Large Deviations and Variational Representations

Freidlin-Wentzell Asymptotics in Infinite Dimensions

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Outline.

1. Background and basic definitions.
2. Classical approaches to small noise LDP.
3. Advantages of the current approach.
4. Main Result.
5. Some applications.
6. Key ingredient in the proof: Variational repn. for inf. dim. BM.
7. Proof sketch of the variational repn.
8. Extensions.
Background and Definitions.

- Concerned with decay rate of probabilities of rare events.
- E.g. consider a $k$-dimensional SDE:

$$dX^\epsilon(t) = b(X^\epsilon(t))dt + \sqrt{\epsilon}a(X^\epsilon(t))dW(t), \quad X^\epsilon(0) = x_0, \quad t \in [0, T],$$

As $\epsilon \to 0$, $X^\epsilon \xrightarrow{P} X^0$ in $C([0, T] : \mathbb{R}^k) \equiv C$, where $X^0$ solves the ODE

$$\dot{X}^0 = b(X^0).$$

- Freidlin-Wentzell theory describes precise asymptotics of probabilities such as

$$\mathbb{P}\left\{\sup_{0 \leq t \leq T} |X^\epsilon(t) - X^0(t)| > a \right\}$$

through a Large Deviation Principle.
Large Deviation Principle

Definition. Consider a sequence $\{X^\varepsilon\}_{\varepsilon > 0}$ of $\mathcal{E}$ valued r.vs. $\mathcal{E}$ - Polish.

(1) A function $I$ from $\mathcal{E}$ to $[0, \infty]$ is called a rate function on $\mathcal{E}$ if for each $M < \infty \{ x \in \mathcal{E} : I(x) \leq M \}$ is compact.

(2) $\{X^\varepsilon\}$ is said to satisfy the large deviation principle on $\mathcal{E}$ (as $\varepsilon \to 0$) with rate function $I$ if:
(a) For each closed subset $F$ of $\mathcal{E}$
$$\limsup_{\varepsilon \to 0} \varepsilon \log P(X^\varepsilon \in F) \leq - \inf_{x \in F} I(x).$$
(b) For each open subset $G$ of $\mathcal{E}$
$$\liminf_{\varepsilon \to 0} \varepsilon \log P(X^\varepsilon \in G) \geq - \inf_{x \in G} I(x).$$
Formally, for small $\varepsilon$:

$$P(X^\varepsilon \in A) \approx \exp \left\{ -\frac{\inf_{x \in A} I(x)}{\varepsilon} \right\}, \ A \in \mathcal{B}(\mathcal{E}).$$
• For the f.d. SDEs above, F-W show that the family, \( \{ X^\varepsilon \} \) of \( C \) valued random elements satisfies LDP with rate function \( I \) given as

\[
I(f) = \inf_{u \in A_f} \frac{1}{2} \int_0^T |u_s|^2 ds,
\]

\[
A_f = \{ u \in L^2([0,T] : \mathbb{R}^m) : \dot{f}_t = b(f_t) + a(f_t) u_t, \text{a.e. } t, f_0 = x_0 \}.
\]

In particular:

\[
\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\{ \| X^\varepsilon - X^0 \| \geq a \} \leq - \inf_{x \in F_a} I(x),
\]

where

\[
F_a = \{ x \in C : \| x - X^0 \| \geq a \}.
\]

• Used in study of exit time and invariant measure asymptotics.
Infinite Dimensional Noise

• Here we consider SDEs with an infinite dim. driving noise.

Standard Approaches

• Build on the ideas of Azencott (1980).
• One approximates by Gaussian systems by freezing the diffusion coefficient and suitable time discretizations.
• ...or one considers approximations by SDEs with f.d. noise.
• LDP for approximations follow from classical results.
• Finally one obtains suitable exponential continuity estimates in order to obtain the LDP for the original non-Gaussian infinite dim. system.
• Feng-Kurtz: proofs based on exponential tightness estimates and uniqueness theory for infinite dimensional HJ equations.
• Exponential continuity and tightness estimates are hard.
• ...sometimes obtained under “sub-optimal” conditions.
Advantages of Current Approach

- No approximations or discretizations.
- Exponential prob. estimates are completely bypassed.
- Proofs of LDP reduce to demonstrating basic qualitative properties of certain perturbations of the original system.

- For f.d. SDEs this amounts to showing:
  (i) For any $\theta \in [0, 1)$, $x \in \mathbb{R}^k$ and any $L^2$–bounded control $u$ the SDE below has a unique solution.

\[
dX^{\theta,u}(t) = b(X^{\theta,u}(t)) dt + \theta a(X^{\theta,u}(t)) dW(t) + a(X^{\theta,u}(t)) u(t) dt, \quad X^{\theta,u}(0) = x
\]

(ii) If $\theta_n \to 0$ and $u_n \Rightarrow u$, where $\{u_n\}$ are uniformly $L^2$–bounded controls, then $X^{\theta_n,u_n} \to X^{0,u}$ in distribution.
Brownian Sheet

- Let $\mathcal{O}$ be a bounded open set in $\mathbb{R}^d$ and 
  $\{B(t, x), (t, x) \in [0, T] \times \mathcal{O}\}$ be a Brownian sheet.

  I.e. it is a mean zero, continuous, Gaussian random field such that

  $Cov\left(B(t, x), B(s, y)\right) = \text{Leb}(A_{t,x} \cap A_{s,y})$, where

  $$A_{t,x} = \{(s, y) : s \in [0, t], y \in \mathcal{O} \cap [0, x]\}.$$ 

- $B$ is a $(\mathcal{C}, \mathcal{B}(\mathcal{C}))$ valued r.v., where $\mathcal{C} = C([0, T] \times \overline{\mathcal{O}} : \mathbb{R})$ and $\mathcal{B}(\mathcal{C})$ the Borel sigma-field.

- Denote by $\mu$ the induced Wiener measure.

- Henceforth $B$ is the canonical process on $(\mathcal{C}, \mathcal{B}(\mathcal{C}), \mu)$. 

A General LDP

For $\varepsilon > 0$ and Polish space $\mathcal{E}$, let $G^\varepsilon : \mathbb{C} \to \mathcal{E}$ be a measurable map.

Interested in LDP (as $\varepsilon \to 0$) for

$$X^\varepsilon \doteq G^\varepsilon(\sqrt{\varepsilon}B).$$

Typical example of $X^\varepsilon$: Solution of a small noise SPDE.
LDP for $X^\varepsilon \doteq \mathcal{G}^\varepsilon(\sqrt{\varepsilon}B)$

- Some Notation: Let
  
  $$S^N \doteq \{ \phi \in H \equiv L^2([0, T] \times \mathcal{O}) : \|\phi\|^2_H \leq N \},$$

  where
  
  $$\|\phi\|^2_H = \int_{[0,T] \times \mathcal{O}} \phi^2(s, x) dsdx.$$

- $S^N$ is a compact Polish space with the weak topology.
- With $\{\mathcal{F}_t\}$ the canonical filtration on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, define
  
  $$\mathcal{P}^N_{2} \doteq \{ u : u \text{ is } \mathcal{P} \otimes \mathcal{B}(\mathcal{O}) \text{ measurable and } u(\omega) \in S^N, \mu - a.s. \}.$$

- For $\phi \in H$, define $\text{Int}(\phi) \in \mathbb{C}$ by
  
  $$\text{Int}(\phi)(t, x) \doteq \int_{A_{t, x}} \phi(s, y) dsdy,$$
LDP for $X^{\varepsilon} \equiv \mathcal{G}^{\varepsilon}(\sqrt{\varepsilon}B)$(contd.)

Assumption. There exists a measurable map $\mathcal{G}^0 : \mathbb{C} \to \mathcal{E}$ such that: For every $N < \infty$:

- Whenever $\{u_n\} \subset \mathcal{P}_2^N$ is such that $u_n \Rightarrow u$ (as $S^N$–valued random elements), and $\varepsilon_n \in [0, 1)$ is such that $\varepsilon_n \to 0$, we have

  $$\mathcal{G}^{\varepsilon_n}\left(\sqrt{\varepsilon_n}B + \text{Int}(u_n)\right) \Rightarrow \mathcal{G}^0\left(\text{Int}(u)\right).$$

Theorem. Suppose Assumption holds. Then, the family $\{X^{\varepsilon}\}$ satisfies LDP on $\mathcal{E}$, with rate function

$$I(f) \equiv \inf\left\{ u \in H : f = \mathcal{G}^0(\text{Int}(u)) \right\} \left\{ \frac{1}{2} \|u\|^2_H \right\}.$$
Extensions.

• A slight strengthening of Assumption 1 gives a uniform LDP.
• Analogous results can be obtained for an infinite sequence of real i.i.d. BMs, a cylindrical BM, and a Hilbert space valued BM.
An Application: A Toy Example.

Nonlinear stochastic cable equation: Let \( L = \alpha I - \beta \frac{d^2}{dx^2} \).

\[
dX^\varepsilon(t,r) = -LX^\varepsilon(t,r)\,dr\,dt + \sqrt{\varepsilon}F(X^\varepsilon(t,r))B(dr\,dt), \quad x \in (0,b), \quad t \in [0,T].
\]

with initial and boundary condition \( X^\varepsilon(0,r) = f(r), \quad r \in [0,b], \)

\[
\frac{\partial}{\partial x} X^\varepsilon(t,0) = \frac{\partial}{\partial x} X^\varepsilon(t,b) = 0, \quad t \in [0,T].
\]

There is a unique continuous mild solution:

\[
X^\varepsilon(t,r) = \int_{[0,b]} G(t, r, q)f(q)\,dq \\
+ \sqrt{\varepsilon} \int_{[0,t] \times [0,b]} G(t-s, r, q)F(X^\varepsilon(s, q))B(dq\,ds).
\]

Thus there is a measurable map \( G^\varepsilon : \mathbb{C} \to \mathbb{C} \) such that

\[
X^\varepsilon = G^\varepsilon(\sqrt{\varepsilon}B).
\]
Verifying Assumption for $G^\varepsilon$

Defining $G^0$: For $\phi \in H$, let $\xi_\phi$ be the unique soln. of

$$
\xi_\phi(t, r) = \int_{[0,b]} G(t, r, q) f(q) dq \\
+ \int_{[0,t] \times [0,b]} G(t - s, r, q) F(\xi_\phi(s, q)) \phi(s, q) dq \, ds.
$$

Define $G^0 : \mathbb{C} \to \mathbb{C}$ as

$$
G^0(v) = \xi_\phi, \quad \text{if } v = \text{Int}(\phi), \quad \text{for some } \phi \in H.
$$

Set $G^0(v) = 0$ otherwise.
Let \( \{u^\varepsilon\} \) be a sequence in \( \mathcal{P}_2^N \) such that \( u^\varepsilon \Rightarrow u \).

Let \( X^{u^\varepsilon} = \mathcal{G}^\varepsilon(\sqrt{\varepsilon}B + \text{Int}(u^\varepsilon)) \) and \( X^0 = \mathcal{G}^0(\text{Int}(u)) \).

Need to show: \( X^{u^\varepsilon} \Rightarrow X^0 \). Can Check \( X^{u^\varepsilon} \) solves:

\[
X^{u^\varepsilon}(t, r) = \int_{[0,b]} G(t, r, q)f(q)dq + \sqrt{\varepsilon} \int G(...) F(X^{u^\varepsilon})B(dqds) + \int G(...) F(X^{u^\varepsilon})u^\varepsilon(s,q)dqds.
\]

Also \( X^0 \) solves

\[
X^0(t, r) = \int_{[0,b]} G(t, r, q)f(q)dq + \int G(...) F(X^0)u(s,q)dq ds.
\]

Weak convergence follows by standard estimates...
Applications


\textit{Conditions relaxed in B.-Dupuis-Maroulas(2008).}
Applications (continued)

- **Stochastic flows of diffeomorphisms. B.-Dupuis-Maroulas (2009).** Prior works include Millet, Nualart and Sanz-Sole (1992), Ben Arous and Castell (1995)—these concern finite dimensional flows.

  – asymptotic relation, in terms of the rate function, between (small noise) Bayesian solution of an image matching problem with the solution of a deterministic variational problem.


Other Applications.

- Rockner, Zhang, Zhang (preprint) Stochastic tamed 3D Navier-Stokes equations.
- Wang and Duan (preprint) Stochastic parabolic PDEs with rapidly varying (random) boundary conditions.
- Bo and Jiang (preprint) Stochastic variational inequalities, reflected SPDEs.
- Bessaih and Millet (preprint) Inviscid shell models.
- Chueshov and Millet (preprint) Stochastic 2D hydrodynamical type systems.
- Du, Duan, Gao (preprint) Two-layer geophysical flows under uncertainty.
- Manna, Sritharan and Sundar (preprint) Stochastic shell model of turbulence.
Proof of the general LDP: $X^\varepsilon \dot{=} G^\varepsilon(\sqrt{\varepsilon} B)$

Suffices to show that Laplace principle holds: For all $h \in C_b(\mathcal{E})$

$$
\lim_{\varepsilon \to 0} -\varepsilon \log \mathbb{E} \left\{ \exp \left[ -\frac{1}{\varepsilon} h(X^\varepsilon) \right] \right\} = \inf_{x \in \mathcal{E}} \{ h(x) + I(x) \}.
$$

A Variational Representation (B.-Dupuis(2000)): Let $f : \mathbb{C} \to \mathbb{R}$ be a bounded measurable map. Let $B$ be a Brownian sheet. Then

$$
- \log \mathbb{E}(\exp\{-f(B)\}) = \inf_{u \in \mathbb{P}_2} \mathbb{E} \left( \frac{1}{2} \|u\|_H^2 + f(B + \text{Int}(u)) \right).
$$

Recall $X^\varepsilon \dot{=} G^\varepsilon(\sqrt{\varepsilon} B)$. Applying repn. with $f = \frac{1}{\varepsilon} h o G^\varepsilon(\sqrt{\varepsilon} \cdot)$ we have

$$
-\varepsilon \log \mathbb{E} \left\{ \exp \left[ -\frac{1}{\varepsilon} h(X^\varepsilon) \right] \right\} = \inf_u \mathbb{E} \left( \frac{1}{2} \|u\|_H^2 + h(X^\varepsilon, u) \right),
$$

where $X^\varepsilon, u = G^\varepsilon(\sqrt{\varepsilon} B + \text{Int}(u))$.
Proof: \( \inf_u \mathbb{E} \left( \frac{1}{2} \| u \|_H^2 + h(X^{\epsilon, u}) \right) \rightarrow \inf_{x \in \mathcal{E}} \{ h(x) + I(x) \} \).

Proof of Upper Bound. Fix \( \delta \in (0, 1) \) and choose for each \( \epsilon \), \( u^\epsilon \) such that

\[
\text{LHS} \geq \mathbb{E} \left( \frac{1}{2} \| u^\epsilon \|_H^2 + h(X^{\epsilon, u^\epsilon}) \right) - \delta.
\]

- WLOG, for some \( N < \infty \), \( \sup_{\epsilon > 0} \| u^\epsilon \|_H^2 \leq N \).

Pick a subsequence along which \( u^\epsilon \) converges in distribution to some \( u \). From Assumption:

\[
\liminf_{\epsilon \to 0} \mathbb{E} \left[ \frac{1}{2} \| u^\epsilon \|_H^2 + h \circ G^\epsilon \left( \sqrt{\epsilon} B + \text{Int}(u^\epsilon) \right) \right] \\
\geq \mathbb{E} \left[ \frac{1}{2} \| u \|_H^2 + h \circ G^0 \left( \text{Int}(u) \right) \right] \\
\geq \inf_{(x,u) \in \mathcal{E} \times H: x = G^0(\text{Int}(u))} \left\{ \frac{1}{2} \| u \|_H^2 + h(x) \right\} \\
\geq \inf_{x \in \mathcal{E}} \{ I(x) + h(x) \}.
\]
Proof: \[
\lim_{\epsilon \to 0} \inf_u \mathbb{E}\left( \frac{1}{2} \|u\|_{H}^2 + h(X^\epsilon, u) \right) = \inf_{x \in \mathcal{E}} \{h(x) + I(x)\}.
\]

Proof of the lower bound. Fix \(\delta > 0\) and let \(x_0 \in \mathcal{E}\) be such that 
\[
I(x_0) + h(x_0) \leq \inf_{x \in \mathcal{E}} \{I(x) + h(x)\} + \frac{\delta}{2}.
\]
Choose \(\tilde{u} \in H\) such that:
\[
\frac{1}{2} \|\tilde{u}\|_{H}^2 \leq I(x_0) + \frac{\delta}{2} \quad \text{and} \quad x_0 = G^0 \left( \int_0^\cdot \tilde{u}(s) ds \right).
\]
Then
\[
\begin{align*}
\text{LHS} & = \limsup_{\epsilon \to 0} \inf_u \mathbb{E} \left[ \frac{1}{2} \|u\|_{H}^2 + h \circ G^\epsilon \left( \sqrt{\epsilon} B + \text{Int}(u) \right) \right] \\
& \leq \limsup_{\epsilon \to 0} \mathbb{E} \left[ \frac{1}{2} \|\tilde{u}\|_{H}^2 + h \circ G^\epsilon \left( \sqrt{\epsilon} B + \text{Int}(\tilde{u}) \right) \right] \\
& = \frac{1}{2} \|\tilde{u}\|_{H}^2 + \mathbb{E} \left[ h \circ G^0 \left( \text{Int}(\tilde{u}) \right) \right] = \frac{1}{2} \|\tilde{u}\|_{H}^2 + h(x_0) \\
& \leq I(x_0) + h(x_0) + \frac{\delta}{2} \leq \inf_{x \in \mathcal{E}} \{I(x) + h(x)\} + \delta.
\end{align*}
\]
Variational Repn.: Sketch of Proof.

\[- \log \mathbb{E}^\mu(\exp\{-f(B)\}) = \inf_{u \in \mathcal{P}_2} \mathbb{E}^\mu \left(\frac{1}{2}||u||^2_H + f(B + \text{Int}(u))\right). \tag{1}\]

Donsker-Varadhan variational formula:

\[- \log \mathbb{E}^\mu(\exp\{-f(B)\}) = \inf_{\gamma \in \mathcal{P}(\mathbb{C})} (R(\gamma \parallel \mu) + \mathbb{E}^\gamma(f(B))). \]

Upper Bound (In (1), LHS \leq RHS): For \( u \in \mathcal{P}_{\text{SIM}} \), let \( \gamma^u \in \mathcal{P}(\mathbb{C}) \) be defined as

\[ d\gamma^u = \exp \left( \int_{[0,T]\times\mathcal{O}} u(s, x)W(dsdx) - \frac{1}{2}||u||^2_H \right) d\mu. \]

Then \( R(\gamma^u \parallel \mu) = \mathbb{E}^\gamma^u \left(\frac{1}{2}||u||^2_H\right). \)
Upper Bound Proof (ctd.)

\[- \log \mathbb{E}^{\mu}(\exp\{-f(B)\}) \leq \inf_{u \in \mathcal{P}_{\text{SIM}}} \mathbb{E}^{\gamma^u}(f(B) + \frac{1}{2}\|u\|_H^2)\]

\[= \inf_{u \in \mathcal{P}_{\text{SIM}}} \mathbb{E}^{\gamma^u}(f(\tilde{B}^u + \text{Int}(u)) + \frac{1}{2}\|u\|_H^2)\]

\[= \inf_{u \in \mathcal{P}_{\text{SIM}}} \mathbb{E}^{\mu}(f(B + \text{Int}(u)) + \frac{1}{2}\|u\|_H^2)\]

\[= \inf_{u \in \mathcal{P}_2} \mathbb{E}^{\mu}(f(B + \text{Int}(u)) + \frac{1}{2}\|u\|_H^2).\]
Lower Bound Proof.

Infimum in the D-V formula attained at $\gamma^0$, with
\[ d\gamma^0 = c \exp(-f(B))d\mu. \]
I.e.
\[ -\log \mathbb{E}^\mu(\exp\{-f(B)\}) = \left( R(\gamma^0 \parallel \mu) + \mathbb{E}^{\gamma^0}(f(B)) \right). \]

Define a martingale $L(t) = E\left( \frac{d\gamma^0}{d\mu} \mid \mathcal{F}_t \right)$. Martingale repn. theorem gives, for some $v \in \mathcal{P}_2$

\[ L(t) = 1 + \int_{[0,t] \times \mathcal{O}} v(s, x)dW(s, x) 
= 1 + \int_{[0,t] \times \mathcal{O}} L(s)u(s, x)dW(s, x). \]

So $L(t) = \exp\left\{ \int_{[0,t] \times \mathcal{O}} u(s, x)dW(s, x) - \frac{1}{2} \int_{[0,t] \times \mathcal{O}} |u(s, x)|^2dsdx. \right\}$

Thus $\gamma^0 = \gamma^u$. Also: $\tilde{B}^u = B - \text{Int}(u)$ is a Brownian sheet under $\gamma^0$. 

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Lower Bound Proof (ctd.)

\[- \log \mathbb{E}^{\mu}(\exp\{-f(B)\}) \quad = \quad \mathbb{E}^{\gamma^u}(f(\tilde{B}^u + \text{Int}(u)) + \frac{1}{2}\|u\|^2_H) \]

\[\geq \quad \inf_{u \in \mathcal{P}_2} \mathbb{E}^{\gamma^u}(f(\tilde{B}^u + \text{Int}(u)) + \frac{1}{2}\|u\|^2_H) \]

\[\geq \quad \inf_{u \in \mathcal{P}_{\text{SIM}}} \mathbb{E}^{\gamma^u}(f(\tilde{B}^u + \text{Int}(u)) + \frac{1}{2}\|u\|^2_H) \]

\[= \quad \inf_{u \in \mathcal{P}_{\text{SIM}}} \mathbb{E}^{\mu}(f(B + \text{Int}(u)) + \frac{1}{2}\|u\|^2_H) \]

\[\geq \quad \inf_{u \in \mathcal{P}_2} \mathbb{E}^{\mu}(f(B + \text{Int}(u)) + \frac{1}{2}\|u\|^2_H). \]
Extensions.

Proof of LDP based on a Variational Representation (B.-Dupuis(2000)):

Let $f : \mathbb{C} \to \mathbb{R}$ be a bounded measurable map. Let $B$ be a Brownian sheet. Then

$$- \log \mathbb{E}(\exp\{-f(B)\}) = \inf_{u \in \mathcal{P}_2} \mathbb{E} \left( \frac{1}{2} ||u||_H^2 + f(B + \text{Int}(u)) \right).$$

A similar repn. now available for general Poisson Random measures (PRM) $N$ (B.-Dupuis-Maroulas(2009)):

$$- \log \mathbb{E}(\exp\{-f(N)\}) = \inf_{\phi \in \mathcal{A}} \mathbb{E} \left( L(\phi) + f(N^\phi) \right).$$

- A different repn obtained in Zhang(2009) - not suitable for large deviation applications.
More generally, can write a similar repn. for functionals of 
$$(W, N) \equiv (BS, PRM)$$. 

- large deviation applications - infinite dimensional jump-diffusions. 
- Another application of repn.: Asymptotics of a large number of interacting (jump) diffusions.