Thresholds for Emmergence of Multi-scale Behavior for the Contact Process on Networks

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Contact process on a network

Let $G = (V, E)$ be a graph.

- Each node is in one of two states: Infected (occupied) or Susceptible (un-occupied)
- Infected sites become susceptible at rate 1
- Susceptible sites become infected at rate λ times the number of infected neighbors

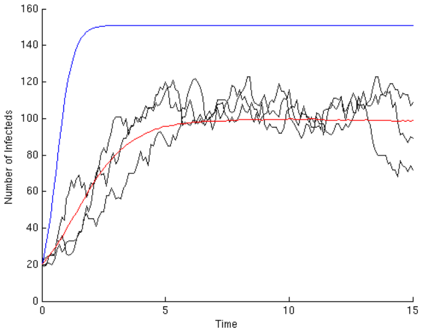
λ is the infection rate across a single edge

Strong convergence to ODE on K_n

If $G = K_n$, the stochastic process converges strongly to compartmental model as $n \longrightarrow \infty$

$$
\begin{aligned}
\frac{dI}{dt} &= \tilde{\lambda} S I - I \\
\frac{dS}{dt} &= -\tilde{\lambda} S I + I
\end{aligned}
$$

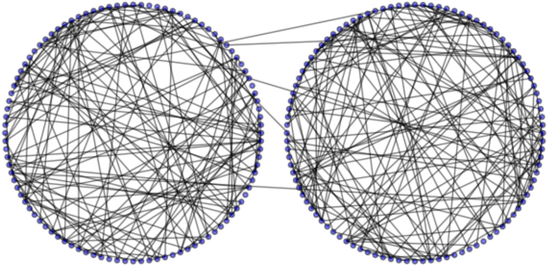
If $G \neq K_n$, the ODE model is an OK approximation if the degree distribution is tight about its mean and there are no bottlenecks.



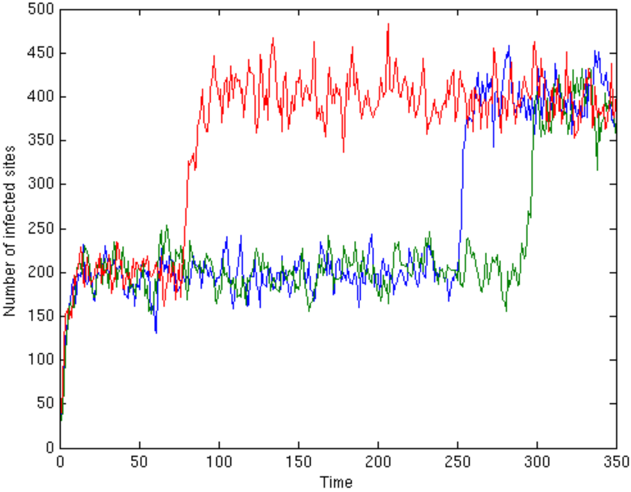
Red: Average of 1000 stochastic realizations
Black: Three sample paths, $\lambda = 2$, G is ER random graph $n = 200, p = .5$.
Blue: Solution to SIS ODE, $N = 200$, $\tilde{\lambda} = 2/98$ rescaled by mean degree of G

Two scale toy network

Bottle-neck: What happens if we start an infection on the left side?



Multi-scale behavior in number of infecteds



Time vs Number of occupied sites. 3 realizations on two-scale block model

Stochastic block model

Modeling bottlenecks, and for now restrict to 2 clusters

- Let $m = 2$ be the number of clusters
- Let n_1, n_2 be the number of nodes in each clusters
- Let p_i be the probability of an edge inside cluster i.
- Let p_{12} be the probability of an edge between clusters 1 and 2

Bottleneck: $p_{ij} \ll p_i$

Questions:

- How strong must clustering be to see multi-scale behavior? Want conditions on n_1, n_2, p_1, p_{12} and λ.
- If there is "stair-step" in number of infecteds, what is distribution of jump times from Cluster 1 to Cluster 2?

Question 2: Distribution of jump times for 2 clusters

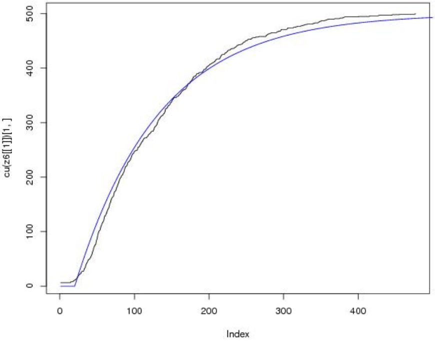
Idea: Times between infection events are exponential AND events *within* clusters happen much more often than events *between* clusters. So pretend to invoke ergodicity and say the fraction of time a node is infected is the same as the fraction of infected nodes at equilibrium.

- Guess: rate at which second cave is infected is roughly exponential with parameter

$$B\lambda \left(1 - \frac{1}{\lambda n_1 p_1}\right)\left(1 - \frac{1}{\lambda n_2 p_2}\right)$$

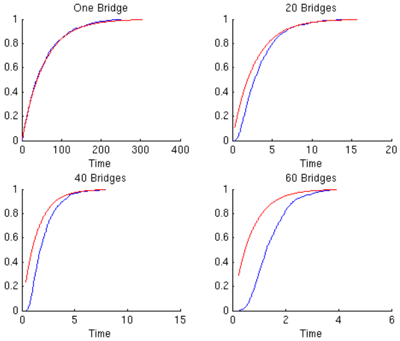
- B: Expected number of bridges between clusters: $B = n_1 n_2 p_{12}$
- λ: Infection rate across one edge
- $1 - \frac{1}{\lambda n_1 p_1}$: Equilibrium number of infected sites in cluster 1
- $1 - \frac{1}{\lambda n_2 p_2}$: Prob. of survival of contact process in cluster 2

How good is the guess?



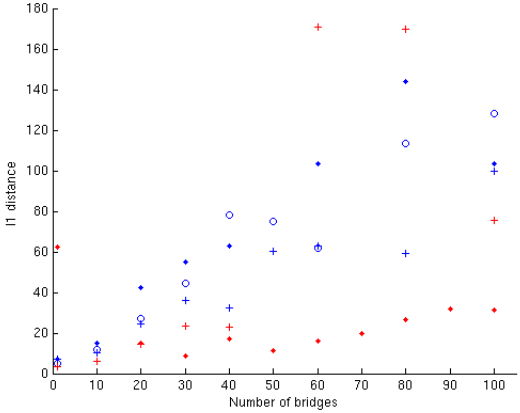
BLACK: “Empirical” CDF of hitting times for 500 simulations
Blue: Exponential CDF with parameter from previous slide

But less good as bridges increase



Infection started in one cave. CDFs of hitting times for the second cave. $N = 200, p = .5, \lambda = (5/2)/(Np)$

Blue: Crosses, $\lambda = (5/3)/(Np)$; Circles, $\lambda = 2/(Np)$; Dots, $\lambda = (5/2)/(Np)$ Red: Crosses, $\lambda = (10/7)/(Np)$; Dots, $\lambda = (5/4)/(Np)$



What can we prove?

For dense Erdos-Renyi random graph

- Probability of survival of contact process from single infected
- From single infected, time to equilibrium is $O(logN)$. (couple with branching process)
- In equilibrium, distribution of infecteds "looks like a product measure"

(with Rick Durrett and David Sivakoff)

Future Concerns

- Proofs (sharp bounds on multi-scale behavior)
- Coupling with meta-population models
- 3 clusters: 2 are supercritical, but there's a subcritical one in the middle