $p = 1, \ldots, n_j$, $h = 1 \ldots, n_{ij}$, is observed, we may easily see that the posterior is

$$\pi(\theta_i, \theta_j|\mathbf{D}) \propto \left[e^{-\phi_i\theta_i}\theta_i^{\mu_i-n_i-n_{ij}-1}\right] I_{[\max_v(g_i^v),\infty)}(\theta_i) \times$$
$$\times \left[e^{-\phi_j\theta_j}\theta_j^{\mu_j-n_j-n_{ij}-1}\right] I_{[\max_p(g_j^p),\infty)}(\theta_j) \times$$
$$\times \prod_{h=1}^{n_{ij}} \left\{2g_{ij}^h \cdot I_{[0,\min(\theta_i,\theta_j)]}(g_{ij}^h) + \min(\theta_i,\theta_j) \times\right.$$
$$\left. \times I_{[\min(\theta_i,\theta_j),\max(\theta_i,\theta_j)]}(g_{ij}^h)\right\}$$

with $\mu_i - n_i - n_{ij} > 0$ and $\mu_j - n_j - n_{ij} > 0$.

We could sample from it using the Metropolis-Hastings algorithm of Appendix A.3.

d) when $\mathbf{D} = \{g_{ijr}^1, \ldots, g_{ijr}^{n_{ijr}}\}$ is observed, the gravity is $g_{ijr} = \max(g_i, g_j, g_r)$. The posterior is obtained as the product of the joint prior and the likelihood as follows

$$\pi(\theta_i, \theta_j, \theta_r|\mathbf{D}) \propto \left[e^{-\phi_i\theta_i}\theta_i^{\mu_i-n_{ijr}-1}\right]\left[e^{-\phi_j\theta_j}\theta_j^{\mu_j-n_{ijr}-1}\right]\left[e^{-\phi_r\theta_r}\theta_r^{\mu_r-n_{ijr}-1}\right] \times$$
$$\times \prod_{c=1}^{n_{ijr}} \left[3(g_{ijr}^c)^2 \cdot I_{[0,\min]}(g_{ijr}^c) + 2g_{ijr}^c \min \times I_{\left[\min, \frac{\theta_i\theta_j\theta_r}{\min\cdot\max}\right]}(g_{ijr}^c) + \right.$$
$$\left. + \frac{\theta_i\theta_j\theta_r}{\max} \cdot I_{\left[\frac{\theta_i\theta_j\theta_r}{\min\cdot\max}, \max\right]}(g_{ijr}^c)\right]$$

with $\mu_i - n_{ijr} - 1 > 0$, $\mu_j - n_{ijr} - 1 > 0$ and $\mu_r - n_{ijr} - 1 > 0$

We could sample from it using the Metropolis-Hastings algorithm of Appendix A.4.

Extensions to other cases are straightforward.

5 An example

We will show now how to compute the project total gravity and cost distributions, while accounting for external risks, through the following case study. We have simplified several management steps for clarity in the exposition of the models introduced previously.

Assuming event independence

For the four events, we get the same expert assessments for the mean (0.09) and mode (0.08) for their probability of occurrence. Therefore, assuming beta distributions, the parameters can be estimated by solving the system equations $E[q_i] = \frac{\alpha_i}{\alpha_i+\beta_i} = 0.09$ and $\frac{\alpha_i - 1}{\alpha_i+\beta_i-2} = 0.08$, $i = 1, \ldots, 4$. Hence, a priori, we have

$$q_i \sim Beta(7.561, 76.447) \quad i = 1, \ldots, 4.$$

For each single event, the posterior distribution's parameters are shown in the second column of Table 3, and their posterior moments are shown in Table 4.

Posterior	Independent		Partitioning		Limiting		Constraints	
	α	β	α	β	α	β	α	β
$q_0\|\mathbf{D}$	$E[q_0\|\mathbf{D}] = 0.635$ $Var[q_0\|\mathbf{D}] = 0.0022$		2.79	4.77	2.81	4.74	2.91	4.64
$q_1\|\mathbf{D}$	9.56	79.45	2.23	5.32	2.21	5.34	2.16	5.39
$q_2\|\mathbf{D}$	8.56	80.45	1.23	6.32	1.21	6.34	1.16	6.39
$q_3\|\mathbf{D}$	10.56	78.45	3.23	4.32	3.21	4.34	3.16	4.39
$q_4\|\mathbf{D}$	9.56	79.45	2.23	5.32	2.21	5.34	2.16	5.39

Table 3: Parameters of the beta posteriors for each approach.

Posterior	Independent		Partitioning		Limiting		Constraints		Copulas	
	Mean	Var	Mean	Var	Mean	Var	Mean	Var	Mean	Var
$q_0\|\mathbf{D}$	0.635	0.0022	0.369	0.0272	0.372	0.0273	0.385	0.0277	0.375	0.0276
$q_1\|\mathbf{D}$	0.107	0.0011	0.295	0.0243	0.293	0.0242	0.286	0.0239	0.292	0.0249
$q_2\|\mathbf{D}$	0.096	0.0010	0.163	0.0159	0.160	0.0157	0.154	0.0152	0.155	0.0155
$q_3\|\mathbf{D}$	0.119	0.0012	0.428	0.0286	0.425	0.0286	0.418	0.0285	0.427	0.0292
$q_4\|\mathbf{D}$	0.107	0.0011	0.295	0.0243	0.293	0.0242	0.286	0.0239	0.291	0.0244

Table 4: Posterior moments for each event following the proposed approaches.

Partitioning the event space

Suppose that the expert provides, following the procedure introduced in Section 2.2, information about all elements in the partition (sixteen in this case) through: $E[q_0^*] = 0.7$, $z_1 = 0.25$, $z_2 = 0.0418$, $z_3 = 0.008$, $z_4 = 0.0002$, and $\varsigma = 0.05$ as the variance of his assessment for the probability that one event happens alone. This implies, for example, that $E[q_i^*] = 0.0625$ or $E[q_{ij}^*] = 0.0069$.

In order to fix the hyperparameters of the Dirichlet prior, we have $a = \sum_{l=0}^{\{1234\}} a_l$ and $a_i = a \cdot E[q_i^*]$. Hence, there are $2^4 + 1$ unknown variables in 2^4 equations, and we have one degree of freedom to fix the spread of the distribution. The variance of any variable X_i in our Dirichlet distribution has the expression

$$Var[X_i] = E[X_i]\left(\frac{a \cdot E[X_i] + 1}{a + 1} - E[X_i]\right)$$

Hence, making $Var[q_i^*] = \varsigma$, we find the parameter a as

$$a = \frac{E[q_i^*]}{\varsigma}\Big(1 - E[q_i^*]\Big) - 1 \ ,$$

In our example, $a = 2.55$, and the prior is

$$q_0^*, q_1^*, \ldots, q_{1234}^* \sim Dir(1.787, 0.1596, \ldots, 0.1596, 0.0178, \ldots, 0.0178, \\ 0.0051, \ldots, 0.0051, 5.1 \cdot 10^{-4}) \ ,$$

Therefore, the posterior is

$$q_0^*, q_1^*, \ldots, q_{1234}^*|\mathbf{D} \sim Dir(2.787, 0.1596, 0.1596, 1.1596, 0.1596, 1.0178, 0.0178, \\ 0.0178, 0.0178, 0.0178, 1.0178, 0.0051, 0.0051, 1.0051, 0.0051, 5.1 \cdot 10^{-4}) \ .$$

For each single event, the posterior distribution's parameters are shown in the third column of Table 3, and their posterior moments are shown in Table 4.

Limiting the interactions

We assume now that the expert provides only information about the cases in which nothing happens, one event arises alone, and two events occur only, along with the variance of his assessment for the probability that only one event arises ς. This implies that all prior information we have is $E[q_0] = 0.7082$, $E[q_i^*] = 0.0625$, $E[q_{ij}^*] = 0.0069$, $Var[q_i^*] = 0.05$, and that higher order interactions are degenerated at zero. Then, the prior distributions are, with $a = 2.55$ obtained as before,

$$\begin{cases} q_i^* & \sim Beta(\alpha_i^{'}, \beta_i^{'}) \qquad i = 0, 1, \dots, 4 \\ q_{ij}^* & \sim Beta(\alpha_{ij}^{'}, \beta_{ij}^{'}) \qquad i \neq j \end{cases}$$

where, for example,

$$\begin{aligned} \alpha_{12}^{'} &= a_{12} = 0.0176 \\ \beta_{12}^{'} &= a_0 + a_1 + a_2 + a_3 + a_4 + a_{13} + a_{14} + a_{23} + a_{24} + a_{34} = 2.5314 \end{aligned}$$

For each single event, the posterior distribution's parameters are shown in the fourth column of Table 3, and their posterior moments are shown in Table 4.

Setting marginals and deterministic constraints

We do not ask now for further information from the expert. Instead, we induce correlation through the deterministic constraints that ensure coherence. This implies that we have $E[q_0] = 0.75$, $E[q_i^*] = 0.0625$, $Var[q_i^*] = 0.05$. Then, the prior distributions are

$$\begin{cases} q_i^* & \sim Beta(\alpha_i^{'}, \beta_i^{'}) \qquad i = 0, 1, \dots, 4 \\ (q_0^*, q_1^*, \dots, q_4^*) & \in \mathbf{B}^* \end{cases}$$

where $\mathbf{B}^*$ is defined as in (1), $\alpha_0^{'} = 1.91$, $\beta_0^{'} = 0.64$, $\alpha_i^{'} = a_i = 0.16$ and $\beta_i^{'} = \sum_{j \neq i} a_j = 2.39$, $i = 1, \dots, 4$. The posterior distributions' parameters are shown in the fifth column of Table 3, and their posterior moments are shown in Table 4.

A copula-based joint density

We use now the approach based on copulas. As before, from the expert's assessments we obtain the following prior distributions: $q_0^* \sim Beta(1.91, 0.64)$, $q_i^* \sim$

$Beta(0.16, 2.39)$, $i = 1, \ldots, 4$. Additionally, he provides information about pairwise correlation among events. Our expert has sufficient statistical training to answer questions such as *Given that, in a project, it is almost sure (with probability 0.95) that labor strike (q_1) has probability not greater that 0.37, what would be your estimation (greater or less/equal than/to 0.95) in the case of the probability of accident* (q_2)? The expert's answer to the question above is 0.9, which corresponds, in the previous priors, to $q_2 = 0.213$. From standard nonparametric regression, we know that

$$E[\Pi_i(q_i)|q_j] = \rho^S_{ij}[\Pi_j(q_j) - 0.5] + 0.5 \qquad i \neq j \ ,$$

and the expert has assessed, given that $\Pi_1(0.37) = 0.95$, $E[\Pi_2(q_2)|0.37] = 0.90 = \Pi_2(0.213)$. Then, we can easily solve for the dependence measure $\rho^S_{12} = 0.9$. Repeating the process for every pair of events, and transforming them into the corresponding linear correlation coefficients, e.g. $r_{12} = 2 \cdot \sin\left(\frac{\pi \cdot \rho^S_{12}}{6}\right) = 0.9$, we find the following correlation matrix, $M = (V^{-1} - \mathbf{I})$,

$$V = \begin{pmatrix} 1 & 0.9 & 0.001 & -0.005 \\ 0.9 & 1 & 0.09 & -0.005 \\ 0.001 & 0.09 & 1 & 0.85 \\ -0.005 & -0.005 & 0.85 & 1 \end{pmatrix} \quad M = \begin{pmatrix} 5.01 & -5.57 & 1.78 & -1.51 \\ -5.57 & 5.21 & -2.0 & 1.7 \\ 1.78 & -2.0 & 3.25 & -3.61 \\ -1.51 & 1.7 & -3.61 & 3.07 \end{pmatrix}$$

Hence, the joint posterior distribution is

$$\begin{aligned} \pi(q_1, \ldots, q_4|V, \mathbf{D}) \propto \exp\left\{-\frac{1}{2} \cdot y' \cdot M \cdot y\right\} \times q_1^{1.16}(1-q_1)^{4.26} \times \\ \times q_2^{0.16}(1-q_2)^{5.26} \times q_3^{2.16}(1-q_3)^{3.26} \times q_4^{1.16}(1-q_4)^{4.26}, \end{aligned}$$

where $y = (\Phi^{-1}[\Pi_1(q_1)], \ldots, \Phi^{-1}[\Pi_4(q_4)])'$. In Figure 1, we represent the corresponding histograms of the posterior samples for each basic event. The corresponding posterior moments are shown in Table 4.

Learning about gravities

We focus now on obtaining insights from gravities observed during similar projects in the past, as shown in last column of Table 2. The expert's assessments about the expected gravities of the four disruptive events were 200, 100, 150 and 50, with variance 100, 25, 75 and 10, respectively. Hence, we obtain the priors

$$\begin{array}{ll} \theta_1 \sim Gamma(400, 2) & \qquad \theta_2 \sim Gamma(400, 4) \\ \theta_3 \sim Gamma(300, 2) & \qquad \theta_4 \sim Gamma(250, 5) \end{array}$$

Figure 1: Histograms of the posterior probabilities for each event using the copula-based joint density approach.

Then, the posterior distribution is

$$\begin{aligned}\pi(\theta_1,\theta_2,\theta_3,\theta_4|\mathbf{D}) \propto \Bigg[& \frac{1}{\theta_3} I_{[g_3,\infty)}(\theta_3)\cdot \frac{1}{\theta_1\theta_2} I_{[g_{12},\infty)}(\theta_1+\theta_2)\cdot \frac{1}{\theta_1\theta_3\theta_4}\times \\ & \times \{\min(g_{12},\theta_1)-\max(0,g_{12}-\theta_2)\}\cdot \frac{1}{\theta_3\theta_4}\cdot \{\min(g_{34},\theta_3)-\max(0,g_{34}-\theta_4)\}\times \\ & \times I_{[g_{134},\infty)}(\theta_1+\theta_3+\theta_4)\cdot \int_{\max(0,g_{134}-\theta_1)}^{\min(g_{134},\theta_3+\theta_4)} \{\min(g_{34},\theta_3)-\max(0,g_{34}-\theta_4)\}\ dg_{34}\Bigg]\times \\ & \times \theta_1^{399}\cdot\theta_2^{399}\cdot\theta_3^{299}\cdot\theta_4^{249}\cdot\exp\{-2\theta_1-4\theta_2-2\theta_3-5\theta_4\}\end{aligned}$$

For future similar projects, the histograms of posterior predictives for g_{12}, g_{13}, g_{24}, g_{34}, g_{134} and g_{234} are shown in Figure 2. Samples $\{\theta_i^l\}_{l=1}^N$, $\{\theta_j^m\}_{m=1}^N$ and $\{\theta_r^s\}_{s=1}^N$, $i,j,r\in\{1,\ldots,4\}$, are obtained from $\pi(\theta_1,\theta_2,\theta_3,\theta_4|\mathbf{D})$ through the following Metropolis-Hastings algorithm

`Given` $(\theta_1^{(t)},\theta_2^{(t)},\theta_3^{(t)},\theta_4^{(t)})$

`Generate`

$\theta_{1t}\sim Gamma(398,2)$; $\theta_{2t}\sim Gamma(399,4)$

Figure 2: Histograms of the posterior predictive gravities if two or three events arise only in the project.

$$\theta_{3t} \sim Gamma(297, 2); \;\; \theta_{4t} \sim Gamma(248, 5)$$

`Take`

$$(\theta_1^{(t+1)}, \theta_2^{(t+1)}, \theta_3^{(t+1)}, \theta_4^{(t+1)}) = \begin{cases} (\theta_{1t}, \theta_{2t}, \theta_{3t}, \theta_{4t}) \text{ with probability } \rho, \\ (\theta_1^{(t)}, \theta_2^{(t)}, \theta_3^{(t)}, \theta_4^{(t)}) \text{ with probability } 1-\rho \end{cases}$$

$$t = t + 1$$

where $\rho = \min \left\{ \frac{H\left(\theta_{1t}, \theta_{2t}, \theta_{3t}, \theta_{4t}\right)}{H\left(\theta_1^{(t)}, \theta_2^{(t)}, \theta_3^{(t)}, \theta_4^{(t)}\right)}, 1 \right\}$, and

$$\begin{aligned} H(\theta_1, \theta_2, \theta_3, \theta_4) &= I_{[315,\infty)}(\theta_1 + \theta_2) \cdot I_{[387,\infty)}(\theta_1 + \theta_3 + \theta_4) \cdot I_{[143,\infty)}(\theta_3) \times \\ &\times \{\min(315, \theta_1) - \max(0, 315 - \theta_2)\} \cdot \{\min(207, \theta_3) - \max(0, 207 - \theta_4)\} \times \\ &\times \left[\int_{\max(0, 387-\theta_1)}^{\min(387, \theta_3+\theta_4)} \{\min(g_{34}, \theta_3) - \max(0, g_{34} - \theta_4)\} \; dg_{34} \right] \end{aligned}$$

Computing the total project cost

Let us assume that, following the ideas introduced in Palomo et al. (2004b), the company has estimated the project cost, under normal circumstances, through (in thousands of euros)

$$\pi(c) \sim N(900, 100).$$

This estimation does not take into account the possible disruptive events that could arise during the development of the project, increasing its final cost. Given that estimation, we now face the decision of whether or not to outsource the construction and pay $970,000$ euros (lowest bid) for it.

We assume that the total gravity observed on each of the past projects is the sum of the gravities arisen, and that it is virtually impossible that more than three events occur simultaneously. Then, using the posteriors obtained as introduced above, the joint gravity is modelled through the mixture

$$g = \sum_{i=1}^{4} q_i^* \cdot g_i + \sum_{i \neq j} q_{ij}^* \cdot g_{ij} + \sum_{i \neq r \neq j} q_{ijr}^* \cdot g_{ijr}$$

The expected project total cost, accounting for external risks, will be

$$E[c_T] = \iint (c+g)\pi(c)\pi(g)\ dc\ dg.$$

As we have to proceed by simulation to compute it, we use the following scheme, for the 'setting marginals and deterministic constraints' approach, $i = 0, 1, \ldots, 4$,

$k = 0$

`Loop` N `times`

  `Loop`

    `Sample` q_i^k `from` $\pi(q_i|\mathbf{D})$

    `Sample` q_{ij}^k `from` $\pi(q_{ij}|\mathbf{D})$ $\quad i \neq j$

    `Sample` q_{ijr}^k `from` $\pi(q_{ijr}|\mathbf{D})$ $\quad i \neq j \neq r$

  `Until` $\{q_i^k, q_{ij}^k, q_{ijr}^k\} \in \mathbf{B} = \{\max(q_1, \ldots, q_4) \leq 1 - q_0 \leq \sum_{i=1}^4 q_i\}$

  `Compute` $\{q_i^*, q_{ij}^*, q_{ijr}^*\}$ `from` $\{q_i^k, q_{ij}^k, q_{ijr}^k\}$

  `Sample` $l \sim \begin{pmatrix} q_1^* & \cdots & q_4^* & \cdots & q_{ij}^* & \cdots & q_{ijr}^* & q_0^* \\ \{1\} & \cdots & \{4\} & \cdots & \{ij\} & \cdots & \{ijr\} & \{0\} \end{pmatrix}$

  `Sample` $\{g_i^k\}$ `from` $\pi(g_i|\mathbf{D})$, $i \in l$

Compute $g^k = \sum_{i \in l} g_i^k$
Sample c^k from $\pi(c)$
$k = k + 1$
Output $\frac{1}{N} \sum_{k=1}^{N} (c^k + g^k)$

In Figure 3, we present the histogram for the total project cost in our case

Figure 3: Histogram of the total cost for the construction project.

study. In order to make the final decision, we model the decision maker's preferences through

$$u(c) = \frac{1 - \exp\left(-\frac{1500-c}{\psi}\right)}{1 - \exp\left(-\frac{1000}{\psi}\right)}$$

where ψ is his risk tolerance. In Figure 4 we show six utility curves representing different risk attitudes of the decision maker, going from risk neutral ($\psi = 150$) to risk aversion ($\psi = 10^{10}$). Also, for each risk tolerance level, the expected utility for the project cost, project total cost estimates, and the best bid are represented with blue dots, red dots and crosses respectively. Notice that, as expected, the more risk averse the decision maker is, the clearer it becomes that the best alternative is to

Figure 4: Expected utility for the project cost estimations (blue dots), project total cost (red dots) and the best bid (black crosses). Risk tolerances $\psi = \{150, 500, 750, 1050, 1500, 10^{10}\}$, from the bottom to the top curves respectively.

outsource the construction. In Figure 4, the red dots are always beneath the crosses. Since, independently of decision maker's risk aversion, the expected utility of paying $970,000$ euros for the project is greater than building its own plant, we conclude that it is better to subcontract the construction. Furthermore, to reinforce such statement we have computed the probability that the project total cost is greater than the best bid received based on the following algorithm

$prob = 0$

`Loop` N `times`

`Sample` $l \sim \begin{pmatrix} q_1^* & \cdots & q_k^* & \cdots & q_{ij}^* & \cdots & q_{ijr}^* & \cdots & q_{1\ldots k}^* & q_0^* \\ \{1\} & \cdots & \{k\} & \cdots & \{ij\} & \cdots & \{ijr\} & \cdots & \{1\ldots k\} & \{0\} \end{pmatrix}$

`Sample` $\{g_i^k\}$ `from` $\pi(g_i|\mathbf{D})$`,` $i \in l$

`Compute` $g^k = \sum_{i \in l} g_i^k$

```
    Sample $c^k$ from $\pi(c)$
      if $g^k + c^k \geq \bar{g}$ then $\{prob = prob + 1\}$
    k=k+1
  $Pr(c_T \geq \bar{g}) \simeq \frac{prob}{N}$,
```

obtaining that it is approximately 0.48.

6 Discussion

Traditional project costing techniques have proven inadequate for taking into account all uncertainties involved in large projects. This has had a negative impact on decisions related to project implementation as shown in many projects who ended up costing much more than what was expected, see CITA. In this paper, we have provided a framework for forecasting project costs when disruptive events arise. This entails three steps: forecasting the probability of occurrence of such events, forecasting their gravities (or increases in project cost, duration, ...), and combining these forecasts for later use in various engineering decisions. Some care must be taken since more than one event might happen simultaneously, and their effects could overlap. We have considered cases in which effect aggregation is either through addition or maximum.

The methodology we have sketched may be implemented to the advantage of the decision maker. Once he has obtained an appropriate project performance forecast, if interested, the company would be ready to participate in a bidding process to get future contracts, as illustrated in Palomo et al. (2004a). We have focused on costs, but our models apply equally to project duration or performance forecasting.

Extensions to the cases were several events occur, with some of the gravities being additive, whereas others are not, are also possible, but more difficult to be dealt within a general fashion. For example, if events E_1, a delay in the arrival of raw materials, E_2, a labor accident, and E_3, a strong rainfall, occur, we could find $g_{12} = g_1 + g_2$ and $g_{13} = \max(g_1, g_3)$, or even more complicated $g_{123} = \max(g_1, g_2) + g_3$.

When events are dependent, we have proposed a multivariate normal copula. However, several other copula families are available, such as t-Student copula or Archimedean copula,... which could better capture the joint behavior of the event probabilities. See Nelsen (1998) for suggestions in the computations. Also, we have used the Dirichlet distribution to model event probabilities, whereas the generalized Dirichlet distribution has a more general covariance structure to model the dependences among events, see Wong (1998) for more details about this distribution. These are topics for further elaboration.

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Appendix

A Metropolis-Hastings algorithm

This algorithm allows to generate, for a density $\pi(\theta)$, known up to a normalizing constant, and a proposal density $\Psi(\theta)$, a chain that, under conditions of irreducibility and aperiodicity, converges to $\pi(\theta)$. See French and Rios Insua (2000), Robert (2001) or Robert and Casella (1999) for further details. A general M-H algorithm, with initial value θ^0, is as follows

Given $\theta^{(t)}$

Generate $\theta_t \sim \Psi(\theta)$

$$\texttt{Take } \theta^{(t+1)} = \begin{cases} \theta_t \texttt{ with probability } \rho\left(\theta^{(t)}, \theta_t\right), \\ \theta^{(t)} \texttt{ with probability } 1-\rho\left(\theta^{(t)}, \theta_t\right) \end{cases}$$

$t = t + 1$

with $\rho\left(\theta^{(t)}, \theta_t\right) = \min\left\{\frac{\pi(\theta_t)\Psi(\theta^{(t)})}{\pi(\theta^{(t)})\Psi(\theta_t)}, 1\right\}$

We describe now the Metropolis-Hastings algorithms proposed for the models introduced in the paper.

A.1 For event probabilities: copula case

We use it in Section 2.2. The proposal densities $\Psi(q_i)$ we use are $Beta(\alpha_i{}' + m_i, \beta_i' + n - m_i)$, $i = 1, \ldots, k$. The acceptance probability is simplified to

$$\rho\Big((q_1^{(t)}, \ldots, q_k^{(t)}), (q_{1t}, \ldots, q_{kt})\Big) = \min\left\{\exp\left\{-\frac{1}{2}\left[\Big(y_t + y^{(t)}\Big)' \cdot (V^{-1} - I) \cdot \Big(y_t - y^{(t)}\Big)\right]\right\}, 1\right\}$$

where

$$y_t = \big(\Phi^{-1}[\Pi_1(q_{1t})], \ldots, \Phi^{-1}[\Pi_k(q_{kt})]\big)'$$
$$y^{(t)} = \Big(\Phi^{-1}[\Pi_1(q_1^{(t)})], \ldots, \Phi^{-1}[\Pi_k(q_k^{(t)})]\Big)'$$

with $\Phi(\cdot)$, V and $\Pi(\cdot)$ is defined in Section 2.2.

A.2 For gravities: Cases when only $n_{ij} > 0$

We use it in Sections 3 and 4. Our proposal generating density is

$$\Psi(\theta_i, \theta_j) = \Psi(\theta_i)\Psi(\theta_j) = Gamma(\mu_i - n_{ij}, \phi_i) \cdot Gamma(\mu_j - n_{ij}, \phi_j)$$

with $\mu_i - n_{ij} > 0$, $\mu_j - n_{ij} > 0$.

A.3 For gravities: Cases when $n_i, n_j, n_{ij}, n_0 > 0$

We use it in Sections 3 and 4. Our proposal generating densities $\Psi(\theta_i, \theta_j) = \Psi(\theta_i)\Psi(\theta_j|\theta_i)$ is, assuming $\mu_i - n_i - n_{ij} > 0$ and $\mu_j - n_j - n_{ij} > 0$:

- In the additive gravities case,

$$\Psi(\theta_i) \sim Gamma(\mu_i - n_i - n_{ij}, \phi_i) I_{[\max_v(g_i^v), \infty)}(\theta_i)$$
$$\Psi(\theta_j|\theta_i) \sim Gamma(\mu_j - n_j - n_{ij}, \phi_j) I_{[\max_l(g_{ij}^l) - \theta_i, \infty)}(\theta_j)$$

- In the maximum gravity case,

$$\Psi(\theta_i) \sim Gamma(\mu_i - n_i - n_{ij}, \phi_i) I_{[\max_v(g_i^v), \infty)}(\theta_i)$$
$$\Psi(\theta_j|\theta_i) \sim Gamma(\mu_j - n_j - n_{ij}, \phi_j) I_{[\max_l(g_{ij}^l), \infty)}(\max(\theta_i, \theta_j))$$

To sample from $\Psi(\theta_i, \theta_j)$, for example, in the additive gravities case, we may use the following algorithm

```
Generate θ_i^t ~ Gamma(μ_i − n_i − n_ij, φ_i)
If θ_i^t ≥ max_v(g_i^v)
    Generate θ_j^t ~ Gamma(μ_j − n_j − n_ij, φ_j)
    If θ_j^t ≥ max_l(g_ij^l) − θ_i^t
        Output {θ_i^t, θ_j^t}
```

A.4 For gravities: Cases when only $n_{ijr} > 0$

We use it in Sections 3 and 4. Our proposal generating density is

$$\Psi(\theta_i, \theta_j, \theta_r) = \Psi(\theta_i)\Psi(\theta_j)\Psi(\theta_r) = Gamma(\alpha_i, \phi_i) \cdot Gamma(\alpha_j, \phi_j) \cdot Gamma(\alpha_r, \phi_r)$$

where $\alpha_s = \mu_s - n_{ijr} > 0$, $s = \{i, j, r\}$.