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# Algebraic Equations for Blocking Probabilities in Asymmetric Networks

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## §1 Introduction

We are interested in multiple solutions to polynomial equations for blocking probabilities in circuit-switched networks where dynamic routing is used and where asymmetry leads to several variables. We establish a region of two solutions for networks where one edge has a capacity parameter $c$ and the others have capacity parameter $d$. The method uses traceforms to count solutions to a polynomial system in three indeterminates and two parameters $c, d$ in the coefficient field. The region is presented explicitly for 3, 4, and 5 nodes, and it is shown that the region for networks with nodes greater than 5 is qualitatively similar. We show that the specified region has exactly two solutions.

The symmetric case, where all links on a complete graph have the same capacity and traffic, has been treated in the literature by Krupp (1982), Kelly (1986), Gibbens, Hunt, Kelly (1990) and others. A fixed point equation in one variable $b$ is equivalent to a cubic polynomial in one variable $b$ for positive solutions, and its solutions are a limit in a rigorous sense of the blocking probability for a large symmetric network. The asymmetric case has been studied little and primarily by simulation. Krupp observed that with four or more nodes, the "algebraic manipulations in an analytic treatment become quite difficult."

For the most general asymmetry on a complete $n$ node graph, where each of the $\binom{n}{2}$ edges $e$ has a different given numerical parameter value $c_e$ and blocking probability solution $b_e$, numerical solutions are not hard for up to five nodes. We have used the software PHC of Jan Verschelde for this calculation. Computations with more than five nodes can take a long time.

The problem with larger networks is that the number of complex solutions is in the thousands, whereas the number of real solutions with each component in the interval $(0, 1)$ is small, apparently either 0, 1, or 2. Furthermore, the numerical approach is not good at resolving the inverse problem of finding parameter values that give certain properties of the solutions, such as multiple probability solutions. For these reasons, an algebraic approach with parameters in a coefficient field gives the most insight. The regions we have found for multiple roots have boundaries given by the 0-contours of coefficients (which are explicit polynomials in the two parameters $c, d$) of the characteristic polynomials of certain traceform matrices. Since the polynomials in the parameters that define the boundaries have high degree, it is useful at the end to use numerical software to understand the boundaries, in particular to find numerically the points (parameter values) where two bounding polynomials intersect and make a corner on the region.

Dynamic routing means that traffic may randomly try a second route if the primary route is blocked. Our networks have complete graphs, with a capacity parameter on each edge, and the primary route is the direct, single edge between two nodes. "Route is blocked" means that an attempted communication on the route finds that some part of the route is already at maximum capacity. The blocking probability $b_{ij}$ on link $\{i, j\}$, which together are the quantities of primary interest here, represents the proportion of time the link is at full capacity. The probability that an attempted transmission over the direct route $\{i, j\}$ will be stopped and rerouted is $b_{ij}$. There is some underlying use of the principle that "arrivals see time averages" in this interpretation, so a rigorous derivation of the equations we propose will have to make some assumptions on the precise nature of the communication process in terms of arrivals and holding times.

In general, the fraction of attempted traffic that reaches its destination is a complicated function of the blocking probabilities and attempted traffic rates. Traffic between each ordered pair of nodes $(i, j)$ will reach its destination if it is not blocked on its first direct attempt, or if it is blocked on its first attempt then both of the subsequent links are not blocked. If heuristic independence assumptions are made, expressions can be written down for the total loss probability. We give the expression for the symmetric network in §2.

Multiple roots are curious mathematically and possibly useful for applications. One of the roots gives total loss probability lower than when there is no rerouting, while the other root gives higher total loss probability, and the actual performance over the long term in an ergodic network is a weighted average of the two. Simulations of Gibbens, Hunt, and Kelly (1990) show that a system starting empty moves first towards the solution with low blocking before attaining equilibrium, when the two solutions correspond to modes of a stationary distribution. This might imply that a sensor network with changing traffic parameters coming up from zero could benefit by having a configuration with two solutions, one of which would give temporary high efficiency. Also, the fundamental method of solving fixed point equations that involve the Erlang blocking formula is used in other applications, such as optical burst switching (Rosberg, Vu *et al.* (2003)). Therefore, algebraic methods that work in the setting of circuit-switched networks should have wider applicability.

**§2 Notation and Equations**

Define $E(r, C)$ to be the Erlang blocking formula for a queue with $C$ servers, no waiting line, and an incoming Poisson process of rate $r$. $E(r, C)$ is the steady state probability that the arrival process sees all $C$ servers busy, and the arrival is then blocked from entering. It is the proportion of time all the $C$ servers are busy. The formula for the steady-state distribution leads to the formula

$$E(r, C) := \frac{r^C/C!}{\sum_{n=0}^{C} r^n/n!} = \frac{1}{\int_0^\infty (y/r+1)^C e^{-y} dy} \tag{1}$$

for the blocking probability. In our application, this is the exact formula for the blocking probability $b_{1,2}$ in a graph with two nodes $1, 2$ and one edge $\{1, 2\}$ with $C$ units of capacity and attempted traffic rates of $r/2$ in each direction.

If $r, C$ are large, there is a limiting approximation that we will use:

$$E(rN, CN) \to \max\{0, 1 - C/r\}$$

as $N \to \infty$. This approximation is described in §5 of Kelly (1986) and other places.

The links will be considered undirected edges in a graph, so traffic on link $\{i, j\}$ may go from $i$ to $j$ or from $j$ to $i$, at originating rates $\nu_{(i,j)}$ and $\nu_{(j,i)}$, and both directions take up capacity. The quantities of interest are blocking probabilities $b_{i,j}$ one for each edge, which together will be denoted $\mathbf{b} = (b_{i,j})$.

In the approximation, the rate $r$ for link $\{1, 2\}$ is the link traffic rate $\lambda_{1,2}$ scaled up by $1 - b_{1,2}$ to get the "attempted" rate used in $E(r, C)$ rather than the "through" rate $\lambda_{1,2}$. Then on each link $\{i,j\}$ we get the equation:

$$b_{i,j} = E(\frac{\lambda_{i,j}}{1-b_{i,j}}, C_{i,j})$$

where $\lambda_{i,j} = \lambda_{i,j}(\mathbf{b})$ and $C_{i,j}$ are fixed parameters.

Solving this system is usually impossible because of the high degree polynomials, but the limit approximation simplifies it further. Using the limit approximation for large $C_{i,j}$ and large $\frac{\lambda_{i,j}}{1-b_{i,j}}$ we try to solve

$$b_{i,j} = \max\{0, 1 - \frac{C_{i,j}}{\lambda_{i,j}/(1-b_{i,j})}\}$$

for variables $b_{i,j}$ on each undirected edge, given parameters $C_{i,j}$ and attempted traffic rates $\nu_{(i,j)}$. This equation is discussed in Kelly (1991) for general loss networks. If there is only one solution $\mathbf{b}$, then iteration of the fixed point equation will find it. When there are two solutions, typically only one of them is stable so more delicate methods are needed to find them both.

Then positive numbers $b_{i,j}$, will solve the system:

$$\lambda_{i,j} = C_{i,j}. \tag{2}$$

where $\lambda_{i,j} = \lambda_{i,j}(\mathbf{b})$ depends on several of the indeterminates $b_{i,j}$ that we seek. This will be a polynomial equation.

The expression for each $\lambda_{i,j}$ is derived from naive probability assuming independence of links. This derivation has been justified rigorously (see Kelly 1995) for the symmetric network under Poisson assumptions on traffic, but not for the asymmetric cases we are studying.

To illustrate, consider the symmetric triangle where the ratios $C_{1,2}/\nu, C_{2,3}/\nu, C_{1,3}/\nu$ are all set equal to a single parameter $c > 0$. All blocking probabilities on the three edges are assumed equal to the same value $b$. With attempted one-way traffic rates of $\nu$, it follows that the actual traffic on each edge $\{i,j\}$ is $\lambda_{i,j} = 2\nu(1-b+2b(1-b)^2)$ because edge traffic can be direct, with rate $\nu(1-b)$, or indirect from blocked attempts on the other two edges and successful rerouting.

Then $b$ satisfies the fixed point equation

$$b = \max\{0, 1 - \frac{c/2}{(1-b+2b(1-b)^2)/(1-b)}\}. \tag{3}$$

There can be one or two solutions in $[0,1]$ to the fixed point equation, depending on the value of $c$. Let $u_s = \mathtt{max}_{[0,1]} 1 - x + 2x(1-x)^2 \leq 1.0671$, and the maximum occurs at $(4-\sqrt{10})/6 \approx .14$. The value of $u_s$ defines the interval $(2, 2\,u_s)$ for the values of $c$ on which there are two positive solutions. For values of $c < 2$, one of the solutions turns negative, and for values of $c > 2\,u_s$, the two real solutions to the polynomial system turn complex and leave only the vanishing solution to the fixed point equation.

With two solutions, the large solution is a stable fixed point and can be found by iterating the fixed point equation. The small solution is not stable and cannot be found this way. With attempted one-way traffic rate $\nu$, the fraction of attempted calls that are routed on the first attempt is $6\nu \times (1-b)/6\nu$, and the fraction that are blocked on the first attempt and successfully rerouted is $6\nu \times b(1-b)^2/6\nu$. The total loss probability is then $1-(1-b)-b(1-b)^2 = b - b(1-b)^2$.

When $c$ is near 2, the smaller solution for $b$ very close to 0 corresponds to a situation where most (proportion $1-b$) of the calls are on the first routing attempt and very few (proportion $b(1-b)^2$) are reroutes. The larger root corresponds to a situation where about fifteen percent ($b(1-b)^2$) of the calls are reroutes.

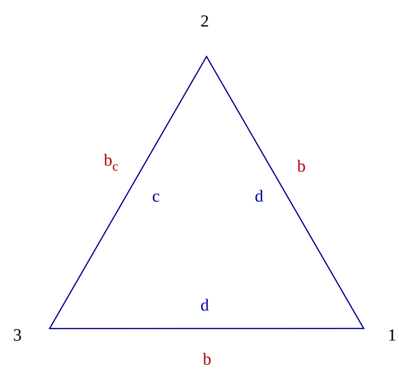
## §3 Asymmetric Graphs

Networks with different capacities on different links and different traffic rates between node pairs $(i,j)$ become very complicated. We will focus on the situation where attempted traffic rates $\nu$ are equal for all node pairs, and edge $\{n-1, n\}$ has parameter $c = C_{n-1,n}/\nu$ while the other edges have parameter $d = C_{i,j}/\nu$. The remaining symmetry results in two blocking probabilities $b_c, b$ for the triangle, where $b_c$ is the blocking probability on the edge with capacity $c$. For four vertices and higher, there are three blocking probabilities $b_c, b_a, b$ where $b_a$ is the blocking probability on the $2(n-2)$ edges (with parameter $d$) connected to edge $\{n-1, n\}$ (with parameter $c$), and $b$ is the blocking probability on the $\binom{n-2}{2}$ edges (of capacity $d$) that are not connected to the special edge.

We seek solutions with all coordinates in the interval $(0,1)$, which can be interpreted as probabilities, and which lead to polynomial systems (2). Computations are done in the ring $Q[b_c, b]$ for the triangle and $Q[b_c, b_a, b]$ for larger graphs, where the parameters $c, d$ are in the coefficient field. We have used the software Singular for most algebraic computations to be able to treat $c, d$ as parameters rather than indeterminates.

Let $h(x) = x(1-x)$ for a real variable $x$. Then $h(x) > 0$ if and only if $x \in (0,1)$. This is a function that can test for values $x$ that may be interpreted as probabilities. Let $S_{h(b_c)}$ be the traceform matrix for $h$ applied to $b_c$, and similarly let $S_{h(b)}$ be the traceform matrix for $h$ applied to $b$. The method for finding where there are two possible blocking probabilities involves finding explicitly the characteristic polynomials of the above traceforms, with coefficients in the form of rational functions in $(c, d)$, and then counting roots using sign changes in the coefficients. It is clear that 0-contours of these coefficients will define boundaries where sign changes in the coefficients take place, and hence will define boundaries where root counts change. The interesting property of these systems is that the boundaries for the two-solution region come from the constant terms of $S_{h(b)}$ and $S_{h(b_c)}$ only. They do not require other coefficients or other traceforms such as $S_{h^2(b)}$, $S_{h^2(b_c)}$.

**Example 3.1. Triangle**




With one-way rates $\nu_{(i,j)} = \nu$, and $d = C_{1,2}/\nu, d = C_{1,3}/\nu, c = C_{2,3}/\nu$ and blocking probabilities $b_c = b_{2,3}, b = b_{1,3} = b_{1,2}$, the fixed point system in variables $b, b_c$ is

$$\begin{pmatrix} b_c \\ b \end{pmatrix} = T_3 \begin{pmatrix} b_c \\ b \end{pmatrix} := \begin{pmatrix} \max\{0, 1 - \frac{c/2}{1+2b(1-b)}\} \\ \max\{0, 1 - \frac{d/2}{1+b(1-b_c)+b_c(1-b)}\} \end{pmatrix}$$

Numerically, the probability solutions to the system with capacities $(c_{1,2}, c_{2,3}, c_{1,3}) = (2, 2, 2)$ are $\mathbf{b} = (.29, .29, .29)$ and $\mathbf{b} = (0, 0, 0)$. With $(c_{1,2}, c_{2,3}, c_{1,3}) = (2, 2, 2.45)$ there is only the one vanishing solution to the fixed point equation.

The interval for $c$ (when $d = 2$) that gives two positive solutions is $(2, u)$ where the upper limit $u \approx 2.430531976$ is the middle real root to the polynomial $29040 - 35232\, c + 15200\, c^2 - 2880\, c^3 + 253\, c^4 - 8\, c^5$. As $c$ goes above $u$, the two real solutions converge (giving multiplicity 2 at precisely $u$) and turn complex, leaving no probability solutions to the polynomial system and one for the fixed point equation. As $c$ goes below 2, one of the solutions becomes negative and we are left with one positive probability solution.

For $c$ in the interval $(2, u)$, only one of the two solutions is a stable fixed point of $\mathbf{b} = T_3(\mathbf{b})$ under iteration.

For positive solutions, the corresponding polynomial system is written:

$$c/2 = (1-b_c) + 2b(1-b_c)(1-b) \quad (4)$$

$$d/2 = (1-b) + b(1-b)(1-b_c) + b_c(1-b)^2 \quad (5)$$

With $c, d$ in the coefficient field and $b_c > b$ in lexicographic term order, there are four standard monomials $1, b, b^2, b_c$, so there are four complex solutions. Let $V_{c,d} \subset \mathbf{R}^2$ be the set of real solutions, and define the set of parameter values

$$P = \{(c,d) \in \mathbf{R}^2 : |\{(b, b_c) \in (0,1)^2 \cap V_{c,d}\}| = 2\}.$$

The following Lemma is useful for identifying the region $P$ using trace-forms.

**Lemma 3.1.** Let $h(x) = x(1-x)$ for $x \in \mathbf{R}$, and define

$$\begin{aligned} Q_1 &:= \{(c,d) \in \mathbf{R}^2 : |\{(b,b_c) \in V_{c,d} : h(b) > 0\}| = 2\} \\ Q_2 &:= \{(c,d) \in \mathbf{R}^2 : |\{(b,b_c) \in V_{c,d} : h(b_c) > 0\}| = 2\} \\ Q_0 &:= \{(c,d) \in \mathbf{R}^2 : |\{(b,b_c) \in V_{c,d} : -h(b)h(b_c) > 0\}| = 0\}. \end{aligned}$$

Then $Q_1 \cap Q_2 \cap Q_0 \subset P$.

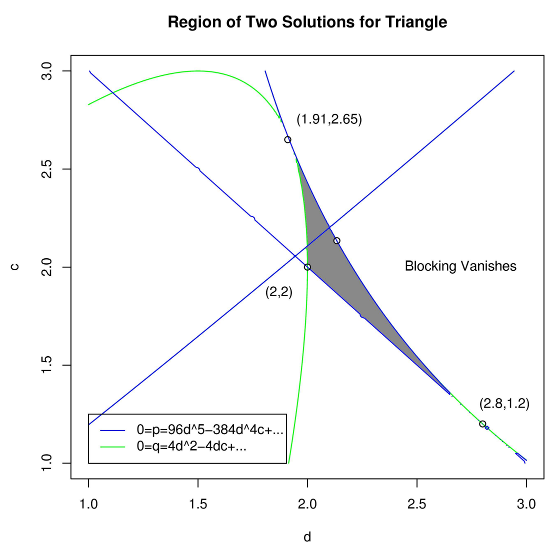
**Proof.** Let $(c,d) \in Q_1 \cap Q_2 \cap Q_0$. Using the property $(c,d) \in Q_1$, let the two elements with first coordinate in $(0,1)$ be denoted $(b_1, b_{c,1})$, and $(b_2, b_{c,2})$. Then $|\{(b,b_c) \in (0,1)^2 \cap V_{c,d}\}| \leq 2$. If we can show that both $b_{c,1}$ and $b_{c,2}$ are in $(0,1)$ using $Q_2 \cap Q_0$, then the two points must be in $P$ and we will be done.

A list of all real solutions $V_{c,d}$ shows exactly two with first coordinate in the interval $(0,1)$ ($h(b) > 0$) and exactly two with second coordinate in $(0,1)$ ($h(b_c) > 0$). That these are the same two solutions comes from the fact that $(c,d) \in Q_0$ makes impossible any real solution $(b, b_c)$ where $h(b) > 0$ and simultaneously $h(b_c) < 0$. Thus the solutions with $h(b) > 0$ and $h(b_c) > 0$ must the same ones. ∎

**Proposition 3.1.** For the system (4) − (5) in variables $b_c, b$, there is an open, bounded region in $\mathbf{R}^2$ of parameter values $(c,d)$ on which the system has exactly two positive solutions $(b_c, b) \in (0,1)^2$, and which contains the line segment $c = d \in (2, 2u_s)$.

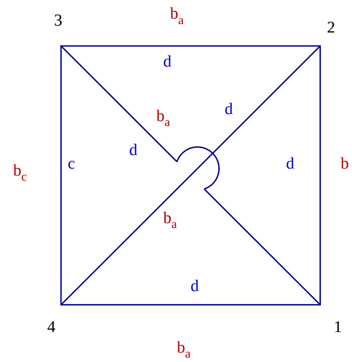
**Remark.** The boundaries of the region are the 0-contours of the constant terms of the characteristic polynomials of the two traceforms $S_{h(b_c)}$ and $S_{h(b)}$.

**Proof.** With $c, d$ in the coefficient field and $b_c > b$ in lexicographic term order, there are four standard monomials: $1, b, b^2, b_c$. Then $S_{h(b_c)}$, $S_{h(b)}$, $S_{h^2(b)}$, and $S_{-h \otimes h}$ are $4 \times 4$ traceform matrices for $h(x) := x(1-x)$ applied to $b_c$ and $b$ respectively. The sign changes on the characteristic polynomials of the four traceforms can be used to count positive eigenvalues and ultimately signatures. The 0-contours of the coefficients of the characteristic polynomials will define regions with a certain number of roots. In particular, the region $Q_1 \cap Q_2$ in the figure below is bounded by the 0-contours of the constant terms in the characteristic polynomials of $S_{h(b_c)}$, $S_{h(b)}$, the other boundaries being redundant. A further calculation shows that all parameters in this region are contained in $Q_0$. By Lemma 3.1, the illustrated region is a set of parameters where there are exactly two solutions $(b, b_c) \in (0,1)^2$ to the system $(4), (5)$. ∎



**Example 3.2. K4**

The setting is a complete graph on 4 vertices, with capacity parameter $c = C_{3,4}/\nu$ on one edge, capacity parameter $d$ on the others.




By symmetry, there are three blocking probabilities: $b_c$ on the one special edge, $b_a$ on the 4 connected edges, and $b$ on the remaining edge that does not connect to the special edge.

Consider the equations on an edge of each type:

$$b_c = \max\{0, 1 - \frac{c/2}{1 + 2b_a(1-b_a)}\} \tag{6}$$

$$b_a = \max\{0, 1 - \frac{d/2}{1 + \frac{b_a(1-b_c)}{2} + \frac{b(1-b_a)}{2} + \frac{b_c(1-b_a)}{2} + \frac{b_a(1-b)}{2}}\} \tag{7}$$

$$b = \max\{0, 1 - \frac{d/2}{1 + 4\frac{b_a(1-b_a)}{2}}\} \tag{8}$$

which can be written as a fixed point equation of a continuous map $T_4$ on the cube $[0,1]^3$. A positive fixed point will satisfy the system

$$c/2 = (1-b_c)(1+2b_a(1-b_a)) \tag{9}$$

$$d/2 = (1-b_a)(1 + b_a(1-b_c)/2 + b(1-b_a)/2 + b_c(1-b_a)/2 + b_a(1-b)/2) \tag{10}$$

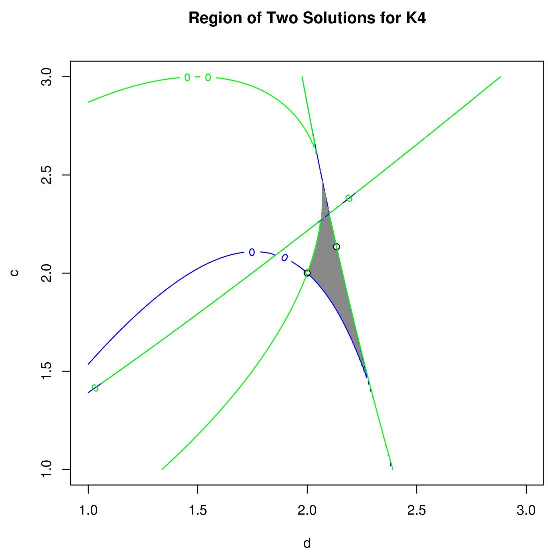
$$d/2 = (1-b)(1+4b_a(1-b_a)/2). \tag{11}$$

**Proposition 3.2.** For the system $(9)-(11)$ in variables $b_c, b_a, b$, there is an open, bounded region in $\mathbf{R}^2$ of parameter values $(c,d)$ on which the system $(6)-(8)$ has exactly two positive solutions $(b_c, b_a, b) \in (0,1)^3$, and which contains the line segment $c = d \in (2, 2u_s)$.

**Remark.** The boundaries of the region are the 0-contours of the constant terms of the characteristic polynomials of the two traceforms $S_{h(b_c)}$ and $S_{h(b)}$.

**Proof.** With $c, d$ in the coefficient field, the system has four standard monomials. Then $S_{h(b_c)}$, $S_{h(b)}$, $S_{h^2(b)}$, and $S_{-h(b)h(b_c)}$ are $4 \times 4$ traceform matrices for $h(x) := x(1-x)$ applied to $b_c$ and $b$ respectively. The sign changes on the characteristic polynomials and a version of Lemma 3.1 applied to variables $(b_c, b)$ yield the region bounded by the 0-contours of the constant terms in the characteristic polynomials of $S_{h(b_c)}$, $S_{h(b)}$, on which there are precisely two real solutions $(b_c, b_a, b)$ that satisfy $(b_c, b_a, b) \in (0,1) \times \mathbf{R} \times (0,1)$. It remains to show that the coordinate $b_a \in (0,1)$ as well.

The equation (8) and $b > 0$ leads to $1 + 2b_a(1 - b_a) > d/2$. Since $d > 2$ on the region, $b_a$ satisfies $b_a(1 - b_a) > 0$, forcing $b_a \in (0,1)$ by the nature of the quadratic in $b_a$. ∎




**Example 3.3. K5 and Kn**

The setting is a complete graph on $n$ vertices, with capacity $c$ on one edge, capacity $d$ on the others.

By symmetry, there are three blocking probabilities: $b_c$ on the one special edge, $b_a$ on the $2n-4$ connected edges, and $b$ on the remaining edges that do not connect to the special edge.

Consider the equations on an edge of each type:

$$b_c = \max\{0, 1 - \frac{c/2}{1 + 2b_a(1 - b_a)}\} \tag{12}$$

$$b_a = \max\{0, 1 - \frac{d/2}{1 + \frac{b_a(1-b_c)}{(n-2)} + (n-3)\frac{b(1-b_a)}{(n-2)} + \frac{b_c(1-b_a)}{(n-2)} + (n-3)\frac{b_a(1-b)}{(n-2)}}\} \tag{13}$$

$$b = \max\{0, 1 - \frac{d/2}{1 + 4\frac{b_a(1-b_a)}{(n-2)} + 2(n-4)\frac{b(1-b)}{(n-2)}}\} \tag{14}$$

which can be written as a fixed point equation of a continuous map $T_n$ on the cube $[0,1]^3$:

$$\begin{pmatrix} b_c \\ b_a \\ b \end{pmatrix} = T_n(\begin{pmatrix} b_c \\ b_a \\ b \end{pmatrix})$$

so there will always be at least one fixed point by the Brouwer fixed point theorem, possibly identically zero.

A positive probability solution to the above will satisfy the polynomial system ($n \geq 5$):

$$c/2 = 1 - b_c + 2b_a(1-b_a)(1-b_c) \tag{15}$$

$$d/2 = 1 - b_a + \frac{b_a(1-b_c)(1-b_a)}{(n-2)} + (n-3)\frac{b(1-b_a)^2}{(n-2)} + \frac{b_c(1-b_a)^2}{(n-2)} + (n-3)\frac{b_a(1-b)(1-b_a)}{(n-2)} \tag{16}$$

$$d/2 = 1 - b + 4\frac{b_a(1-b_a)(1-b)}{(n-2)} + 2(n-4)\frac{b(1-b)^2}{(n-2)} \tag{17}$$

For all $n \geq 5$, this system has the same 14 standard monomials and so computations for $n = 5$ are representative of all large graphs. The detailed proof of the result below was done with $n = 5$. We need the following variation on Lemma 3.1. Define $R := \{(c,d) \in \mathbf{R}^2 : |\{(b, b_a, b_c) \in (0,1) \times \mathbf{R} \times (0,1) \cap V_{c,d}\}| = 2\}$, the set of parameter values which give exactly two real solutions to (15) $-$ (17) with coordinates $b$ and $b_c$ in $(0,1)$. The proof is almost the same as the proof of Lemma 3.1.

**Lemma 3.2.** Let $h(x) = x(1-x)$ for $x \in \mathbf{R}$, and define

$$\begin{aligned} R_2 &:= \{(c,d) \in \mathbf{R}^2 : |\{(b, b_a, b_c) \in V_{c,d} : h(b) > 0\}| = 2\} \\ R_3 &:= \{(c,d) \in \mathbf{R}^2 : |\{(b, b_a, b_c) \in V_{c,d} : h(b_c) > 0\}| = 3\} \\ R_1 &:= \{(c,d) \in \mathbf{R}^2 : |\{(b, b_a, b_c) \in V_{c,d} : -h(b)h(b_c) > 0\}| = 1\}. \end{aligned}$$

Then $R_2 \cap R_3 \cap R_1 \subset R$.

**Proposition 3.3.** For the system (15) $-$ (17) with $n = 5$ there is an open, bounded region in $\mathbf{R}^2$ of parameter values $(c,d)$ on which there are two positive solutions $(b_c, b_a, b) \in (0,1)^3$, and which contains the line segment $c = d \in (2, 2u_s)$.

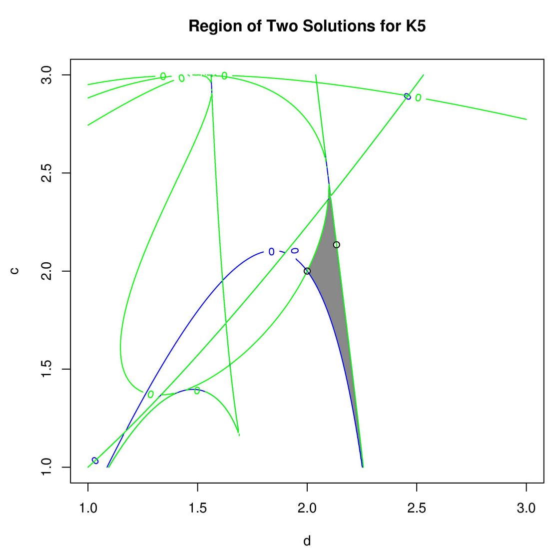
**Remark.** The boundaries of the region are the 0-contours of the constant terms of the characteristic polynomials of the two traceforms $S_{h(b_c)}$ and $S_{h(b)}$.

**Proof.** Using traceforms and the variables $b_c, b$, one can find the region bounded by the 0-contours of the constant terms of the characteristic polynomials of $S_{h(b)}, S_{h(b_c)}$ and show that parameter values $(c, d)$ in this region yield six real solutions. Two of the real solutions have $b \in (0,1)$, three have $b_c \in (0,1)$, and using $S_{-h(b)h(b_c)}$ one finds that exactly one solution satisfies $-h(b)h(b_c) > 0$, meaning $b_c \notin (0,1)$ and $b \in (0,1)$. By Lemma 3.2, there are exactly two solutions $(b_c, b_a, b)$ in $(0,1) \times \mathbf{R} \times (0,1)$. Again, as with the graph K4, one must show that $b_a \in (0,1)$ as well.

From (14), it follows that $b \leq 1 - \frac{d/2}{1+1/3+2/3\times(1/4)}$ by maximizing the denominator with $b = 1/2$. Hence $b \leq 1 - d/3$ and on the region specified $d > 2$ so $b \leq 1 - 2/3 = 1/3$.

Also, knowing that $b_c \in (0,1)$, equation (12) implies that $1+2b_a(1-b_a) = (c/2)/(1-b_c) > 0$, so $b_a(1-b_a) > -1/2$ and $b_a < 1/2 + \sqrt{3}/2$.

A lex Gröbner basis for (15) − (17) with order $c > d > b_c > b > b_a$ includes the equation $f(b_a) := (2b_c + 1)b_a^2 + (-3b_c - 2b - 4)b_a + (b_c - 2b^3 + 4b^2 + 3b)$, which must be satisfied by the two candidates for $b_a$ at the possible values of $(b_c, b) \in (0,1)^2$. Now $f(0) = b_c - 2b^3 + 4b^2 + 3b > 0$ and $f(1) = 1 - 2b - 4 - 2b^3 + 4b^2 + 3b < 0$ over all $b \in (0,1)$. Thus there is exactly one root for $b_a \in (0,1)$, by the intermediate value theorem and since the equation is quadratic. Furthermore, $f(1/2 + \sqrt{3}/2) < 0$ over all $b \in (0, 1/3]$, which covers all possible values of $b$. Therefore the one root for $b_a$ in $(0,1)$ is the only possible solution consistent with $(b_c, b) \in (0,1)^2$. ∎



## §4 Graphs with one special edge: limiting case

The polynomial system (15)−(17) of §3 for positive blocking probabilities has a limit as $n \to \infty$:

$$\begin{aligned} c/2 &= 1 - b_c + 2b_a(1-b_a)(1-b_c) && (18) \\ d/2 &= 1 - b_a + b(1-b_a)^2 + b_a(1-b)(1-b_a) && (19) \\ d/2 &= 1 - b + 2b(1-b)^2. && (20) \end{aligned}$$

These can be solved for $b, b_a, b_c$. The last univariate equation in $b$ may have 0, 1, 2 or solutions, which can be found numerically. The second quadratic equation implies that

$$b_a = b, \frac{2b^2 - 2b - 1}{1 - 2b}$$

but only $b_a = b$ can be a probability solution. Finally, $b_c$ depends linearly on $c$ and ranges from 0 to 1:

$$b_c = 1 - \frac{c/2}{2b(1-b)+1} \in [0, 1]$$

with $b_{c=0} = 1, b_{c/2=2b(1-b)+1} = 0$.

Using the last equation, it follows that: if $b_a > 0$ and $b > 0$ and if $c \geq d/(1-b)$, then the blocking probability $b_c = 0$. The implication is that for a large network, one may attain a blocking probability of 0 on the single edge with capacity $c$ by increasing $c$ to $d/(1-b)$, where $b$ is the nonzero blocking probability on the non-adjacent edges.

The limiting fixed point equations for a transformation $T_\infty = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$ are

$$\begin{aligned} b_c &= t_1(\mathbf{b}) := \max\{0, 1 - \frac{c/2}{1 + 2b_a(1-b_a)}\} && (21) \\ b_a &= t_2(\mathbf{b}) := \max\{0, 1 - \frac{d/2}{1 + b(1-b_a) + b_a(1-b)}\} && (22) \\ b &= t_3(\mathbf{b}) := \max\{0, 1 - \frac{d/2}{1 + 2b(1-b)}\} && (23) \end{aligned}$$

By the considerations above, there can be at most two solutions in $[0, 1]^3$ to the above system, say $S_\infty = \{\mathbf{b}_1, \mathbf{b}_2\}$. To see this, consider first the case where $b_a = 0$. Then $b_c = \max\{0, 1 - c/2\}$, and $b$ satisfies the last equation that admits one or two solutions always. Now take the other case where

$b_a > 0$. Only one of the two roots of the quadratic for $b_a$ gives a possible probability value, and leads to a single value for $b_c$ and possibly two values for $b$.

Furthermore, one can show that if $b_a = 0$, then it is not possible for $b > 0$, and if $b_a > 0$, the quadratic implies that $b_a = b$, so in all cases $b_a = b$. Then the nature of the possible solutions is quite simple and is determined by the last equation in $b$:

1. If $r \in [0, 1]$ is the only solution to (23) for fixed $d$, then the only fixed point for (21) – (23) is $b = r, b_a = r, b_c = \max\{0, 1 - \frac{c/2}{1+r(1-r)}\}$.

2. If $r_1, r_2 \in [0, 1]$ are two solutions to equation (23), then the system (21) – (23) has two solutions: $b = r_i, b_a = r_i, b_c = \max\{0, 1 - \frac{c/2}{1+r_i(1-r_i)}\}$.

With this description of the roots, one can prove a result about stable fixed points that seems to hold more generally. The implication is that in a large network, only one of two solutions can be found by iterating the fixed point equation, so algebraic methods are necessary for finding the second solution.

**Proposition 4.1.** If the system (21) – (23) has two positive fixed points, only one of them is stable under iteration.

**Proof.** We must look at the eigenvalues of the derivative matrix of the transformation $T_\infty = (t_1, t_2, t_3)'$ at the two fixed points above. If we order the variables $b_c, b_a, b$, the derivative is upper triangular:

$$DT_\infty := \begin{pmatrix} \partial_{b_c} t_1 & \partial_{b_a} t_1 & \partial_b t_1 \\ \partial_{b_c} t_2 & \partial_{b_a} t_2 & \partial_b t_2 \\ \partial_{b_c} t_3 & \partial_{b_a} t_3 & \partial_b t_3 \end{pmatrix} = \begin{pmatrix} 0 & \partial_{b_a} t_1 & 0 \\ 0 & \partial_{b_a} t_2 & \partial_b t_2 \\ 0 & 0 & \partial_b t_3 \end{pmatrix}$$

Since $t_3$ is just the univariate symmetric equation, we know it has two roots $.5 > r_1 > r_2 > 0$, the first of which is stable with $|\partial_b t_3(r_1)| < 1$, while the second is unstable with $|\partial_b t_3(r_2)| > 1$. Thus the solution corresponding to $r_2$ is already unstable. For the solution corresponding to $r_1$, the other eigenvalue is $\partial_{b_a} t_2(r_1) = \frac{-2r_1(d/2)}{(1+r_1(1-r_1))^2} = 2r_1(1 - r_1)/(1 + 2r_1(1 - r_1)) \in (-1/2, 0)$. Thus, for $r_1$, all eigenvalues are in $(-1, 1)$ proving local stability. ∎

## §5 Further Problems

Numerical work indicates that much more is true about the nature of solutions to these systems than we have proved. We conjecture that the regions we have identified for exactly two solutions are exactly the regions for any number of multiple solutions, and that there can always be at most two probability solutions. Furthermore, networks with more than two parameters seem to have at most two solutions as well. Finally, rigorous work in stochastic processes beyond the symmetric network to justify the heuristic probabilistic reasoning that leads to these equations would be interesting.

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