Key ideas

1. Random walks on the integers
2. Recurrence, transience, Markov property
3. Scaling: laws of large numbers, CLT, iterated logarithm
4. Definition of Brownian motion
5. Sample path properties of Brownian motion
6. Brownian motion as a scaling limit of a random walk
What is a *random walk*?

A *random walk* is a walk where the direction and/or size of each step is chosen at random.

Random walks are sometimes evocatively known as “drunken strolls.”

Mathematically, we think of a random walk as a collection of *random variables* which describe the size and direction of each step the walker takes.

Suppose you left your favorite restaurant and decided to move one step to the north, south, east, or west, with the direction chosen at random. Where might you end up after 500 such steps?
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Suppose you left your favorite restaurant and decided to move one step to the north, south, east, or west, with the direction chosen at random. Where might you end up after 500 such steps?
A random variable is a function which depends on the outcome of some experiment: for instance, the number of heads in three tosses of a coin.

In general, the particular value a random variable will take cannot be known for certain until the experiment is performed.

However, the outcomes of the experiment may exhibit some statistical regularity—for instance, if a fair coin is tossed many times, we expect to see heads about half the time.

The probability $P$ of a given outcome is the proportion of times that outcome is observed in a large number of trials of the experiment.
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The probability $P$ of a given outcome is the proportion of times that outcome is observed in a large number of trials of the experiment.
Let $X$ denote a random variable and $x$ a particular value that $X$ might assume.

We define the **cumulative distribution function of $X$** $F(x)$ as

$$F(x) = P(X \leq x).$$

If $X$ can only assume discretely many values, we define the **probability mass function** $p(x)$ as

$$p(x) = P(X = x).$$

Many random variables have a continuum of possible values. In such cases, there may be a function $f(t)$ so that

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Expected values, variance, and independence

For a random variable $X$ with a density $f$, the expected value of $X$ is

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx.$$  

The variance of $X$ is $V(X) = E([X - E(X)]^2)$.  

Two random variables with the same distribution functions are identically distributed.

Two random variables $X$ and $Y$ are independent if for any two sets of possible outcomes $A$ and $B$,

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A continuous random variable is called Gaussian or normal with parameters $\mu$ and $\sigma$ if it has a density given by

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

The expected value of $X$ is $\mu$, and the standard deviation $\sqrt{V(X)}$ is $\sigma$. 
In the words of the famed physicist Lord Kelvin:

“A mathematician is someone to whom
\[ \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, dx = \sqrt{2\pi} \]

is as obvious as 2 + 2 = 4 is to you and me.”
Starting at 0, suppose you toss a fair coin. If heads, take a step from 0 to +1 (to the right); if tails, take a step from 0 in −1 (to the left).

Let $S_n$ denote your position after $n$ such steps.

1. How is $S_n$ distributed—who are you likely to be after $n$ steps?
2. How far will you have moved on average?
3. What is the probability of coming back to zero (i.e., where you started)?
Symmetric random walk on $\mathbb{Z}$

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Some calculations

First, $S_n = \sum_i^n X_i$, where each $X_i = \pm 1$ and $X_i$ are i.i.d. with $E(X_i) = 0$, $V(X_i) = 1$.

Second, each path is a sequence of +1s and -1s; If $S_n = k$, we need to have $k$ more +1s than -1s. To count the number of such paths, we need to count the number of ways to choose $(n + k)/2$ objects from a set of $n$ objects: $[n \text{ C } (n + k)/2]$.

Each step to the right occurs with probability $p$, to the left with $q$; we get

$$\left( \frac{n}{(n+k)/2} \right) p^{n+k/2} q^{n-k/2}$$

which is especially simple in the symmetric case $p = q = 1/2$. 
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Random walks and Brownian motion
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What’s the average value of $S_n$? Since the walk is symmetric, $E(S_n) = 0$.

What about the probability of eventual return to the origin?

Let $p^n_{00}$ be the probability of returning to the origin in $n$ steps.

Claim. If $\sum_1^{\infty} p^n_{00}$ diverges, then the random walk returns to the origin in finite time with probability one, and conversely.
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To calculate $p_{00}^n$: // Note that $p_{00}^n = 0$ if $n$ is odd.

$$p_{00}^{2n} = \binom{2n}{n} p^n q^n.$$ 

Stirling’s formula says $n! \sim \sqrt{2\pi n} (n/e)^n$ for large $n$, and if $p = q = 1/2$, then we get

$$p_{00}^{2n} > \frac{1}{2\sqrt{2\pi n}},$$

so that $\sum_1^\infty p_{00}^n$ diverges.
Let $T$ be the random time of first return to the origin. For the symmetric random walk, we have seen that

$$P(T < \infty) = 1.$$ 

However, for the asymmetric random walk, this is not the case!

We say that a random walk is *recurrent* if the first return time to any given position is finite with probability one. Otherwise, we say the random walk is *transient*.

The symmetric random walk on $\mathbb{Z}$ is recurrent; the asymmetric one is not.
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The symmetric random walk on $\mathbb{Z}$ is recurrent; the asymmetric one is not.
If \( p > q \), what can we expect about the asymmetric random walk?

**Theorem** (The strong law of large numbers, SLLN). Let \( X_i, i = 1, \cdots, n \), be i.i.d random variables with \( E(X_i) = 0, V(X_i) = 1 \). Then with probability one,

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\lim_{n \to \infty} \frac{S_n}{n} = 0.
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**Theorem** (The Central Limit Theorem, CLT). Let \( X_i \) be as above. Then

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\lim_{n \to \infty} P \left( \frac{S_n}{\sqrt{n}} \leq x \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt
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The strong law of large numbers (SLLN) says that the sample mean \( \frac{S_n}{n} \) converges to the true mean with probability one.

The Central Limit Theorem (CLT), on the other hand, says that if we scale differently—i.e. look at \( S_n/\sqrt{n} \), we get a non-deterministic limit—indeed, the sequence of random variables looks approximately like a Gaussian random variable.

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What about when the scaling is “in-between” that of the SLLN and the CLT?

One such result is the

**Theorem (The law of the iterated logarithm).** Let $X_i$ and $S_n$ be as before. Then with probability one,

$$\limsup_{n \to \infty} \frac{S_n}{\sqrt{n \log(\log n)}} = \sqrt{2}.$$

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The strong law illustrates why the asymmetric random walk is transient: $S_n$, our position after time $n$, looks approximately like $nE(X_i)$, and $E(X_i)$ is either positive or negative in the asymmetric case.

Furthermore, even in the symmetric case, it can be shown that

*Theorem.* A symmetric random walk in one or two dimensions is recurrent; in higher dimensions it is transient.

Moral of the story: get drunk only in two dimensions.
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Our random walk also has the *Markov property*—your position at time $n + 1$ depends only on your position at time $n$, and *not on what path you took to get there*.

Furthermore, the *increments* $S_n - S_{n-1}$ are independent.

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Both these properties figure prominently in the behavior of Brownian Motion.
The random walk $S_n$ was a discrete-time, discrete-space stochastic process—a sequence of random variables indexed by some set. The indexing set for $S_n$ is the positive integers.

We can also consider continuous-time stochastic processes—a collection of random variables $X_t$ indexed by an interval $[0, T]$, say, or the whole real line, which take their values in $\mathbb{R}$.

It’s helpful to think of a continuous-time stochastic process as a random trajectory, i.e. a random choice of a function from $[0, T]$ to $\mathbb{R}$. 
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Brownian motion is named after physicist Robert Brown, who observed molecules moving and colliding with one another in random fashion. The formal mathematical construction of such a process is due, in part, to Norbert Wiener.

Definition. A stochastic process $B_t$ with index set $\mathbb{R}$ is called a Wiener process or Brownian motion if the following conditions hold:

1. $B_t$ is a continuous function with probability one;
2. The increments $B_t - B_s$ are independent and have Gaussian distribution with mean zero and variance $t - s$;
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Brownian motion/Wiener process
Some properties of Brownian motion

1. The trajectories are *continuous* with probability one, but are *nowhere differentiable*!

2. Brownian motion has a *fractal* or *self-similarity* property: for any $c > 0$, $\tilde{B}_t = \frac{1}{\sqrt{c}} B_{ct}$ is also Brownian motion.

3. The zeroes of Brownian motion form a *nowhere dense* and *perfect* set (like the Cantor set).

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4. Brownian motion satisfies the Markov property.
Despite the path-wise complexities, it is possible to define an integral with respect to Brownian increments—and this is the basis for stochastic calculus, which is used to model an extensive array of natural phenomena:

1. Fluctuations of stock prices;
2. Random perturbations of physical systems;
3. Mechanisms of disease transmission;

...just to name a few!
Actually constructing Brownian motion, however, presents considerable technical difficulties:

First, how do we construct a stochastic process that satisfies the requirements of independent increments, normality, and continuity? Is there a way to connect this to something more concrete?

Indeed: one can think of Brownian motion as a certain kind of limit of a random walk.
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**Theorem.** Brownian motion can be constructed as a scaling limit of a symmetric random walk:

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W_t = \lim_{n \to \infty} \frac{1}{\sqrt{n}} \sum_{1}^{\lfloor nt \rfloor} X_i
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where \(X_i\) are i.i.d, mean-zero, variance-one random variables, and the limit is convergence in distribution.

The central limit theorem makes this plausible!

Imagine a random walk in which you take steps more and more frequently.
The punchline: Brownian motion as a random-walk limit

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The central limit theorem makes this plausible!

Imagine a random walk in which you take steps more and more frequently.
A simple one-dimensional random walk can be the basis for complicated stochastic processes—different scalings lead to quite different limits.

Thank you!