Rare-event Analysis and Simulations for Gaussian and Its Related Processes

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Overview

- Gaussian process and its related functions: supreme, general convex functions, more complicated structured functions.
- Asymptotic analysis
- Rare-event simulations

Gaussian Random Field

- Probability space $(\Omega, \mathcal{F}, P)$
- $f : T \times \Omega \to \mathbb{R}$, $f(t, \omega)$, short form: $f(t)$.
- $(t_1, ..., t_n) \subset T$, $(f(t_1), ..., f(t_n))$ is a multivariate Gaussian random vector.

Interesting quantities

- The tail probabilities of functions of $\Gamma(f(\cdot))$
- The supremum norm

$$\Gamma(f) = \sup_{t \in T} f(t)$$

- General convex functions, for instance,

$$\Gamma(f) = \int_{t \in T} e^{f(t)} dt$$

- Solutions to differential equations with coefficients driven by $f(t)$.

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The analysis

- Bounds and asymptotic bounds
- Asymptotic approximations
  - Tail probability

$$\lim_{b\to\infty} \frac{P(\Gamma(f) > b)}{\alpha(b)} = 1, \qquad \lim_{b\to\infty} \frac{\log P(\Gamma(f) > b)}{\log \alpha(b)} = 1$$

  - Local results: approximations of the density functions, $g_\Gamma(x)$
- Simulation of the tail probability
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A brief summary of the results

	Approx.	Sim.	Cond. Dist.
$\sup_T f(t)$	A lot	A lot	limited
$\int_T e^{f(t)} dt$	limited	limited	very limited
SPDE	very limited	very limited	???

Asymptotic approximations of $P(\Gamma(f) > b)$

Asymptotic Analysis of $\Gamma(f) = \sup_T f(t)$

- Logarithmic approximation

$$\lim_{u\to\infty} -\frac{\log P(\sup_T f(t) > u)}{u^2} = \frac{1}{2\sup_T \sigma^2(t)}.$$

- Sharp asymptotics under regularity conditions

$$P(\sup_T f(t) > u) = (1 + o(1)) \times C(T) \times u^{\beta} \times P(Z > u)$$

- Cramer and Leadbetter (1967), Pickands (1969), Adler (1981), Sun (1993), Piterbarg (1995), Azais and Wschebor (2005).

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Asymptotic Analysis of $\Gamma(f) = \int_T e^{f(t)} dt$ (L. 2011)

- f is homogeneous.
- $f \in C^3(T)$.

Then,

$$P\left(\int_T e^{\sigma f(t)} dt > b\right) = (1 + o(1)) \cdot H \cdot mes(T) \cdot u^{d-1} \exp(-u^2/2),$$

as $b \to \infty$, where $mes(T)$ is the Lebesgue measure of T and u is the solution to $\left(\frac{2\pi}{\sigma}\right)^{d/2} u^{-d/2} e^{\sigma u} = b$.

The connections

- $P(\sup_t f(t) > u) \approx P(\int_T e^{f(t)} dt > b)$
- if $\left(\frac{2\pi}{\sigma}\right)^{d/2} u^{-d/2} e^{\sigma u} = b$.

With a varying mean function (L. and Xu 2011)

With $\mu(t)$ being the mean function, then

$$
\begin{aligned}
& P\left(\int_T e^{\sigma f(t)+\mu(t)} dt > b\right) \\
& \quad = (1+o(1))\, u^{d-1} \int_T H(\mu, \sigma, t) \cdot \exp\left\{-\frac{\left(u-\sigma^{-1}\mu(t)\right)^2}{2}\right\} dt.
\end{aligned}
$$

$H(\mu, \sigma, t)$ is defined as

$$\frac{|\Gamma|^{-\frac{1}{2}}}{(2\pi)^{\frac{(d+1)(d+2)}{4}}} e^{\frac{\mathbf{1}^T \mu_{22}\mathbf{1}+\sum_i \partial^4_{iiii} C(0)}{8\sigma^2} + \frac{d\cdot\mu_\sigma(t)+Tr(\Delta\mu_\sigma(t))}{2\sigma} + |\partial\mu_\sigma(t)|^2}.$$

$$\times \int_{z\in R^{d(d+1)/2}} e^{-\frac{1}{2}\left[\frac{|\mu_{20}\mu_{22}^{-1}z|^2}{1-\mu_{20}\mu_{22}^{-1}\mu_{02}} + \left|\mu_{22}^{-1/2}z - \frac{\mu_{22}^{1/2}\mathbf{1}}{2\sigma}\right|^2\right]} dz,$$

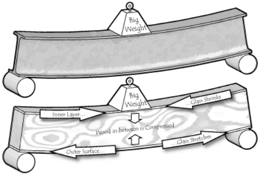
and

$$\mathbf{1} = (\underbrace{1, ..., 1}_{d}, \underbrace{0, ..., 0}_{d(d-1)/2})^\top.$$

Material Failure – one dimensional example

Physical meaning

- $u(x)$: the shape of the material
- $\nabla u(x)$: strain
- $p(x)$: pressure
- $a(x)$: material-specific coefficients



[1]

[1] The picture is published at http://www.guillemot-kayaks.com

Material Failure

- The partial differential equation: $x \in T$

$$\begin{cases} -\nabla \cdot \sigma(x) = p(x) \\ \sigma(x) = a(x)\nabla u(x) \end{cases}$$

- The ordinary differential equation: $x \in [0, 1]$

$$(a(x)u'(x))' = -p(x)$$

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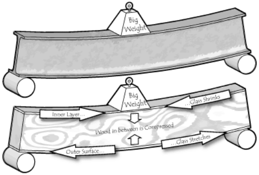

- Composite material characterized by the tensor $a(x)$
- Spatial variation: $a(x) = e^{f(x)}$, where $f(x)$ is a Gaussian process.

Material Failure

- Question: whether and where the material breaks.

The failure probability

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$$P\left(\sup_{x\in T}|\nabla u(x)| > b\right)$$

- The displacement $u(x)$ depends on the process $a(x)$.

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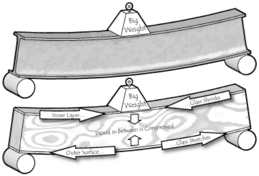
Material Failure – Dirichlet condition



- Dirichlet condition: $u(0) = u(1) = 0$
- The solution:
  $u(x) = \int_0^x F(y)a^{-1}(y)dy - \frac{\int_0^1 F(y)a^{-1}(dy)dy}{\int_0^1 a^{-1}(dy)} \int_0^x a^{-1}(y)dy,$
  where $F(x) = \int_0^x p(y)dy$.

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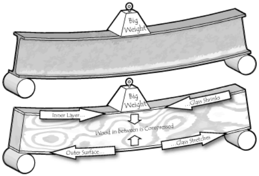

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  where $F(x) = \int_0^x p(y)dy$.

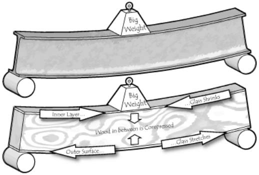
Material Failure – Dirichlet condition

- The strain

$$\begin{aligned} u'(x) &= a^{-1}(x)\left(F(x) - \frac{\int_0^1 F(y)a^{-1}(y)dy}{\int_0^1 a^{-1}(y)dy}\right) \\ &= a^{-1}(x)[F(x) - E_f(F(Y))] \end{aligned}$$

where $a^{-1}(x) = e^{f(x)}$.

The external force



- Delta external force: $p(x) = \delta_{x_*}(x), \quad F(x) = I(x \geq x_*)$.
- Continuous external force $p(x)$: $x_* = \arg\sup_{x \in T} |p(x)|$.

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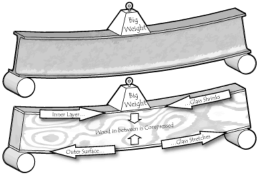

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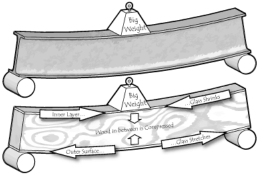
Theorem: approximation of the Delta function (L. and Zhou 2011)

- Homogeneous, mean zero, and $C^3(T)$
- The covariance $C(t) = 1 - \frac{1}{2}t^2 + O(|t|^4)$.
- The external $F(x) = I(x \geq x_*)$, $\qquad p(x) = \delta_{x_*}(x)$.

Use Z to denote a standard normal random variable. Define $H(x) = -\frac{x^2}{2} + \log P(Z \leq x)$, and $\kappa = \sup H(x)$. Let $r = \log b - \kappa$. Then, we have the approximation

$$P\left(\sup_{x\in[0,1]} |u'(x)| > b\right) \sim D \times P(Z > r).$$

Key components of the conditional distribution



- Questions about the conditional distribution
  - Where does the break occur or $\arg\sup u'(x) = ?$
  - Where does $f(x)$ attain it maximum?
  - At what level?

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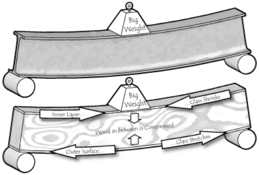

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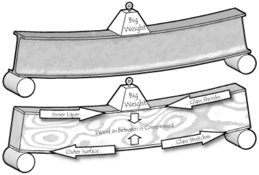

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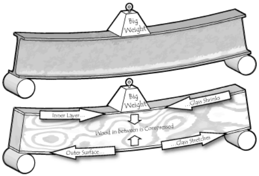

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Theorem: approximation for continuous force (L. and Zhou 2011)

The external force $p(x)$ is a continuously differentiable function. Then, we have the approximation

$$P(\sup_{x\in[0,1]} |u'(x)| > b)$$
$$\sim P(|u'(0)| > b) + P(|u'(1)| > b) + P(\sup_{|x-x_*|<\varepsilon} |u'(x)| > 0).$$

Exact asymptotic approximation for continuous body force

- Let $p(x_*) r^{-1} e^{r-\frac{1}{2}} = b$. Then,

$$P(\sup_{|x-x_*|<\varepsilon} |u'(x)| > 0) \sim \kappa_* \times r^{-1/2} \exp\{-r^2/2\}.$$

- Let $H_0 r_0^{-1/2} e^{r_0} = b$. Then,

$$P(|u'(0)| > b) = \kappa_0 \times r_0^{-1} e^{-r_0^2/2}$$

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The density function of $\Gamma(f)$

General results

- Let $\alpha(b)$ an approximation

$$\frac{P(\Gamma(f) > b)}{\alpha(b)} \to 1$$

- The density function of $\Gamma(f)$

$$\frac{g_\Gamma(b)}{\alpha'(b)} \to 1$$

based on bounds of $\alpha''(b)$.

- Bias control of the simulation algorithms.

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Theorem (L. and Xu 2012a)

- $\Gamma(f) = \log \int_T e^{f(t)+\mu(t)} dt$
- $\mu(t) \in C^3(T)$.

Then,

$$g_\Gamma(b) = (1 + o(1)) u^d \int_T \exp\left\{-\frac{(u - \mu(t))^2}{2}\right\} \cdot H(\mu, t) dt,$$

where $(2\pi)^{\frac{d}{2}} u^{-\frac{d}{2}} e^u = e^b \cdot \int_T e^{\mu(t)} dt.$

The computation of $P(\Gamma(f) > b)$

Computation of $P(\Gamma(f) > b)$

- For $\varepsilon, \delta > 0$ fixed

$$P\left(\left|\frac{Z_b}{P(\Gamma(f) > b)} - 1\right| > \varepsilon\right) < \delta$$

- The total complexity of computing Z_b is $O(\log P(\Gamma(f) > b))^n$
- The relative error of the importance sampling estimator is $O(\log P(\Gamma(f) > b))^m$
- The complexity of generating one sample with controlled bias is $O(\log P(\Gamma(f) > b))^m$

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The change-of-measure-based analysis

- Let P be the original measure.
- The change of measure Q

$$\frac{dQ}{dP} = \int_{t \in T} \frac{g_t(f(t))}{\varphi_t(f(t))} h(t) dt$$

where $\varphi_t(x)$ is the marginal density of $f(t)$, $h(t)$ is a density on T, and $g_t(x)$ is an alternative density.

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Simulation from the change of measure

- Simulate $\tau \in T$

$$\tau \sim h(t)$$

- Simulate $f(\tau)$ according to $g_t(x)$
- Simulate $\{f(t) : t \neq \gamma\}$ given $f(\tau)$ under P
- Simulation and computation

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The interpretations

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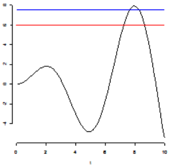
Computing $P(\sup_T f(t) > b)$ (Adler, Blanchet, and L. 2011)

- Random index
  $h(t) = \frac{P(f(t)>b-1/b)}{\int_T P(f(t)>b-1/b)dt} \propto P(f(t) > b - 1/b)$
- The distribution $g_t(x) = \frac{I(x>b-1/b)}{P(x>b-1/b)}\varphi_t(x)$
- The likelihood ratio:

$$\frac{dQ}{dP} = \frac{mes(A_{b-1/b})}{\int_T P(f(t) > b - 1/b)dt}$$

where $A_\gamma = \{t : f(t) > \gamma\}$.



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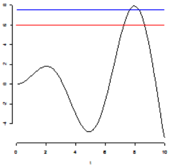

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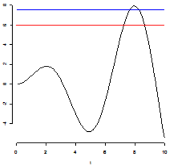
$$\frac{dQ}{dP} = \frac{mes(A_{b-1/b})}{\int_T P(f(t) > b - 1/b)dt}$$

where $A_\gamma = \{t : f(t) > \gamma\}$.



Computing $P(\int_T e^{f(t)}dt > b)$ (L. and Xu 2011, 2012b)

- Random index $h(t) \sim Uniform(T)$
- The distribution

$$g_t(x) \sim N(u, 1)$$

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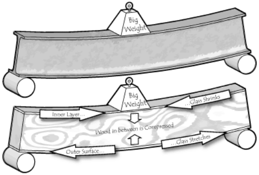

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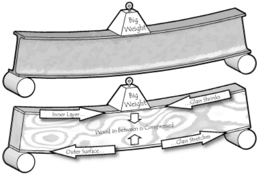

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Summary

- Gaussian process and its related functions:
  - supreme
  - integrals of exponential functions
  - stochastic partial/ordinary differential equations.
- Asymptotic analysis
- Rare-event simulations

Acknowledgement

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