Pairwise Closure Approximations in Epidemic Models on Regular Networks

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Outline

- Introduction to Epidemic models
- Reduction of SIRS models and pairwise closure approximation
- Sustained oscilations in a stochastic PCA-SIRS model
- Power spectrum

Epidemics

1. SIR

$$\mathbf{S} \xrightarrow{\lambda} \mathbf{I} \xrightarrow{\delta} \mathbf{R}$$

Epidemics

1. SIR

$$\mathbf{S} \xrightarrow{\lambda} \mathbf{I} \xrightarrow{\delta} \mathbf{R}$$

2. SIS

$$\mathbf{S} \xrightarrow{\lambda} \mathbf{I} \xrightarrow{\delta} \mathbf{S}$$

Epidemics

1. SIR

$$\mathbf{S} \xrightarrow{\lambda} \mathbf{I} \xrightarrow{\delta} \mathbf{R}$$

2. SIS

$$\mathbf{S} \xrightarrow{\lambda} \mathbf{I} \xrightarrow{\delta} \mathbf{S}$$

3. SIRS

$$\mathbf{S} \xrightarrow{\lambda} \mathbf{I} \xrightarrow{\delta} \mathbf{R} \xrightarrow{\gamma} \mathbf{S}$$

SIRS models

$$\mathbf{S} \xrightarrow{m_x\lambda} \mathbf{I} \xrightarrow{\delta} \mathbf{R} \xrightarrow{\gamma} \mathbf{S}$$

SIRS epidemics on n-regular networks are given by

$$\begin{aligned}
\frac{dP_{S_x}}{dt} &= -\lambda \sum_{y\in N(x)} P_{S_x,I_y} + \gamma P_{R_x} \\
\frac{dP_{I_x}}{dt} &= \lambda \sum_{y\in N(x)} P_{S_x,I_y} - \delta P_{I_x} \\
\frac{dP_{S_x,I_y}}{dt} &= \gamma P_{R_x,I_y} - (\lambda+\delta) P_{S_x,I_y} + \lambda \sum_{z\in N(y)-x} P_{S_x,S_y,I_z} - \lambda \sum_{z\in N(x)-y} P_{I_z,S_x,I_y} \\
\frac{dP_{S_x,R_y}}{dt} &= \delta P_{S_x,I_y} + \gamma P_{R_x,R_y} - \gamma P_{S_x,R_y} - \lambda \sum_{z\in N(x)-y} P_{I_z,S_x,R_y} \\
\frac{dP_{R_x,I_y}}{dt} &= -(\gamma+\delta) P_{R_x,I_y} + \delta P_{I_x,I_y} + \lambda \sum_{z\in N(y)-x} P_{R_x,S_y,I_z}
\end{aligned}$$

PA SIRS models on $n-$regular graphs

Let

$$P_A = \sum_x P_{A_x}, \quad P_{AB} = \frac{1}{n} \sum_{y \in N(x)} P_{A_x, B_y},$$

$$P_{ABC} = \frac{1}{n-1} \sum_{z \in N(y)-x} P_{A_x, B_y, C_z}.$$

We have

$$\begin{aligned}
\frac{dP_S}{dt} &= -n\lambda P_{SI} + \gamma P_R \\
\frac{dP_I}{dt} &= n\lambda P_{SI} - \delta P_I \\
\frac{dP_{SI}}{dt} &= \gamma P_{RI} - (\lambda + \delta) P_{SI} + (n-1)\lambda (P_{SSI} - P_{ISI}) \\
\frac{dP_{SR}}{dt} &= \delta P_{SI} + \gamma P_{RR} - \gamma P_{SR} - (n-1)\lambda P_{ISR} \\
\frac{dP_{RI}}{dt} &= -(\gamma + \delta) P_{RI} + \delta P_{II} + (n-1)\lambda P_{ISR}
\end{aligned}$$

SIRS models

Apply the pairwise closure approximation (PCA)

$$P_{ABC} \sim P_{AB}\frac{P_{BC}}{P_B}.$$

We get

$$\begin{aligned}
\frac{dP_S}{dt} &= -n\lambda P_{SI} + \gamma(1 - P_S - P_I) \\
\frac{dP_I}{dt} &= n\lambda P_{SI} - \delta P_I \\
\frac{dP_{SI}}{dt} &= \gamma P_{RI} - (\lambda + \delta)P_{SI} + (n-1)\lambda(P_{SS}P_{SI} - P_{IS}^2)/P_S \\
\frac{dP_{SR}}{dt} &= \delta P_{SI} + \gamma(1 - P_S - P_I - P_{RI} - 2P_{SR}) - (n-1)\lambda P_{IS}P_{SR}/P_S \\
\frac{dP_{RI}}{dt} &= -(\gamma + \delta)P_{RI} + \delta(P_I - P_{SI} - P_{RI}) + (n-1)\lambda P_{IS}P_{SR}/P_S
\end{aligned}$$

Reduction of SIRS models

$$\mathbf{S} \xrightarrow{m_x\lambda} \mathbf{I} \xrightarrow{\delta} \mathbf{R} \xrightarrow{\boxed{\gamma = \mathbf{0}}} \mathbf{S}$$

- $\gamma = 0$: An SIRS model can be reduced to an SIR model. In particular, In the context of pairwise closure approximations, we have

$$\begin{aligned}
\frac{dP_S}{dt} &= -n\lambda P_{SI} \\
\frac{dP_I}{dt} &= n\lambda P_{SI} - \delta P_I \\
\frac{dP_{SI}}{dt} &= -(\lambda + \delta)P_{SI} + (n-1)\lambda(P_S - 2P_{SI} - P_{SR})P_{SI}/P_S \\
\frac{dP_{SR}}{dt} &= \delta P_{SI} - (n-1)\lambda P_{SI}P_{SR}/P_S \\
\frac{dP_{RI}}{dt} &= -\delta P_{RI} + \delta(P_I - P_{SI} - P_{RI}) + (n-1)\lambda P_{SI}P_{SR}/P_S
\end{aligned} \tag{1}$$

Reduction of SIRS models,ctd

$$\mathbf{S} \xrightarrow{m_x \lambda} \mathbf{I} \xrightarrow{\delta} \mathbf{R} \xrightarrow{\boxed{\gamma = \infty}} \mathbf{S}$$

- $\gamma = \infty$: an SIRS model can be reduced to an SIS model.

$$\begin{aligned} \frac{1}{\gamma}\frac{dP_{SR}}{dt} &\simeq 0 \\ \frac{1}{\gamma}\frac{dP_{RI}}{dt} &\simeq 0 \\ \frac{1}{\gamma}\frac{dP_{RR}}{dt} &\simeq 0 \end{aligned} \quad (2)$$

Reduction of SIRS models, ctd

Notice that

$$\frac{1}{\gamma}\frac{dP_{RI}}{dt} \simeq 0,$$

which implies

$$-(1+\frac{\delta}{\gamma})P_{RI} + \frac{\delta}{\gamma}P_{II} + (n-1)\frac{\lambda}{\gamma}P_{ISR} \simeq 0. \qquad (3)$$

If we assume that $\delta, \lambda, n = O(1)$, in the limit $\gamma \to \infty$, $P_{ISR} \ll P_{II}$ and $(n-1)\lambda/\gamma P_{ISR} \ll P_{RI}$. Equation (3) yields

$$P_{RI} \simeq \frac{\delta}{\gamma}P_{II}. \qquad (4)$$

Reduction of SIRS models, ctd

Hence we have

$$\frac{dP_I}{dt} = n\lambda P_{SI} - \delta P_I$$
$$\frac{dP_{SI}}{dt} \simeq \delta P_{II} - (\lambda + \delta)P_{SI} + \lambda(n-1)(P_{SSI} - P_{ISI}). \quad (5)$$

In the context of the closure approximation,

$$\frac{dP_I}{dt} = -\delta P_I + \lambda n P_{SI}$$
$$\frac{dP_{SI}}{dt} = \delta P_{II} - (\lambda + \delta)P_{SI} + \lambda(n-1)\frac{(1 - P_I - 2P_{SI})P_{SI}}{1 - P_I}. \quad (6)$$

PCA SIS models on $n-$regular graphs by Eames-Keeling

Recall that

$$\frac{d[I]}{dt} = -\delta[I] + \lambda[SI]$$
$$\frac{d[SI]}{dt} = \lambda[SSI] + \delta[II] - \lambda[SI] - \lambda[ISI] - \delta[SI] \qquad (7)$$

They employed closure approximation

$$[ABC] \sim [AB](n-1)\frac{[BC]}{n[B]}.$$

to get

$$\frac{d[I]}{dt} = -\delta[I] + \lambda[SI] \qquad (8)$$
$$\frac{d[SI]}{dt} = \delta[II] - (\lambda+\delta)[SI] - (n-1)\lambda\frac{(n(N-[I]) - 2[SI])[SI]}{N-[I]}$$

PCA SIS models on $n-$regular graphs by Eames-Keeling

Let $P_A = \dfrac{[A]}{N}$, $P_{AB} = \dfrac{[AB]}{nN}$ and $P_{ABC} = \dfrac{[ABC]}{n(n-1)N}$.

1. Closure approximation
$$[ABC] \sim [AB](n-1)\frac{[BC]}{n[B]}$$
is equivalent to
$$P_{ABC} \sim P_{AB}\frac{P_{BC}}{P_B}.$$
2. (8) becomes
$$\frac{dP_I}{dt} = -\delta P_I + \lambda n P_{SI}$$
$$\frac{dP_{SI}}{dt} = \delta P_{II} - (\lambda+\delta)P_{SI} + \lambda(n-1)\frac{(1-P_I-2P_{SI})P_{SI}}{1-P_I}.$$

Damped oscillations in PCA-SIRS model

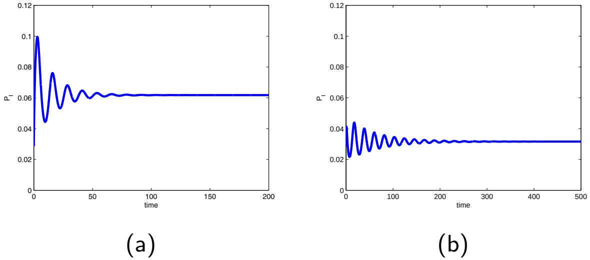


Figure: The time evolution of the fraction of infective population shows the appearance of damped oscillations in PCA-SIRS model. Here $\delta = 1, n = 4$.(a)$\lambda = 2.5, \gamma = 0.1$.(b)$\lambda = 1.5, \gamma = 0.06$.

Damped oscillations in PCA-SIRS model,ctd

In Figure (a), the damped oscillation converges to a non-trivial fixed point, p , for which

$$(P_S, P_I, P_{SI}, P_{SR}, P_{RI}) = (0.32064, 0.06176, 0.006176, 0.18583, 0.039252)$$

If we linearize the PCA-SIRS at this fixed point, the associate Jacobian matrix J has five eigenvalues:
$-\lambda_1 \pm i\,\omega_1 = -0.060244 \pm i\,0.498571, -\lambda_2 = -0.654906$ and
$-\lambda_3 \pm i\,\omega_3 = -1.846917 \pm i\,0.521526.$
Namely, J can be diagonalized into a matrix

$$\begin{bmatrix} -0.0605 & 0.4989 & 0 & 0 & 0 \\ -0.4989 & -0.0605 & 0 & 0 & 0 \\ 0 & 0 & -0.6547 & 0 & 0 \\ 0 & 0 & 0 & -1.8467 & 0.5214 \\ 0 & 0 & 0 & -0.5214 & -1.8467 \end{bmatrix}$$

Damped oscillations in PCA-SIRS model,ctd

In Figure (b), the diagonalized Jacobian matrix is

$$\begin{bmatrix} -0.0156 & 0.2952 & 0 & 0 & 0 \\ -0.2952 & -0.0156 & 0 & 0 & 0 \\ 0 & 0 & -0.3240 & 0 & 0 \\ 0 & 0 & 0 & -1.4227 & 0 \\ 0 & 0 & 0 & 0 & -1.8299 \end{bmatrix}$$

In both examples, there is a principle component of the the linearized PAC-SIRS, and its eigenvalues of the form $-\lambda \pm i\,\omega$ and has the property $\lambda \ll \omega$.

Stochastic PCA-SIRS models

Employ Kurtz diffusion approximation [Kurtz, 1978]. For large population sizes, each density-dependent process converges to a Gaussian diffusion processes.

$$
\begin{aligned}
dP_S &= (-n\lambda P_{SI} + \gamma(1 - P_S - P_I))\,dt + \sigma_N\sqrt{\gamma(1 - P_S - P_I)}dW_1 - \sigma_N\sqrt{n\lambda P_{SI}}dW_2 \\
dP_I &= (n\lambda P_{SI} - \delta P_I)\,dt + \sigma_N\left(\sqrt{n\lambda P_{SI}}dW_2 - \sqrt{\delta P_I}dW_3\right) \\
dP_{SI} &= (\gamma P_{RI} - (\lambda + \delta)P_{SI} + (n-1)\lambda(P_S - 2P_{SI} - P_{SR})P_{SI}/P_S)\,dt \\
&\quad + \sigma_N\left(\sqrt{(n-2)\lambda P_{SI}}dW_2 + \sqrt{\gamma P_{RI}}dW_4 - \sqrt{\delta P_{SI}}dW_5\right) \\
&\quad - \sigma_N\left(\sqrt{(n-1)\lambda P_{SR}P_{SI}/P_S}dW_6 + \sqrt{2(n-1)\lambda P_{SI}^2/P_S}dW_7\right) \\
dP_{SR} &= (\delta P_{SI} + \gamma(1 - P_S - P_I - P_{RI} - 2P_{SR}) - (n-1)\lambda P_{SI}P_{SR}/P_S)\,dt \\
&\quad + \sigma_N\left(\sqrt{\delta P_{SI}}dW_5 + \sqrt{\gamma(1 - P_S - P_I)}dW_1 - \sqrt{\gamma P_{RI}}dW_4\right) \\
&\quad - \sigma_N\left(\sqrt{2\gamma P_{SR}}dW_8 + \sqrt{(n-1)\lambda P_{SI}P_{SR}/P_S}dW_6\right) \\
dP_{RI} &= (-(\gamma + \delta)P_{RI} + \delta(P_I - P_{SI} - P_{RI}) + (n-1)\lambda P_{SI}P_{SR}/P_S)\,dt \\
&\quad + \sigma_N\left(\sqrt{\delta P_I}dW_3 - \sqrt{\delta P_{SI}}dW_5 - \sqrt{2\delta P_{RI}}dW_9 - \sqrt{\gamma P_{RI}}dW_4\right) \\
&\quad + \sigma_N\sqrt{(n-1)\lambda P_{SI}P_{SR}/P_S}dW_6
\end{aligned}
$$

where $\sigma_N = 1/\sqrt{N}$

Stochastic PCA-SIRS models, ctd

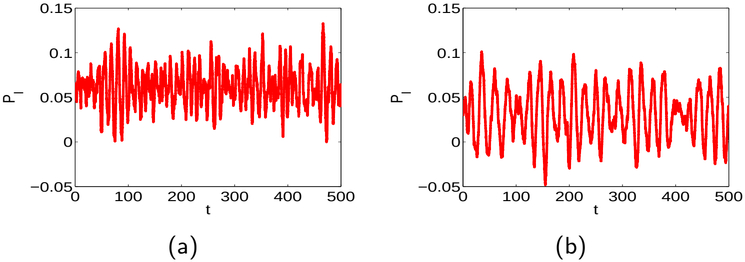


Figure: Sustained oscillation in the PCA-SIRS model. Here $\delta = 1, n = 4$. (a)$\lambda = 2.5, \gamma = 0.1$.(b)$\lambda = 1.5, \gamma = 0.06$.

Stochastic PCA-SIRS models, ctd

Linearize the system at the fixed point.

$$dX_t = AX_t dt + Cd\mathbb{W}_t \tag{9}$$

In the example shown in Figure (a),

$$A = \begin{bmatrix} -0.0605 & 0.4989 & 0 & 0 & 0 \\ -0.4989 & -0.0605 & 0 & 0 & 0 \\ 0 & 0 & -0.6547 & 0 & 0 \\ 0 & 0 & 0 & -1.8467 & 0.5214 \\ 0 & 0 & 0 & -0.5214 & -1.8467 \end{bmatrix}$$

$$C = \begin{bmatrix} -0.0007 & 0.0005 & 0.0014 & 0.0001 & 0.0003 \\ -0.0005 & 0.0008 & 0.0025 & -0.0012 & 0.0008 \\ -0.0003 & 0.0004 & 0.0010 & 0.0005 & 0.0002 \\ -0.0004 & 0.0007 & 0.0014 & 0.0001 & -0.0002 \\ -0.0005 & 0.0001 & 0.0023 & -0.0003 & 0.0013 \end{bmatrix}$$

Stochastic PCA-SIRS models, ctd

In the example shown in Figure (b),

$$A = \begin{bmatrix} -0.0156 & 0.2952 & 0 & 0 & 0 \\ -0.2952 & -0.0156 & 0 & 0 & 0 \\ 0 & 0 & -0.3240 & 0 & 0 \\ 0 & 0 & 0 & -1.4227 & 0 \\ 0 & 0 & 0 & 0 & -1.8299 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0.0004 & -0.0004 & -0.0008 & -0.0002 & -0.0002 \\ 0.0006 & -0.0008 & -0.0023 & 0.0008 & -0.0007 \\ -0.0002 & 0.0003 & 0.0008 & 0.0003 & 0.0002 \\ -0.0004 & 0.0001 & 0.0019 & -0.0003 & 0.0010 \\ 0.0002 & 0.0002 & -0.0012 & 0.0002 & -0.0010 \end{bmatrix}$$

Stochastic PCA-SIRS models, ctd

In general, consider a $d-$dimsional process

$$dX_t = A\,X_t dt + C\,d\mathbb{W}_t \tag{10}$$

where spectrum of A,

$$Spec(A) = \{-\lambda \pm \omega i, -\lambda_j \pm \omega_j i, -\lambda_k\}_{j=1,2,\ldots,p,\ k=p+1,..,(d-1-p)}$$

with $\min\{\lambda, \lambda_j, \lambda_k, \omega_j\} > 0$.
Let Q be A $d \times d$ matrix such that

$$Q^{-1}AQ = \begin{bmatrix} -\lambda & \omega & 0 & 0 & \cdots & 0 \\ -\omega & -\lambda & 0 & 0 & \cdots & 0 \\ 0 & 0 & -\lambda_1 & \omega_1 & \cdots & 0 \\ 0 & 0 & -\omega_1 & -\lambda_1 & \cdots & 0 \\ & \cdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & \lambda_{(d-1-p)} \end{bmatrix} \tag{11}$$

Stochastic PCA-SIRS models, ctd

Let

$$D = CC^* = (d_{ij})_{i,j=1,\dots,d},\, D_0 = (d_{ij})_{i,j=1,2}$$
$$Q = (q_{ij})_{i,j=1,\dots,d},\, Q_0 = (q_{ij})_{i=1,\dots,d,j=1,2}.$$

We define

$$k = \sqrt{\frac{trace(D_0)}{2\lambda}}$$
$$R_t = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$$
$$\tilde{R}_t = \begin{bmatrix} R(t) & \mathbf{0}_{2\times(d-2)} \\ \mathbf{0}_{(d-2)\times 2} & \mathbf{1}_{d-2} \end{bmatrix}$$
$$(\tilde{R}_{\omega t/\lambda} DD^* \tilde{R}_{\omega t/\lambda}) = (v_{ij}(t))$$
$$\frac{d\tilde{v}_{ij}}{dt} = v_{ij}(t) - \bar{v}_{ij}.$$

Stochastic PCA-SIRS models, ctd

Theorem (W. and Greenwood)

For each fixed T *and* $x \in \mathbb{R}_d$, *as* $\|(\tilde{v}_{ij}(t))_{ij}\|_2 \to 0$, $\lambda/\lambda_l \to 0$ *for* $l \neq 1$, X_t *converges weakly to* $kQ_0 R_{-\omega t} S_{\lambda t}$ *where* $\{S_t : 0 \leq t \leq T\}$ *is the 2-dimesional OU process generated by the SDE*

$$dS_t = -S_t\,dt + dW_t \tag{12}$$

with $S_0 = x$

Power spectrum

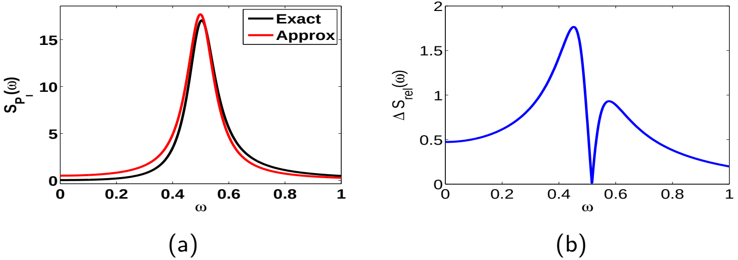
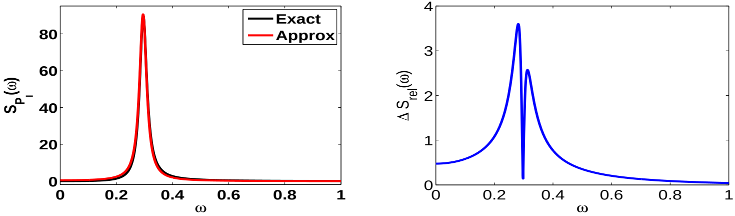


Figure: Power spectrum density function in the PCA-SIRS model. $\lambda/\omega = 0.12$. (a) exact powers spectrum vs. approximation (b) absolute error

Power spectrum



(a) (b)

Figure: Power spectrum density function in the PCA-SIRS model. $\lambda/\omega = 0.05$ (a) exact powers spectrum vs. approximation (b) absolute error

Sketch of the proof

The idea is to use stochastic averaging methods.

- Tightness is standard.

- Uses martingale problem approach. Uniqueness is shown by finding a suitable perturbation for each test function.

Thanks!