Nonintrusive Polynomial Chaos and Stochastic Collocation Methods for Uncertainty Analysis and Design

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Automated refinement: uniform and dimension-adaptive p- and h-refinement

Gradient enhancement: PCE regression, piecewise Hermite interpolation

Building on UQ foundations: mixed UQ, optimization under uncertainty

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Polynomial Chaos Expansions (PCE)

Approximate response w/ spectral proj. using orthogonal polynomial basis fns

i.e. $$R = \sum_{j=0}^{P} \alpha_j \Psi_j(\boldsymbol{\xi})$$ using

$$\begin{aligned}
\Psi_0(\xi) &= \psi_0(\xi_1)\,\psi_0(\xi_2) = 1\\
\Psi_1(\xi) &= \psi_1(\xi_1)\,\psi_0(\xi_2) = \xi_1\\
\Psi_2(\xi) &= \psi_0(\xi_1)\,\psi_1(\xi_2) = \xi_2\\
\Psi_3(\xi) &= \psi_2(\xi_1)\,\psi_0(\xi_2) = \xi_1^2 - 1\\
\Psi_4(\xi) &= \psi_1(\xi_1)\,\psi_1(\xi_2) = \xi_1\xi_2\\
\Psi_5(\xi) &= \psi_0(\xi_1)\,\psi_2(\xi_2) = \xi_2^2 - 1
\end{aligned}$$

- Nonintrusive: estimate α_j using sampling, regression, tensor-product quadrature, sparse grids, or cubature

$$\alpha_j = \frac{\langle R, \Psi_j \rangle}{\langle \Psi_j^2 \rangle} = \frac{1}{\langle \Psi_j^2 \rangle} \int_{\Omega} R\, \Psi_j\, \varrho(\boldsymbol{\xi})\, d\boldsymbol{\xi}$$

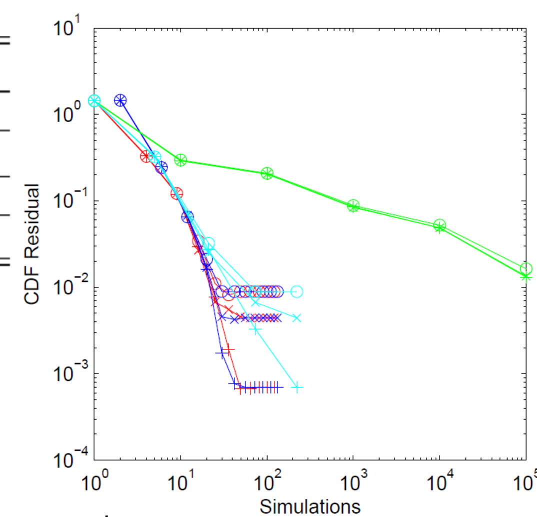
$$\langle \Psi_j^2 \rangle = \prod_{i=1}^{n} \langle \psi_{m_i^j}^2 \rangle$$

Generalized PCE (Wiener-Askey + numerically-generated)

- Tailor basis: selection of basis orthogonal to input PDF avoids additional nonlinearity

Distribution	Density function	Polynomial	Weight function	Support range
Normal	$\frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$	Hermite $He_n(x)$	$e^{\frac{-x^2}{2}}$	$[-\infty, \infty]$
Uniform	$\frac{1}{2}$	Legendre $P_n(x)$	1	$[-1, 1]$
Beta	$\frac{(1-x)^{\alpha}(1+x)^{\beta}}{2^{\alpha+\beta+1}B(\alpha+1,\beta+1)}$	Jacobi $P_n^{(\alpha,\beta)}(x)$	$(1-x)^{\alpha}(1+x)^{\beta}$	$[-1, 1]$
Exponential	e^{-x}	Laguerre $L_n(x)$	e^{-x}	$[0, \infty]$
Gamma	$\frac{x^{\alpha}e^{-x}}{\Gamma(\alpha+1)}$	Generalized Laguerre $L_n^{(\alpha)}(x)$	$x^{\alpha}e^{-x}$	$[0, \infty]$

Additional bases generated numerically (discretized Stieltjes + Golub-Welsch)



- Tailor expansion form:
  - Dimension p-refinement: anisotropic TPQ/SSG, generalized SSG
  - Dimension & region h-refinement: local bases with global & local refinement

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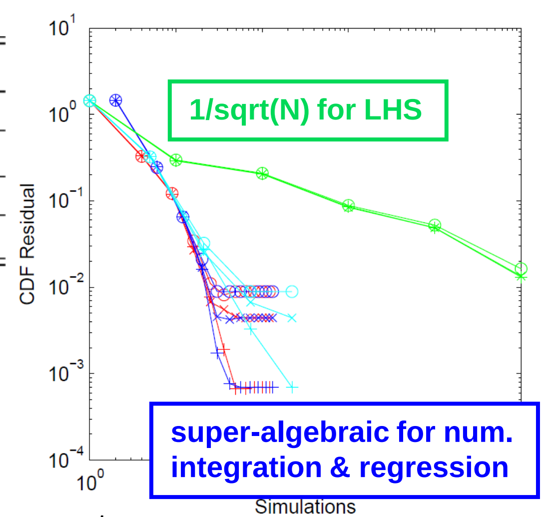

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Stochastic Collocation
(based on interpolation polynomials)

Instead of estimating coefficients for known basis functions, form <u>interpolants</u> for known coefficients

$$R = \sum_{j=1}^{N_p} r_j \boldsymbol{L}_j(\boldsymbol{\xi})$$

- ***Global:*** Lagrange (values) or Hermite (values+derivatives)
- ***Local:*** linear (values) or cubic (values+gradients) splines

$$L_i = \prod_{\substack{j=1 \\ j \neq i}}^{m} \frac{x - x_j}{x_i - x_j}$$

$$R(\boldsymbol{\xi}) \cong \sum_{j_1=1}^{m_{i_1}} \cdots \sum_{j_n=1}^{m_{i_n}} r\left(\xi_{j_1}^{i_1}, \ldots, \xi_{j_n}^{i_n}\right) \left(L_{j_1}^{i_1} \otimes \cdots \otimes L_{j_n}^{i_n}\right)$$

Sparse interpolants formed using Σ of tensor interpolants

Advantages relative to PCE:

- Somewhat simpler (no expansion order to manage separately)
- Often less expensive (no integration for coefficients)
- Expansion only formed for sampling → probabilities (estimating moments of any order is straightforward)
- Adaptive h-refinement with hierarchical surpluses; explicit gradient-enhancement

Disadvantages relative to PCE:

- Less flexible/fault tolerant → structured data sets (tensor/sparse grids)
- Expansion variance not guaranteed positive (important in opt./interval est.)
- No direct inference of spectral decay rates

With sufficient care on PCE form, PCE/SC performance is essentially identical for many cases of interest (tensor/sparse grids with standard Gauss rules)

Approaches for forming PCE/SC Expansions

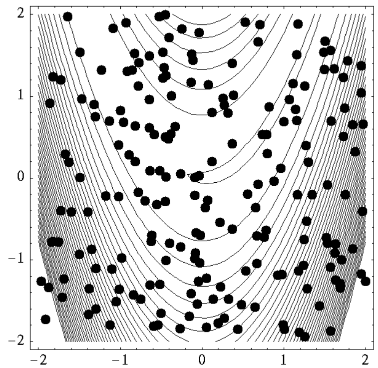
Random sampling: PCE

Expectation (sampling):

- Sample w/i distribution of ξ
- Compute expected value of product of R and each Ψ_j

Linear regression ("point collocation"):

- Sample w/i distribution of ξ
- Solves least squares data fit for all coefficients at once:



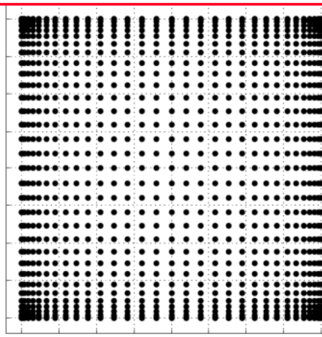
$$\Psi\alpha = R$$

Tensor-product quadrature: PCE/SC

$$\mathscr{U}^i(f)(\xi) = \sum_{j=1}^{m_i} f(\xi_j^i)\, w_j^i$$

$$\mathcal{Q}_{\mathbf{i}}^n f(\xi) = (\mathscr{U}^{i_1}\otimes\cdots\otimes\mathscr{U}^{i_n})(f)(\xi) = \sum_{j_1=1}^{m_{i_1}}\cdots\sum_{j_n=1}^{m_{i_n}} f\left(\xi_{j_1}^{i_1},\dots,\xi_{j_n}^{i_n}\right)\left(w_{j_1}^{i_1}\otimes\cdots\otimes w_{j_n}^{i_n}\right)$$

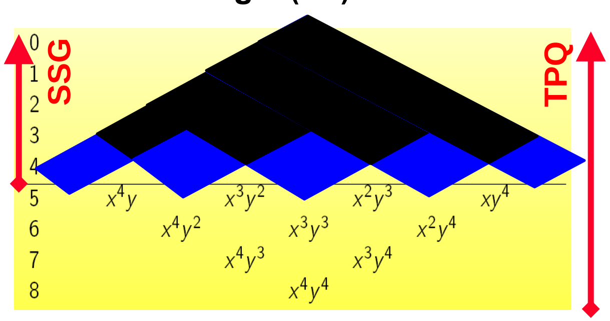
- **Every combination of 1-D rules**
- **Scales as m^n**
- **1-D Gaussian rule of order m → integrands to order $2m-1$**
- **Assuming $R\,\Psi_j$ of order $2p$, select $m = p + 1$**

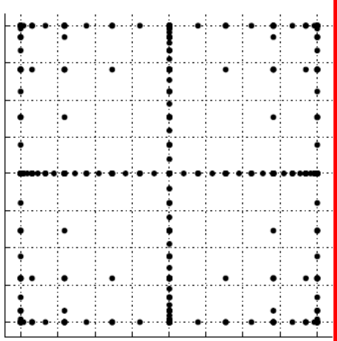


Smolyak Sparse Grid: PCE/SC

$$\mathscr{A}(\mathrm{w},n) = \sum_{\mathrm{w}+1\le|\mathbf{i}|\le\mathrm{w}+n} (-1)^{\mathrm{w}+n-|\mathbf{i}|}\binom{n-1}{\mathrm{w}+n-|\mathbf{i}|}\cdot\left(\mathscr{U}^{i_1}\otimes\cdots\otimes\mathscr{U}^{i_n}\right)$$

Pascal's triangle (2D):



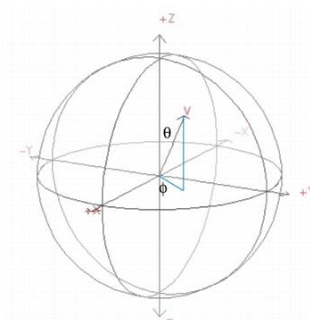


Cubature: PCE

Stroud and extensions (Xiu, Cools)

→ Low order PCE

→ global SA, anisotropy detection



Gaussian $i = 2$ → $p = 1$

$$x_{k,2r-1} = \sqrt{2}\cos\frac{2rk\pi}{n+1}, \qquad x_{k,2r} = \sqrt{2}\sin\frac{2rk\pi}{n+1}$$

Arbitrary PDF

$$t^{(k)} = \frac{1}{\gamma}\left[\sqrt{\gamma c_1}\, x^{(k)} - \delta\right]$$

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PCE/SC Expansions: Smolyak Sparse Grids

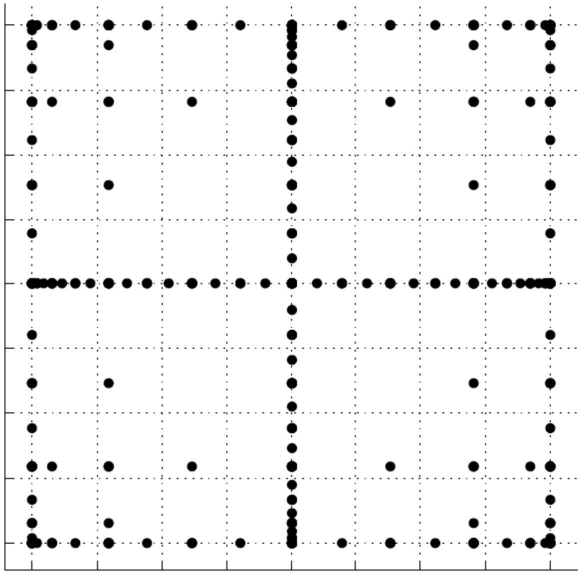
Numerical Integration (Smolyak sparse grids):

$$\mathscr{A}(\mathrm{w},n) = \sum_{|\mathbf{i}| \le \mathrm{w}+n} \left(\Delta^{i_1} \otimes \cdots \otimes \Delta^{i_n}\right)$$

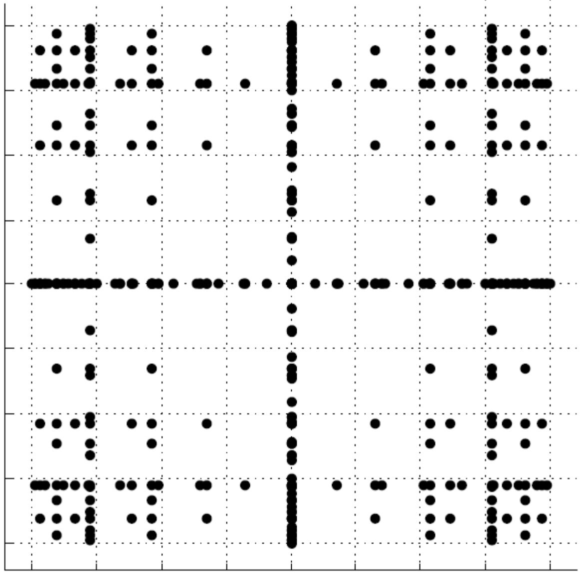
for

$$\Delta^i := \mathscr{U}^i - \mathscr{U}^{i-1}$$

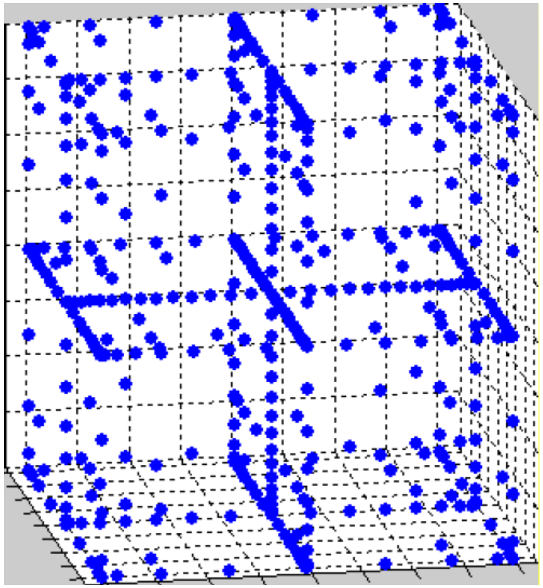
$$\mathscr{A}(\mathrm{w},n) = \sum_{\mathrm{w}+1 \le |\mathbf{i}| \le \mathrm{w}+n} (-1)^{\mathrm{w}+n-|\mathbf{i}|} \binom{n-1}{\mathrm{w}+n-|\mathbf{i}|} \cdot \left(\mathscr{U}^{i_1} \otimes \cdots \otimes \mathscr{U}^{i_n}\right)$$



2D Clenshaw-Curtis sparse grid
(less optimal, more nesting)



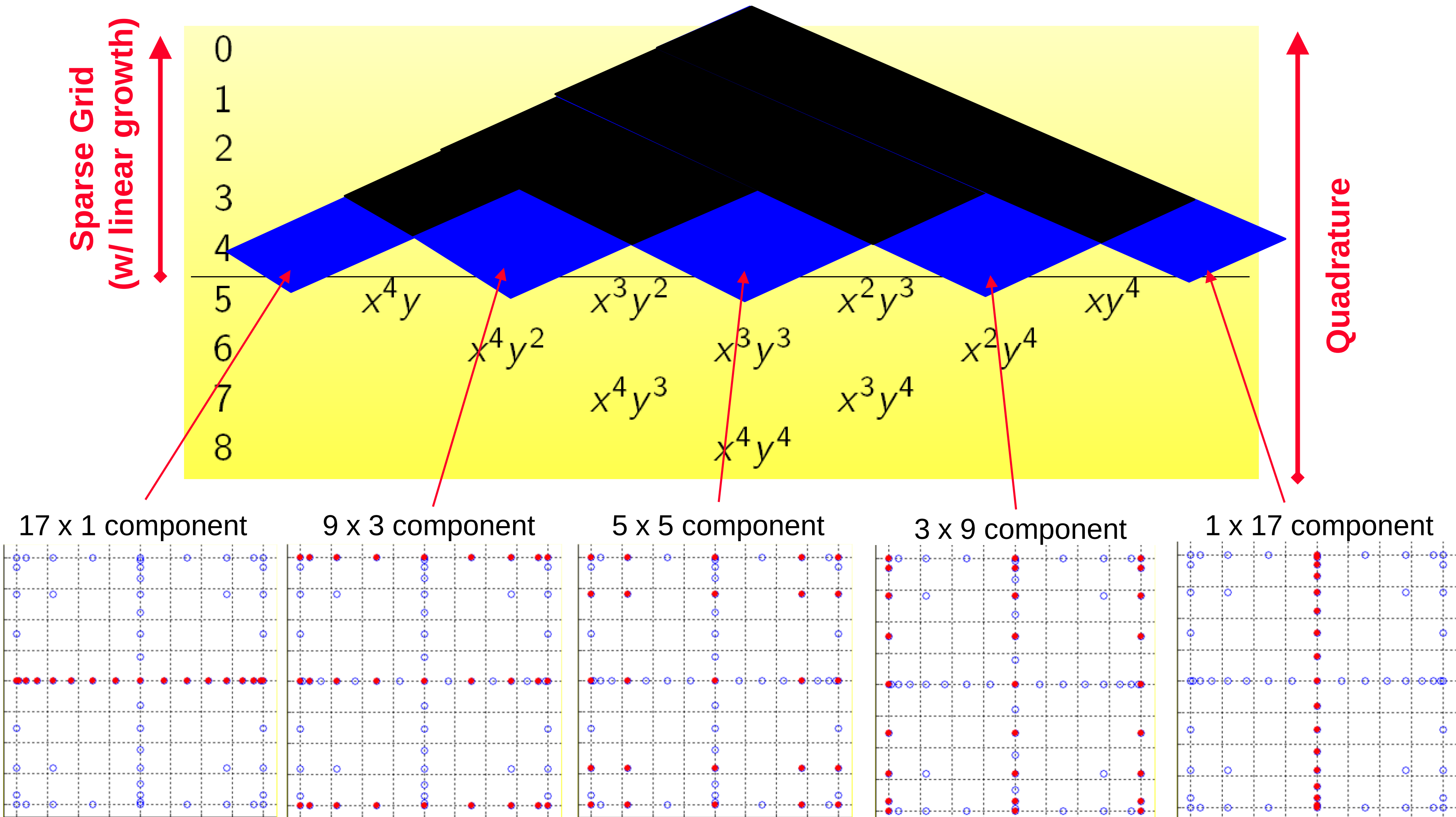
2D Gauss-Legendre sparse grid
(more optimal, less nesting)



3D Clenshaw-Curtis sparse grid

PCE/SC Expansions: Smolyak Sparse Grids (cont.)

Key difference between quadrature & sparse grids: **polynomial order coverage**



From J. Burkardt, "A Low Level Introduction to High Dimensional Sparse Grids."

Adaptive Collocation Methods

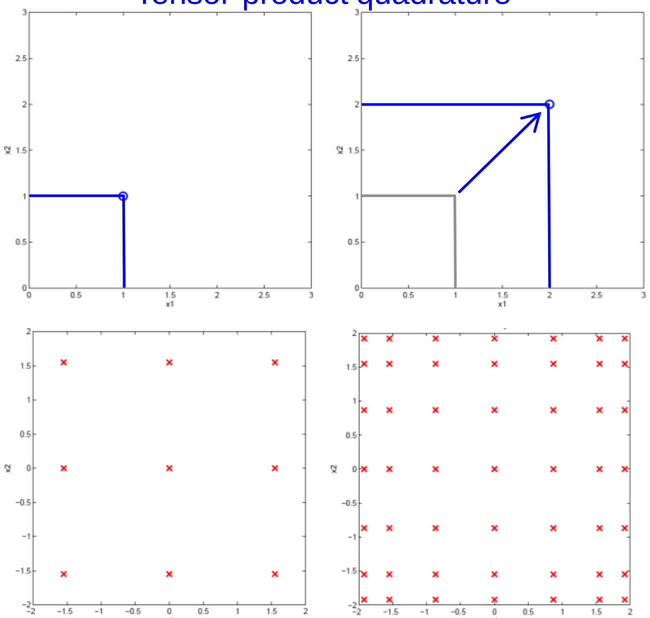
Drivers: Efficiency, robustness, *scalability* → adaptive methods, adjoint enhancement

Polynomial order (p-) refinement approaches:

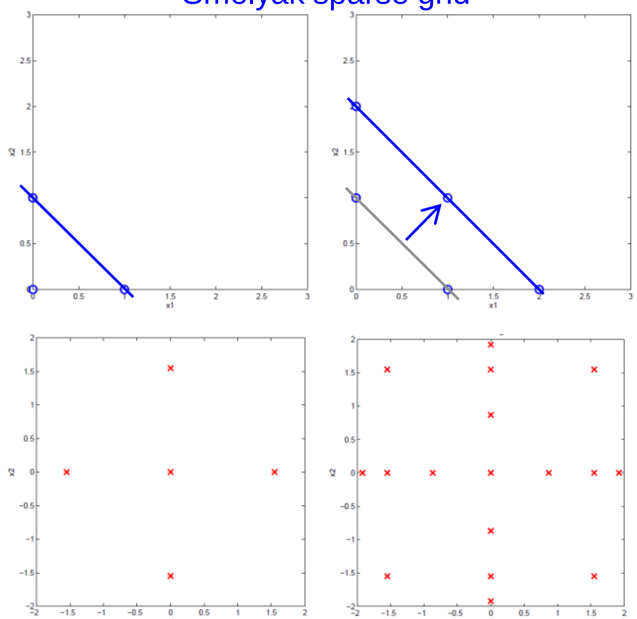
- **Uniform:** *isotropic* tensor/sparse grids
  - *Increment grid:* increase order/level, ensure change (restricted growth in nested rules)
  - *Assess convergence:* L^2 change in response covariance

$$\mathrm{w}+1 \leq |\mathbf{i}| \leq \mathrm{w}+n$$

Tensor-product quadrature



Smolyak sparse grid



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- **Dimension-adaptive:** *anisotropic* tensor/sparse grids
  - **PCE/SC:** variance-based decomp. → total Sobol' indices → anisotropy (dimension preference)

$$
\begin{aligned}
f_{PC}(\boldsymbol{x}) = f_0 &+ \sum_{i=1}^{n} \sum_{\boldsymbol{\alpha}\in\mathcal{I}_i} f_{\boldsymbol{\alpha}} \Psi_{\boldsymbol{\alpha}}(x_i) \\
&+ \sum_{1\le i_1<i_2\le n} \sum_{\boldsymbol{\alpha}\in\mathcal{I}_{i_1,i_2}} f_{\boldsymbol{\alpha}} \Psi_{\boldsymbol{\alpha}}(x_{i_1}, x_{i_2}) + \dots \\
&+ \sum_{1\le i_1<\dots<i_s\le n} \sum_{\boldsymbol{\alpha}\in\mathcal{I}_{i_1,\dots,i_s}} f_{\boldsymbol{\alpha}} \Psi_{\boldsymbol{\alpha}}(x_{i_1}, \dots, x_{i_s}) \\
&+ \dots + \sum_{\boldsymbol{\alpha}\in\mathcal{I}_{1,2\dots,n}} f_{\boldsymbol{\alpha}} \Psi_{\boldsymbol{\alpha}}(x_1, \dots, x_n)
\end{aligned}
$$

$$SU_{i_1\dots i_s} = \sum_{\boldsymbol{\alpha}\in\mathcal{I}_{i_1,\dots,i_s}} f_{\boldsymbol{\alpha}}^2 \,\mathrm{E}\left[\Psi_{\boldsymbol{\alpha}}^2\right] / D_{PC}$$

Main, interaction indices

$$SU^T_{j_1,\dots,j_t} = \sum_{(i_1,\dots,i_s)\in\mathcal{J}_{j_1,\dots,j_t}} SU_{i_1,\dots,i_s}$$

Total indices

Average Sobol indices over response set

$$\mathrm{w}\underline{\gamma} < \mathbf{i}\cdot\gamma \le \mathrm{w}\underline{\gamma} + |\gamma|$$

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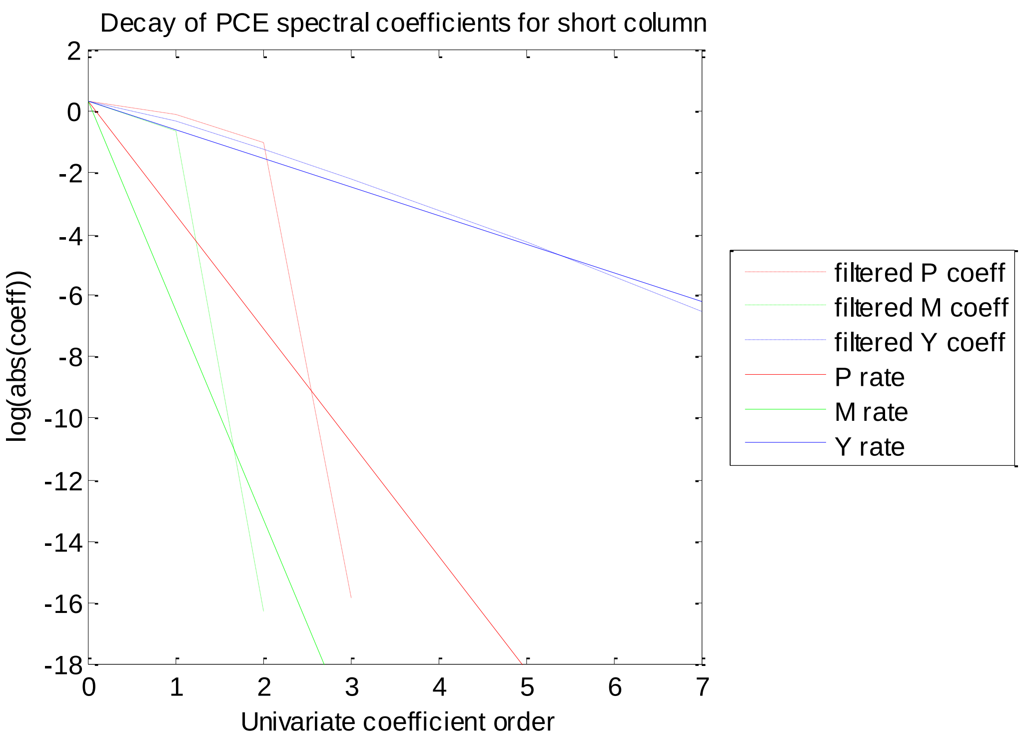

$$\mathrm{w}\underline{\gamma} < \mathbf{i}\cdot\underline{\gamma} \leq \mathrm{w}\underline{\gamma} + |\underline{\gamma}|$$

Min rates over response set

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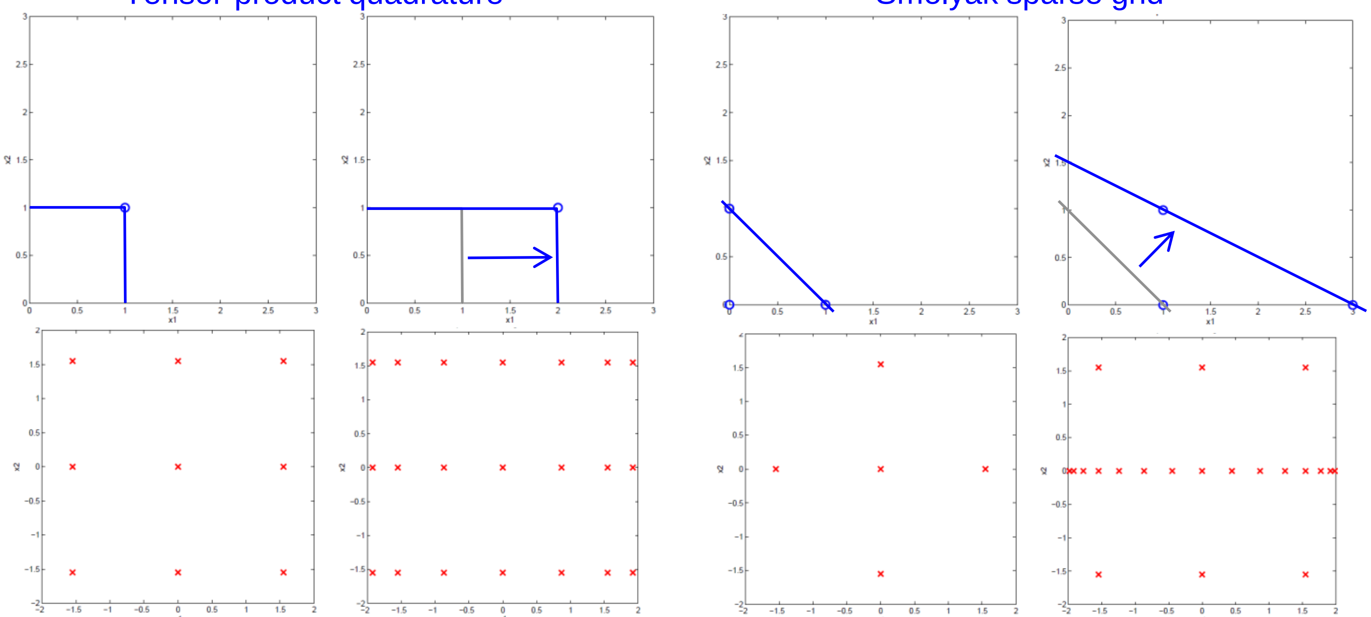

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$$w\underline{\gamma} < \mathbf{i}\cdot\gamma \leq w\underline{\gamma} + |\gamma|$$

  - **PCE/SC:** variance-based decomp. → total Sobol' indices → anisotropy
  - **PCE:** spectral coefficient decay rates → anisotropy
- **Goal-oriented dimension-adaptive:** *generalized* sparse grids
  - **PCE/SC:** change in QOI induced by trial index sets on active front

1. *Initialization:* Starting from reference grid (often w = 0 grid), define active index sets using admissible forward neighbors of all old index sets.

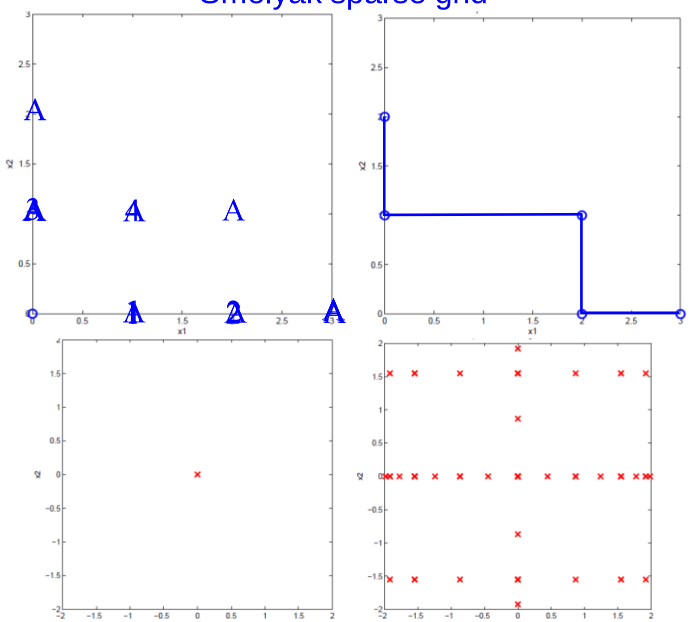
2. *Trial set evaluation:* For each trial index set, evaluate tensor grid, form tensor expansion, update combinatorial coefficients, and combine with reference expansion. Perform necessary bookkeeping to allow efficient restoration.

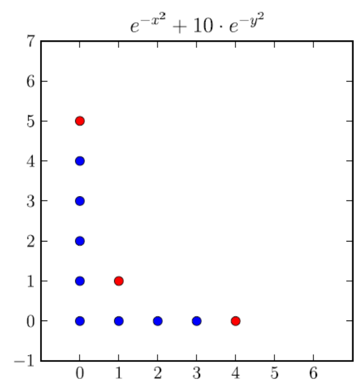
3. *Trial set selection:* Select trial index set that induces largest change in statistical QOI.

4. *Update sets:* If largest change > tolerance, then promote selected trial set from active to old and compute new admissible active sets; return to 2. If tolerance is satisfied, advance to step 5.

5. *Finalization:* Promote all remaining active sets to old set, update combinatorial coefficients, and perform final combination of tensor expansions to arrive at final result for statistical QOI.

Smolyak sparse grid





(Gerstner, 2003)

Fine-grained control: frontier not limited by prescribed shape of index set constraint

Initial Observations

- **Sobol' anisotropic grids:** target largest contributors to response variance
  - Strengths
    - Most broadly applicable technique: PCE or SC, TPQ or SSG
    - Global variance metrics are robust (i.e., reduces non-monotonicity/drop out issues)
  - Weaknesses
    - Dimensions contributing most to variance may be low order (and vice versa → spectral decay)
    - Refinement logic is not entirely self-consistent, since variance increases with resolution (i.e., reduced refinement pressure for under-resolved dimensions; increased for over-resolved)
- **Spectral decay anisotropic grids:** target dimensions where convergence is slow
  - Strengths
    - Refinement logic is more self-consistent: when under-resolved, rate is low → preference is high and when fully resolved (e.g., a polynomial relationship), rate is higher → preference is reduced
  - Weaknesses
    - PCE only (TPQ or SSG)
    - Vice versa: dimensions with higher order relationships not necessarily dominant QOI contributors
    - Non-monotone drop-outs in univariate coefficients may block refinement
- **Generalized sparse grids:** target index sets that are largest contributors to QOI
  - Strengths
    - Unstructured, fine-grained: resolution of interactions not pre-determined by univariate weights
    - Goal-oriented refinement: index set selection governed by general statistical QOI
    - Start from `w` = 0 (don't need to estimate anisotropy; unimportant interactions not explored → scalability)
  - Weaknesses
    - SSG only (PCE or SC), although a greedy TPQ variant could be constructed
    - Non-monotone drop outs at index set level block refinement
    - Increments are smaller than other methods → precision can be an issue → hierarchical basis

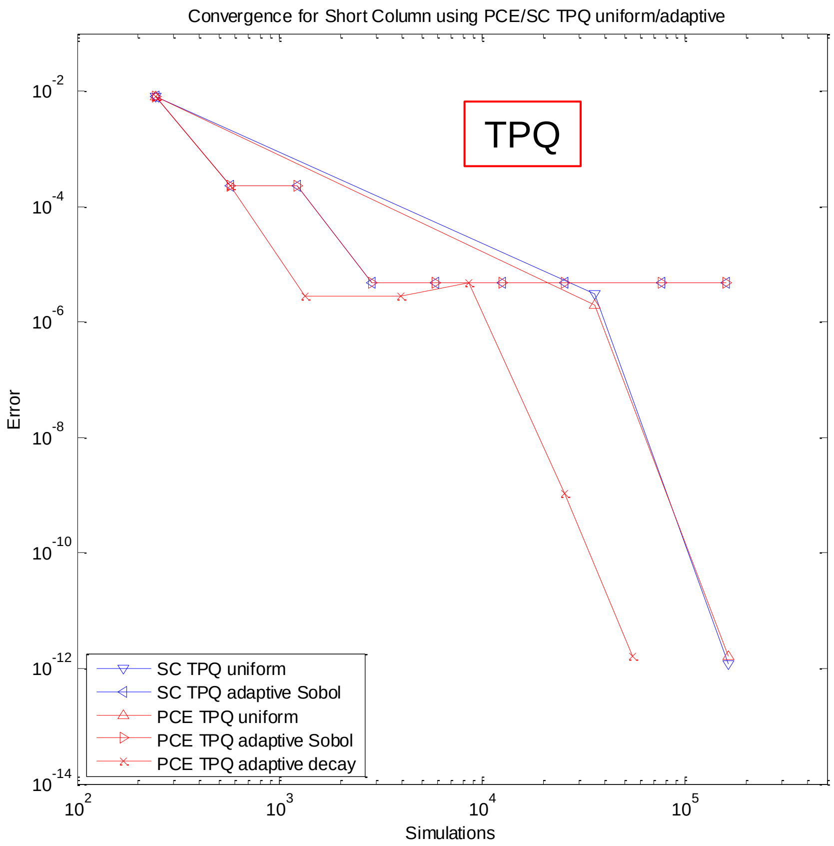
Numerical Experiments
Short Column Test Problem (n=5)

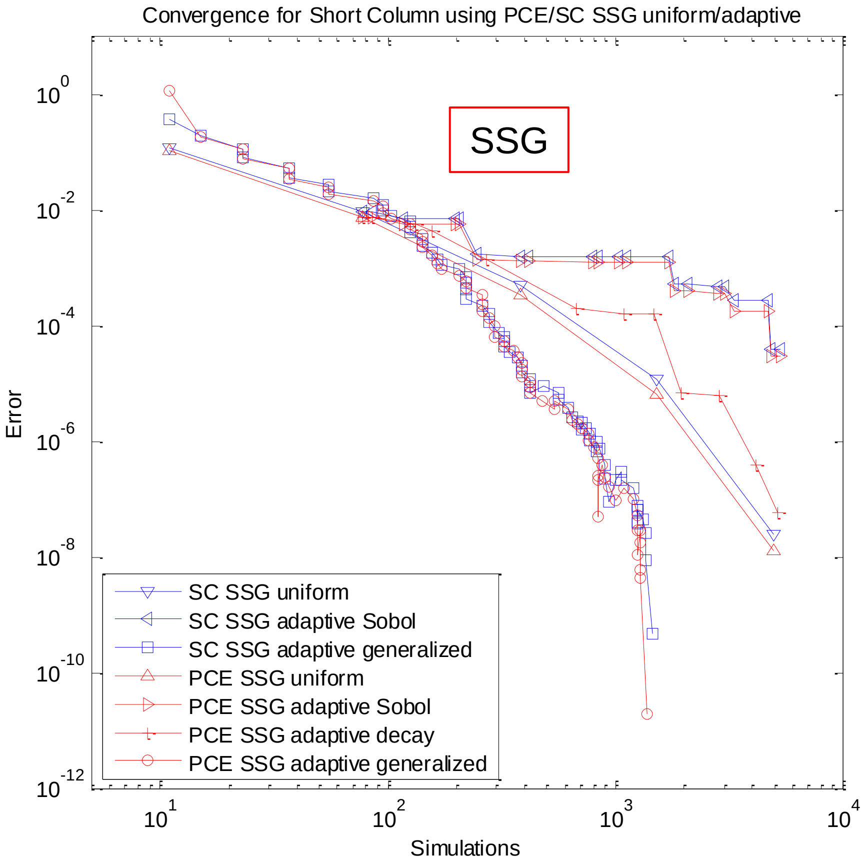
$$g(\mathbf{x}) = 1 - \frac{4M}{bh^2Y} - \frac{P^2}{b^2h^2Y^2}$$

Kuschel & Rackwitz, 1997

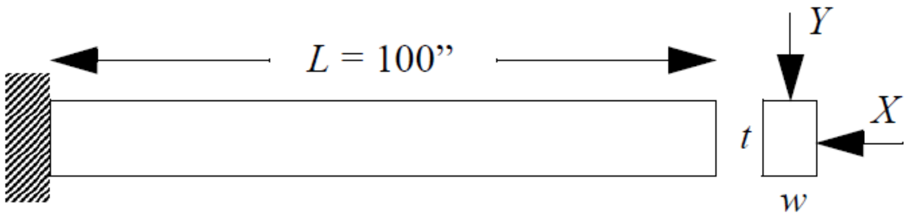
$b = U[5,15]$, $h = U[15,25]$,
$P = N(500, 100)$,
$M = N(2000, 400)$, $\rho_{P,M} = 0.5$,
$Y = logN(5, 0.5)$

$$\begin{aligned} \min \quad & bh \\ \text{s.t.} \quad & \beta \geq 2.5 \\ & 5.0 \leq b \leq 15.0 \\ & 15.0 \leq h \leq 25.0 \end{aligned}$$





Numerical Experiments
Cantilever Beam Test Problem (n=6)

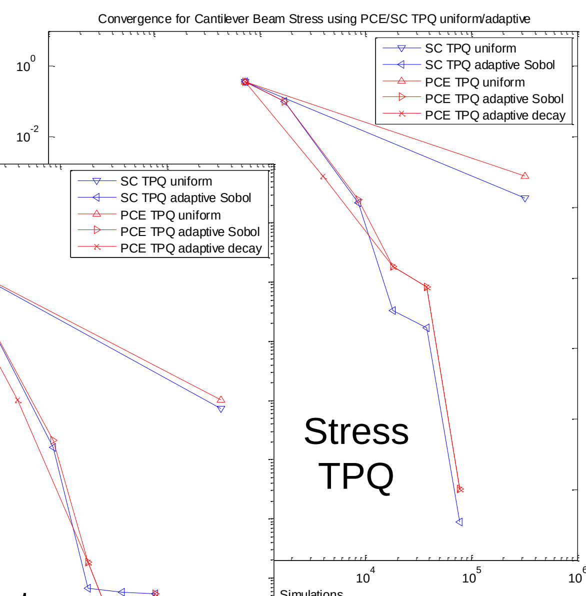


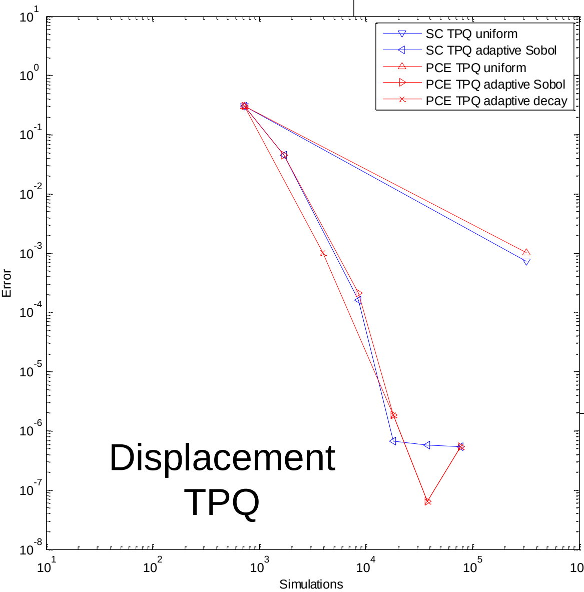
w, t, R, E, X, Y: $U[1,10]$, $U[1,10]$, $N(4E4, 2E3)$, $N(2.9E7, 1.45E6)$, $N(500, 100)$, $N(1E3, 100)$;
$D_0 = 2.2535"$

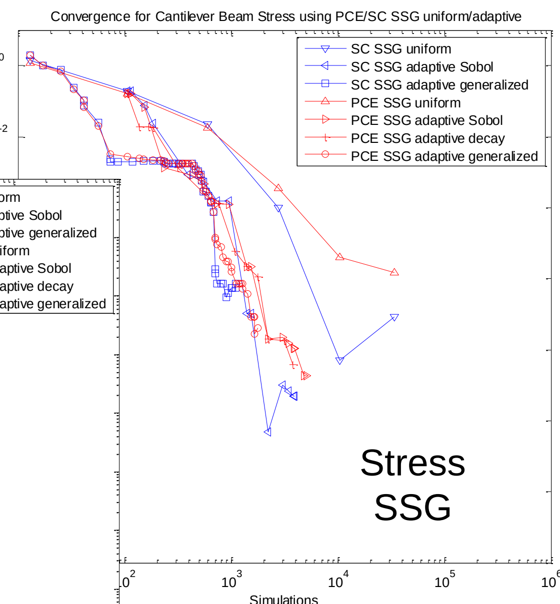
$$S = \frac{600}{wt^2}Y + \frac{600}{w^2t}X \leq R$$

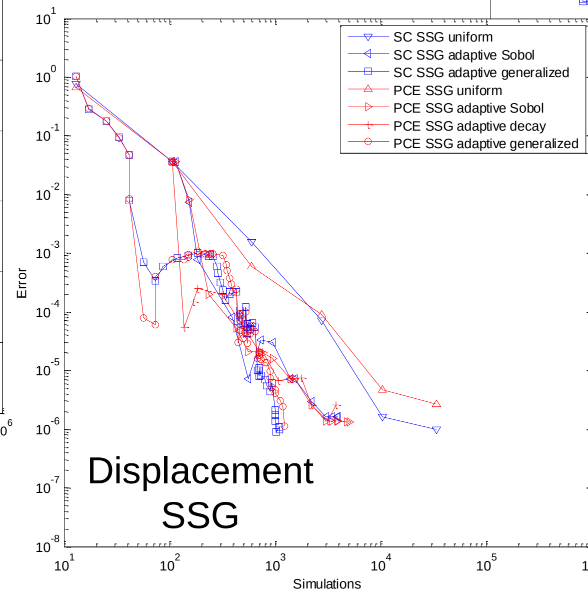
$$D = \frac{4L^3}{Ewt}\sqrt{\left(\frac{Y}{t^2}\right)^2 + \left(\frac{X}{w^2}\right)^2} \leq D_0$$

$$\begin{aligned} \min \quad & wt \\ \text{s.t.} \quad & \beta_S \geq 3.0 \\ & \beta_D \geq 3.0 \\ & 1.0 \leq w \leq 4.0 \\ & 1.0 \leq t \leq 4.0 \end{aligned}$$









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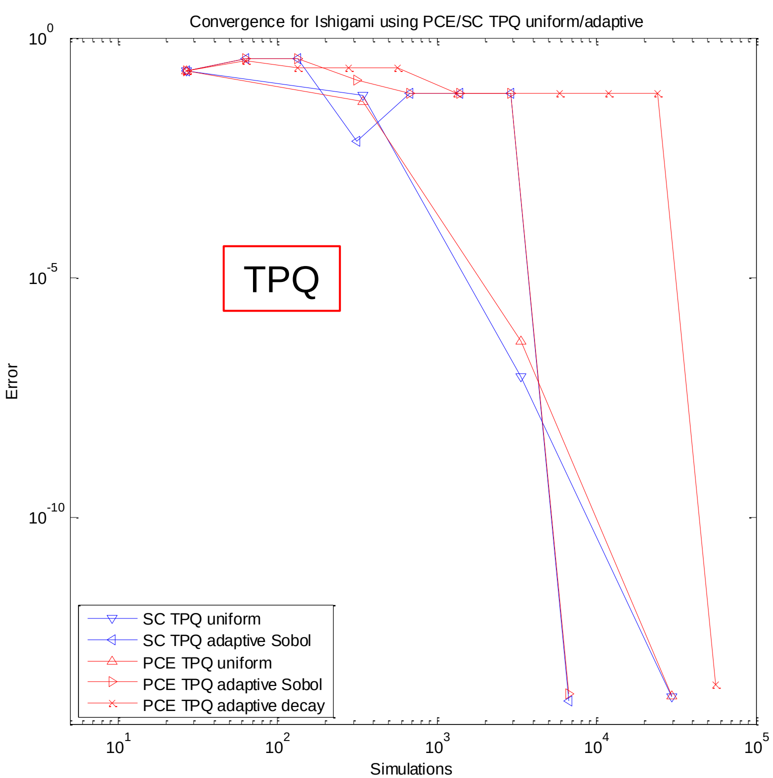
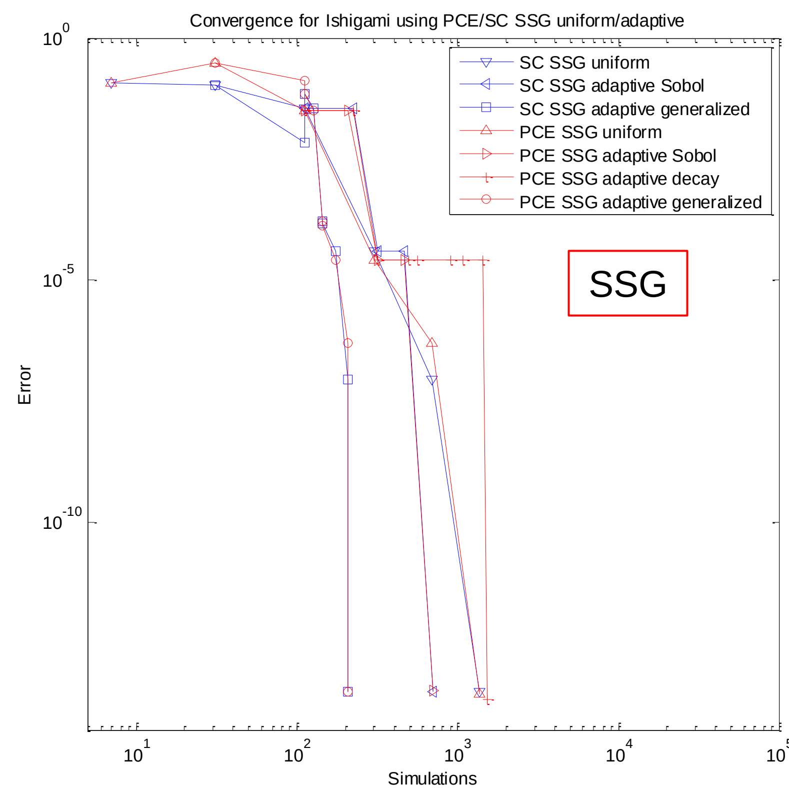
Numerical Experiments
Ishigami Test Problem (n=3)

$$f(\mathbf{x}) = \sin(2\pi x_1 - \pi) + 7\sin^2(2\pi x_2 - \pi) + 0.1(2\pi x_3 - \pi)^4 \sin(2\pi x_1 - \pi)$$

x_1, x_2, x_3: *iid* $U[0, 1]$

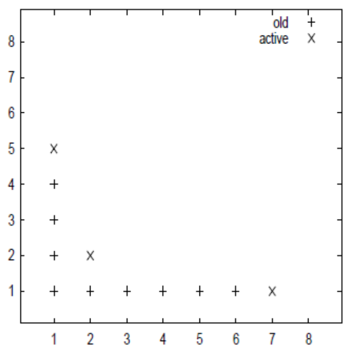
- Specifically designed to be challenging for global SA: term cancellations at mid-point and bounds
- Premature convergence in adaptive methods → start from higher-order grid





Adaptive Collocation Methods
Scalable Gerstner Test Problems (n=10)

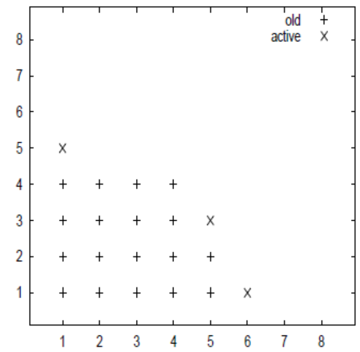
$e^{-x^2} + 10 \cdot e^{-y^2}$



$$f = \sum_{i=1}^{n/2} \left[e^{-x_{2i-1}^2} + 10e^{-x_{2i}^2} \right]$$

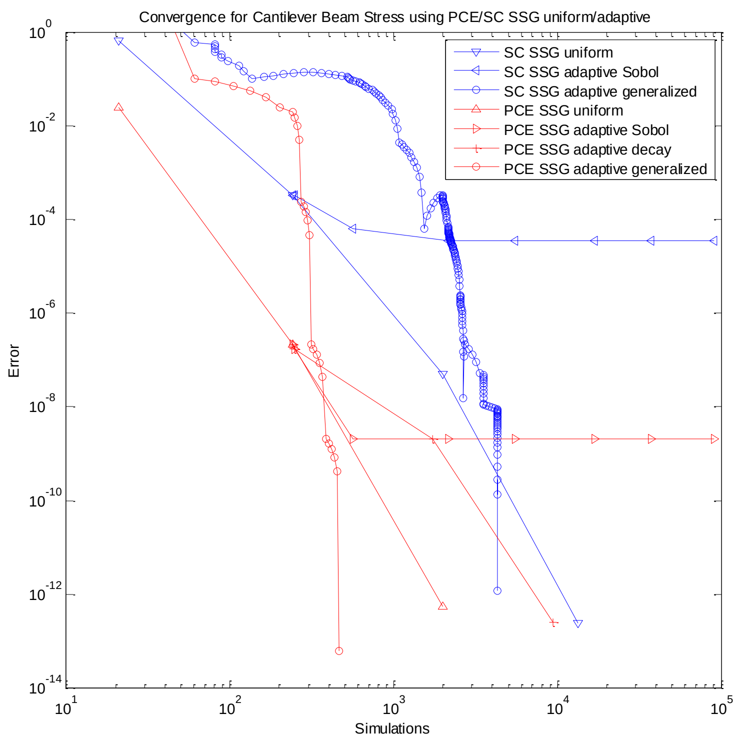
$x_i \; iid \; U[0, 1]$

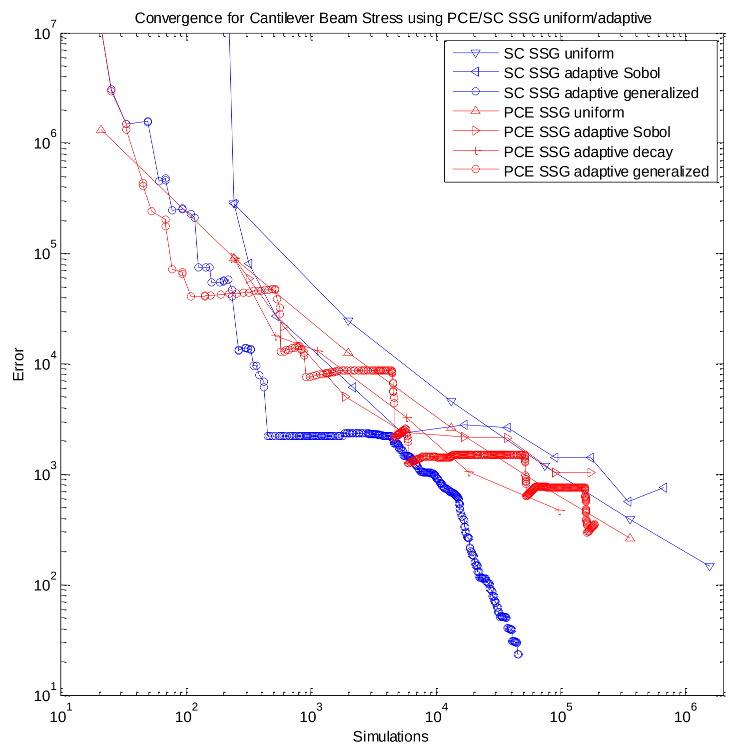
$e^{-10x^2-5y^2}$



$$f = exp\left(-\sum_{i=1}^{n/2} \left[10x_{2i-1}^2 + 5x_{2i}^2 \right] \right)$$

$x_i \; iid \; U[0, 1]$





Remarks on p-refinement

Adaptive p-refinement in nonintrusive stochastic expansions

- PCE and SC with TPQ and SSG
  - Dimension-adaptive p-refinement
    - Uniform p-refinement with *isotropic* tensor/sparse grids
    - Adaptive p-refinement with *anisotropic* tensor/sparse grids
      - VBD → total Sobol' indices → dimension preference
      - PCE spectral coefficient decay rate estimation → index weights
    - Goal-oriented adaptive p-refinement with *generalized* sparse grids
  - Generalized > anisotropic > isotropic

Areas for further study

- Tuning
  - Effect of combining anisotropy/refinement metrics over a response set
  - Scaling of anisotropic preference (VBD/decay rates often too severe)
- Capability extensions
  - Point-wise refinement (finer grain within active index sets)
  - Progression in interpolants: local and global Hermite interpolants (gradient-enhanced)

Extend Scalability through Adjoint Derivative-Enhancement

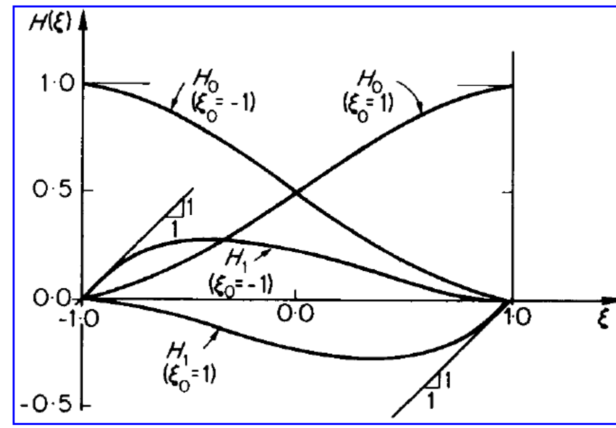
PCE:

- **Linear regression with derivatives**
  - **Gradients/Hessians → addtnl. eqns.**

$$\begin{bmatrix} \vdots & \vdots & & \vdots \\ \pi_{0,j}(\vec{\xi}_i) & \pi_{1,j}(\vec{\xi}_i) & \cdots & \pi_{P,j}(\vec{\xi}_i) \\ \frac{\partial \pi_{0,j}}{\partial \xi_1}(\vec{\xi}_i) & \frac{\partial \pi_{1,j}}{\partial \xi_1}(\vec{\xi}_i) & \cdots & \frac{\partial \pi_{P,j}}{\partial \xi_1}(\vec{\xi}_i) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \pi_{0,j}}{\partial \xi_{n_\xi}}(\vec{\xi}_i) & \frac{\partial \pi_{1,j}}{\partial \xi_{n_\xi}}(\vec{\xi}_i) & \cdots & \frac{\partial \pi_{P,j}}{\partial \xi_{n_\xi}}(\vec{\xi}_i) \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{pmatrix} \vdots \\ \vec{u}^{(m,j)} \\ \vec{u}^{(m+1,j)} \\ \vdots \\ \vec{u}^{(m+n_\xi,j)} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \vec{u}_i \\ \frac{\partial \vec{u}_i}{\partial \xi_1} \\ \vdots \\ \frac{\partial \vec{u}_i}{\partial \xi_{n_\xi}} \\ \vdots \end{pmatrix}$$

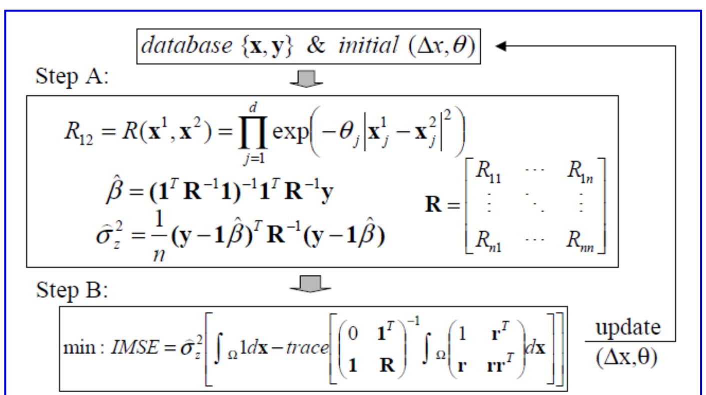
SC:

- **Gradient-enhanced interpolants**
  - **Local: cubic Hermite splines**
  - **Global: Hermite interpolation polynomials**



EGRA:

- **Gradient-enhanced kriging/cokriging**
  - **Interpolates function values and gradients**
  - **Scaling: n^2 → n**



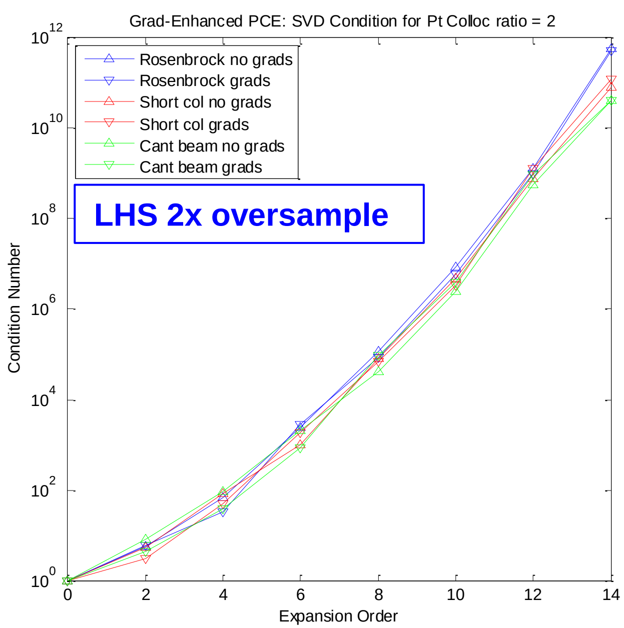
Gradient-Enhanced PCE

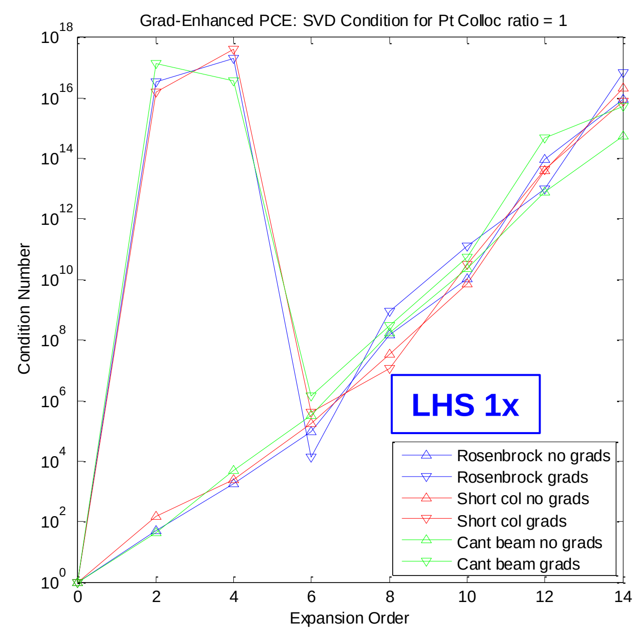
Straightforward regression approach implemented in DAKOTA v5.1:

$$
\begin{bmatrix}
\vdots & \vdots & & \vdots \\
\pi_{0,j}(\vec{\xi}_i) & \pi_{1,j}(\vec{\xi}_i) & \cdots & \pi_{P,j}(\vec{\xi}_i) \\
\frac{\partial \pi_{0,j}}{\partial \xi_1}(\vec{\xi}_i) & \frac{\partial \pi_{1,j}}{\partial \xi_1}(\vec{\xi}_i) & \cdots & \frac{\partial \pi_{P,j}}{\partial \xi_1}(\vec{\xi}_i) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial \pi_{0,j}}{\partial \xi_{n_\xi}}(\vec{\xi}_i) & \frac{\partial \pi_{1,j}}{\partial \xi_{n_\xi}}(\vec{\xi}_i) & \cdots & \frac{\partial \pi_{P,j}}{\partial \xi_{n_\xi}}(\vec{\xi}_i) \\
\vdots & \vdots & & \vdots
\end{bmatrix}
\begin{pmatrix}
\vdots \\
\vec{u}^{(m,j)} \\
\vec{u}^{(m+1,j)} \\
\vdots \\
\vec{u}^{(m+n_\xi,j)} \\
\vdots
\end{pmatrix}
=
\begin{pmatrix}
\vdots \\
\vec{u}_i \\
\frac{\partial \vec{u}_i}{\partial \xi_1} \\
\vdots \\
\frac{\partial \vec{u}_i}{\partial \xi_{n_\xi}} \\
\vdots
\end{pmatrix}
$$

- unweighted LLS by SVD (LAPACK GELSS)
- equality constrained LLS by QR (LAPACK GGLSE) when under-determined by values alone

Vandermonde-like systems known to suffer from ill-conditioning

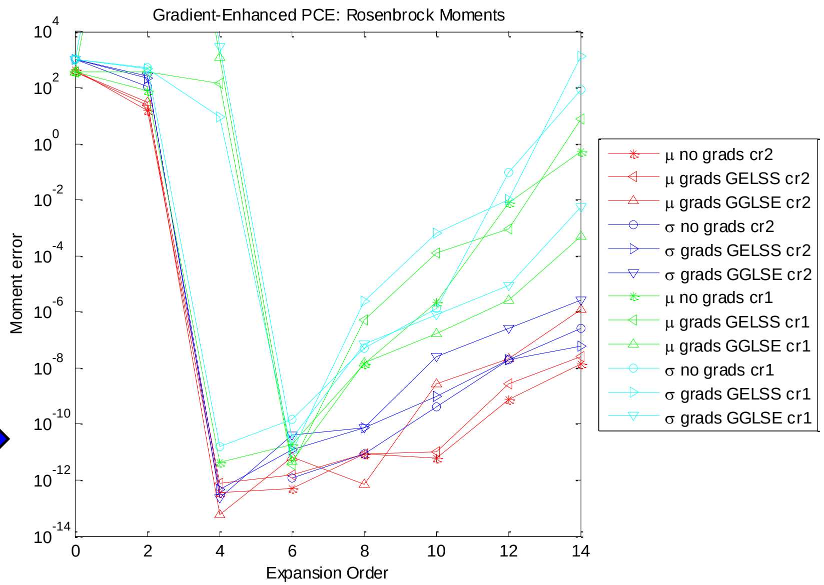


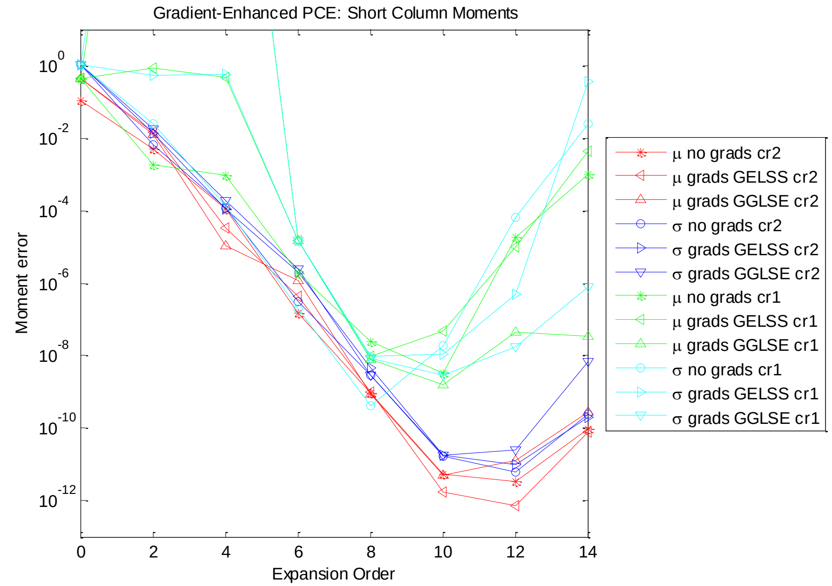


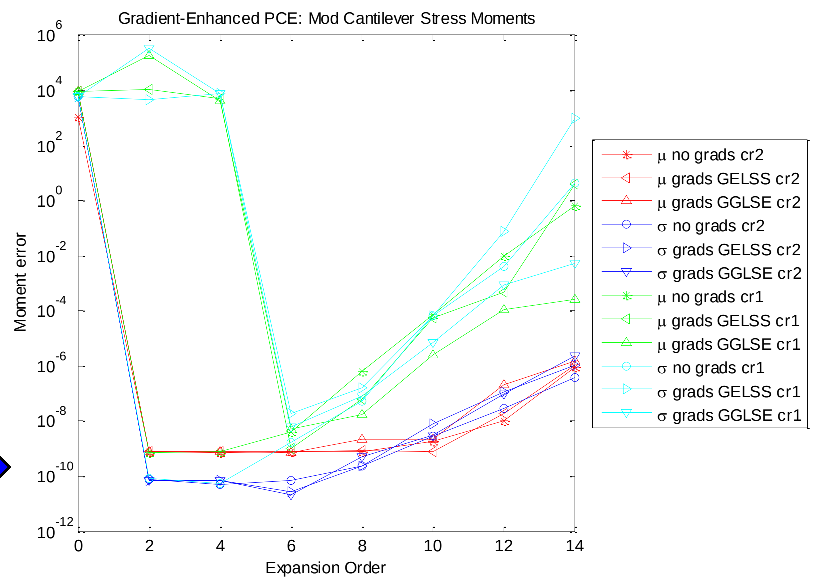
Gradient-Enhanced PCE: “Point Collocation”

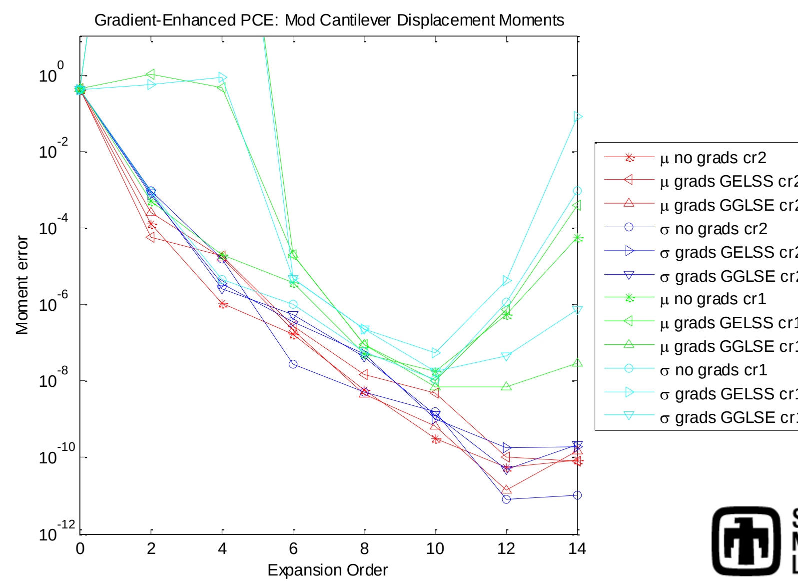
LHS with & without gradients, oversample ratio = 1 or 2

Conditioning issues evident as we over-resolve exact solutions









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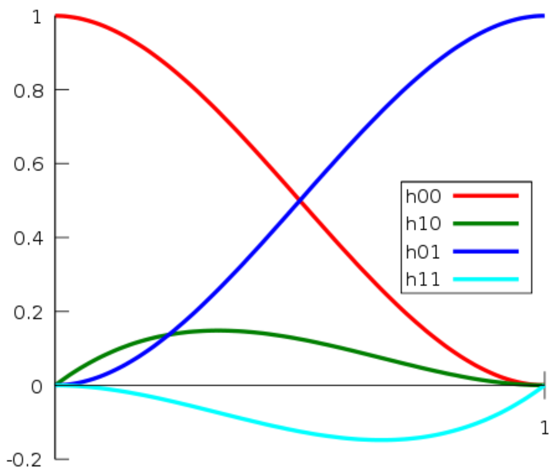
h-refinement with Gradient-Enhanced Interpolants

Dimension-adaptive h-refinement: local interpolant bases within global sparse grids:

- Values only: linear spline
- Values only: Quadratic hierarchical spline
- Values+gradients: cubic Hermite spline ⇨ $\boldsymbol{p}(t) = h_{00}(t)\boldsymbol{p}_0 + h_{10}(t)\boldsymbol{m}_0 + h_{01}(t)\boldsymbol{p}_1 + h_{11}(t)\boldsymbol{m}_1$

Shape functions – type 1 (value) and type 2 (gradient)

- Rearrange cubic Hermite spline into "hat" functions
- Value (type 1): h01 left half, h00 right half
- Gradient (type 2): h11 left half, h10 right half



Hermite interpolation

- 1D to first-order:

$$W_n(x) = \sum_{\nu=1}^{n} f_\nu \, h_\nu^{(1)}(x) + \sum_{\nu=1}^{n} f'_\nu \, h_\nu^{(2)}(x)$$

- Multivariate tensor product to arbitrary order (Lalescu):

$$s^{(n)}(x_1, x_2, \ldots, x_D) = \sum_{l_1,\ldots,l_D=0}^{m} \sum_{i_1,\ldots,i_D=0,1} f^{(l_1,\ldots,l_D)}(i_1,\ldots,i_D) \prod_{k=1}^{D} \alpha_{i_k}^{(n,l_k)}(x_k)$$

Global Hermite interpolants also of interest → gradient-enhanced p-refinement

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Gradient-Enhanced Interpolation:
Explicit formulation eliminates regression error

$$s^{(n)}(x_1, x_2, \ldots, x_D) = \sum_{l_1,\ldots,l_D=0}^{m} \sum_{i_1,\ldots,i_D=0,1} f^{(l_1,\ldots,l_D)}(i_1,\ldots,i_D) \prod_{k=1}^{D} \alpha_{i_k}^{(n,l_k)}(x_k)$$

Example: *n=3, N pts, 1st-order Hermite*

$$f = \sum_{i=1}^{N} f_i H_i^{(1)}(x_1) H_i^{(1)}(x_2) H_i^{(1)}(x_3) + \sum_{i=1}^{N} \frac{df_i}{dx_1} H_i^{(2)}(x_1) H_i^{(1)}(x_2) H_i^{(1)}(x_3) + \sum_{i=1}^{N} \frac{df_i}{dx_2} H_i^{(1)}(x_1) H_i^{(2)}(x_2) H_i^{(1)}(x_3) + \sum_{i=1}^{N} \frac{df_i}{dx_3} H_i^{(1)}(x_1) H_i^{(1)}(x_2) H_i^{(2)}(x_3)$$

$$\mu = \sum_{i=1}^{N} f_i w_i^{(1)} w_i^{(1)} w_i^{(1)} + \sum_{i=1}^{N} \frac{df_i}{dx_1} w_i^{(2)} w_i^{(1)} w_i^{(1)} + \sum_{i=1}^{N} \frac{df_i}{dx_2} w_i^{(1)} w_i^{(2)} w_i^{(1)} + \sum_{i=1}^{N} \frac{df_i}{dx_3} w_i^{(1)} w_i^{(1)} w_i^{(2)}$$

and similar for higher-order moments

Investigated several variants:

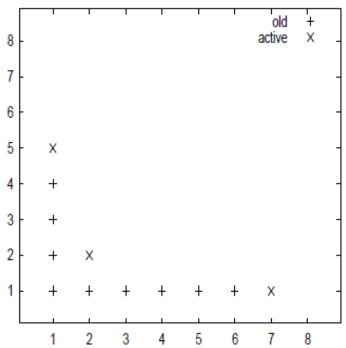
- PW linear vs. PW cubic vs. global
- Newton-Cotes vs. Clenshaw-Curtis
- Restricted growth/delayed sequences vs. unrestricted

Sparse grid gradient-enhanced interpolation of Rosenbrock (w=3, uniform over [-2,2])

```
Colloc pt   1: truth value  =  3.6090000000e+03 interpolant = 3.6090000000e+03 error =  0.0000000000e+00
               truth grad_1 = -9.6120000000e+03 interpolant = -9.6120000000e+03 error =  0.0000000000e+00
               truth grad_2 = -2.4000000000e+03 interpolant = -2.4000000000e+03 error =  0.0000000000e+00
Colloc pt   2: truth value  =  2.5090000000e+03 interpolant = 2.5090000000e+03 error =  0.0000000000e+00
               truth grad_1 = -8.0120000000e+03 interpolant = -8.0120000000e+03 error =  0.0000000000e+00
               truth grad_2 = -2.0000000000e+03 interpolant = -2.0000000000e+03 error =  0.0000000000e+00
Colloc pt   3: truth value  =  1.6090000000e+03 interpolant = 1.6090000000e+03 error =  0.0000000000e+00
               truth grad_1 = -6.4120000000e+03 interpolant = -6.4120000000e+03 error =  0.0000000000e+00
               truth grad_2 = -1.6000000000e+03 interpolant = -1.6000000000e+03 error =  0.0000000000e+00
Colloc pt   4: truth value  =  9.0900000000e+02 interpolant = 9.0900000000e+02 error =  0.0000000000e+00
               truth grad_1 = -4.8120000000e+03 interpolant = -4.8120000000e+03 error =  0.0000000000e+00
               truth grad_2 = -1.2000000000e+03 interpolant = -1.2000000000e+03 error =  0.0000000000e+00
Colloc pt   5: truth value  =  4.0900000000e+02 interpolant = 4.0900000000e+02 error =  0.0000000000e+00
               truth grad_1 = -3.2120000000e+03 interpolant = -3.2120000000e+03 error =  0.0000000000e+00
               truth grad_2 = -8.0000000000e+02 interpolant = -8.0000000000e+02 error =  0.0000000000e+00
Colloc pt   6: truth value  =  5.1250000000e+02 interpolant = 5.1250000000e+02 error =  0.0000000000e+00
               truth grad_1 = -2.7100000000e+03 interpolant = -2.7100000000e+03 error =  0.0000000000e+00
               truth grad_2 = -9.0000000000e+02 interpolant = -9.0000000000e+02 error =  0.0000000000e+00
Colloc pt   7: truth value  =  9.0400000000e+02 interpolant = 9.0400000000e+02 error =  0.0000000000e+00
               truth grad_1 = -2.4080000000e+03 interpolant = -2.4080000000e+03 error =  0.0000000000e+00
               truth grad_2 = -1.2000000000e+03 interpolant = -1.2000000000e+03 error =  0.0000000000e+00
Colloc pt   8: truth value  =  1.0400000000e+02 interpolant = 1.0400000000e+02 error =  0.0000000000e+00
               truth grad_1 = -8.0800000000e+02 interpolant = -8.0800000000e+02 error =  0.0000000000e+00
               truth grad_2 = -4.0000000000e+02 interpolant = -4.0000000000e+02 error =  0.0000000000e+00
Colloc pt   9: truth value  =  1.0400000000e+02 interpolant = 1.0400000000e+02 error =  0.0000000000e+00
               truth grad_1 =  7.9200000000e+02 interpolant = 7.9200000000e+02 error =  0.0000000000e+00
               truth grad_2 =  4.0000000000e+02 interpolant = 4.0000000000e+02 error =  0.0000000000e+00
Colloc pt  10: truth value  =  8.5000000000e+00 interpolant = 8.5000000000e+00 error =  0.0000000000e+00
               truth grad_1 = -1.0600000000e+02 interpolant = -1.0600000000e+02 error =  0.0000000000e+00
               truth grad_2 = -1.0000000000e+02 interpolant = -1.0000000000e+02 error =  0.0000000000e+00
Colloc pt  11: truth value  =  4.0100000000e+02 interpolant = 4.0100000000e+02 error =  0.0000000000e+00
               truth grad_1 = -4.0000000000e+00 interpolant = -4.0000000000e+00 error =  0.0000000000e+00
               truth grad_2 = -8.0000000000e+02 interpolant = -8.0000000000e+02 error =  0.0000000000e+00
Colloc pt  12: truth value  =  2.2600000000e+02 interpolant = 2.2600000000e+02 error =  0.0000000000e+00
               truth grad_1 = -4.0000000000e+00 interpolant = -4.0000000000e+00 error =  0.0000000000e+00
               truth grad_2 = -6.0000000000e+02 interpolant = -6.0000000000e+02 error =  0.0000000000e+00
Colloc pt  13: truth value  =  1.0100000000e+02 interpolant = 1.0100000000e+02 error =  0.0000000000e+00
               truth grad_1 = -4.0000000000e+00 interpolant = -4.0000000000e+00 error =  0.0000000000e+00
               truth grad_2 = -4.0000000000e+02 interpolant = -4.0000000000e+02 error =  0.0000000000e+00
Colloc pt  14: truth value  =  2.6000000000e+01 interpolant = 2.6000000000e+01 error =  0.0000000000e+00
               truth grad_1 = -4.0000000000e+00 interpolant = -4.0000000000e+00 error =  0.0000000000e+00
               truth grad_2 = -2.0000000000e+02 interpolant = -2.0000000000e+02 error =  0.0000000000e+00
Colloc pt  15: truth value  =  1.0000000000e+00 interpolant = 1.0000000000e+00 error =  0.0000000000e+00
               truth grad_1 = -4.0000000000e+00 interpolant = -4.0000000000e+00 error =  0.0000000000e+00
               truth grad_2 =  0.0000000000e+00 interpolant = 0.0000000000e+00 error =  0.0000000000e+00
Colloc pt  16: truth value  =  2.6000000000e+01 interpolant = 2.6000000000e+01 error =  0.0000000000e+00
               truth grad_1 = -4.0000000000e+00 interpolant = -4.0000000000e+00 error =  0.0000000000e+00
               truth grad_2 =  2.0000000000e+02 interpolant = 2.0000000000e+02 error =  0.0000000000e+00
Colloc pt  17: truth value  =  1.0100000000e+02 interpolant = 1.0100000000e+02 error =  0.0000000000e+00
               truth grad_1 = -4.0000000000e+00 interpolant = -4.0000000000e+00 error =  0.0000000000e+00
               truth grad_2 =  4.0000000000e+02 interpolant = 4.0000000000e+02 error =  0.0000000000e+00
Colloc pt  18: truth value  =  2.2600000000e+02 interpolant = 2.2600000000e+02 error =  0.0000000000e+00
               truth grad_1 = -4.0000000000e+00 interpolant = -4.0000000000e+00 error =  0.0000000000e+00
               truth grad_2 =  6.0000000000e+02 interpolant = 6.0000000000e+02 error =  0.0000000000e+00
...
```

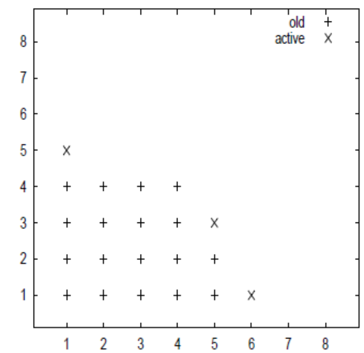
Smooth Test Problems
PWL/PWC bases on Gerstner tests (n=2)

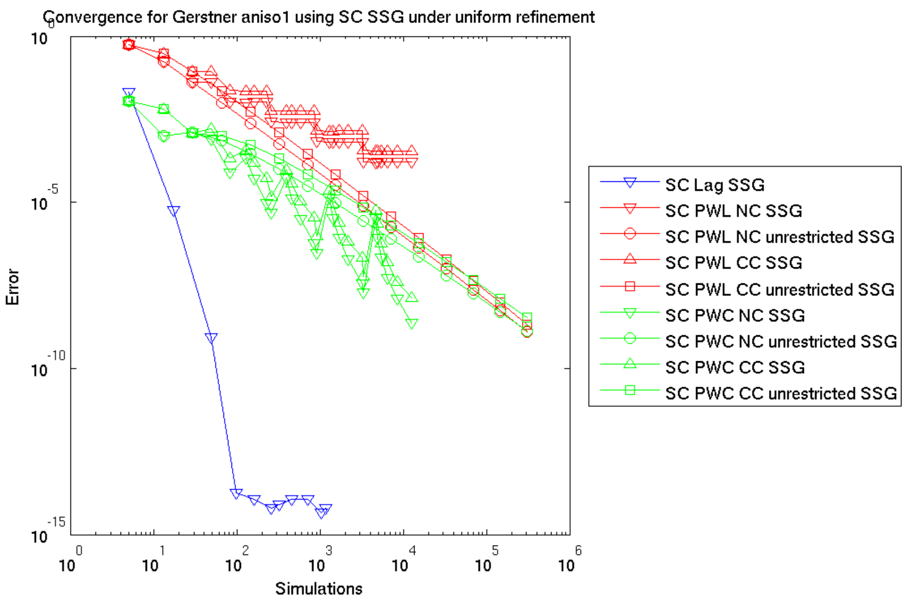
$$e^{-x^2} + 10 \cdot e^{-y^2}$$

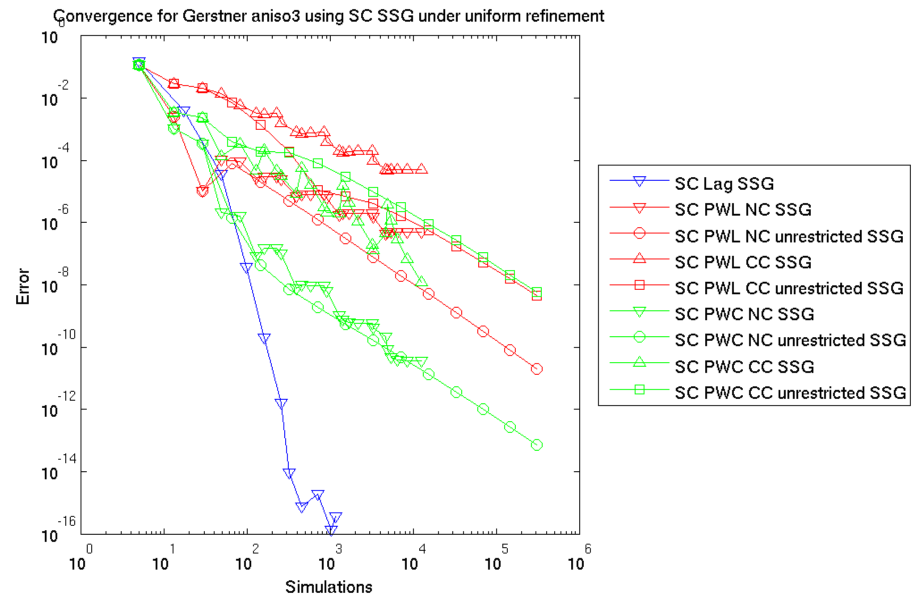


x_i *iid* $U[0, 1]$

$$e^{-10x^2-5y^2}$$

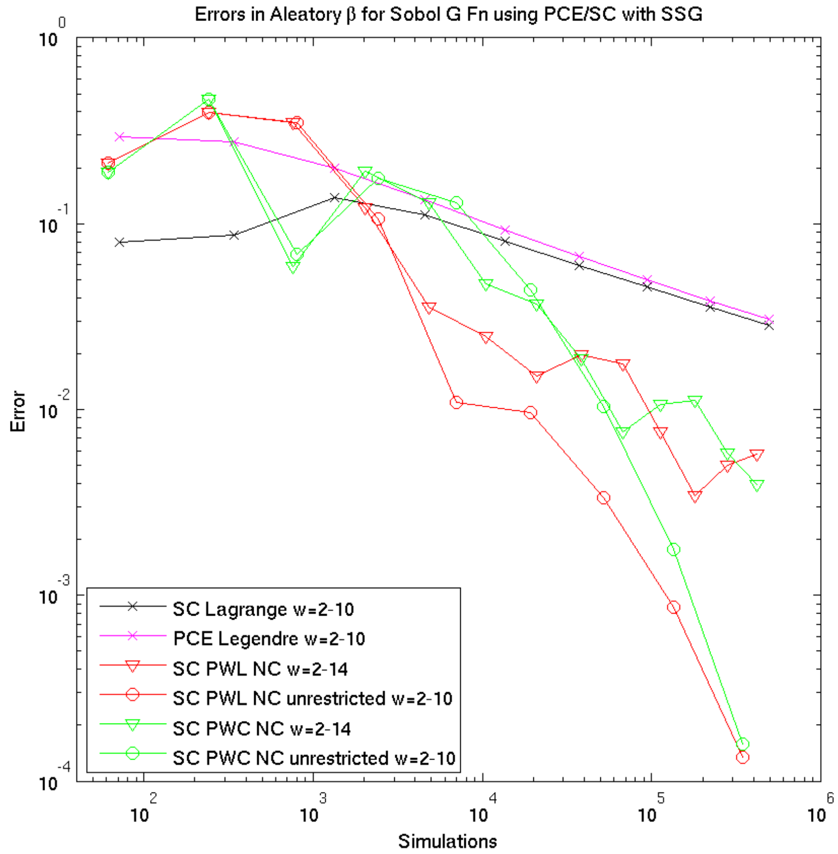






Nonsmooth Test Problem
PWL/PWC bases on Sobol's g fn (n=5)

$$f(\mathbf{x}) = 2 \prod_{j=1}^{5} \frac{|4x_j - 2| + a_j}{1 + a_j}; \quad a = [0,1,2,4,8]$$



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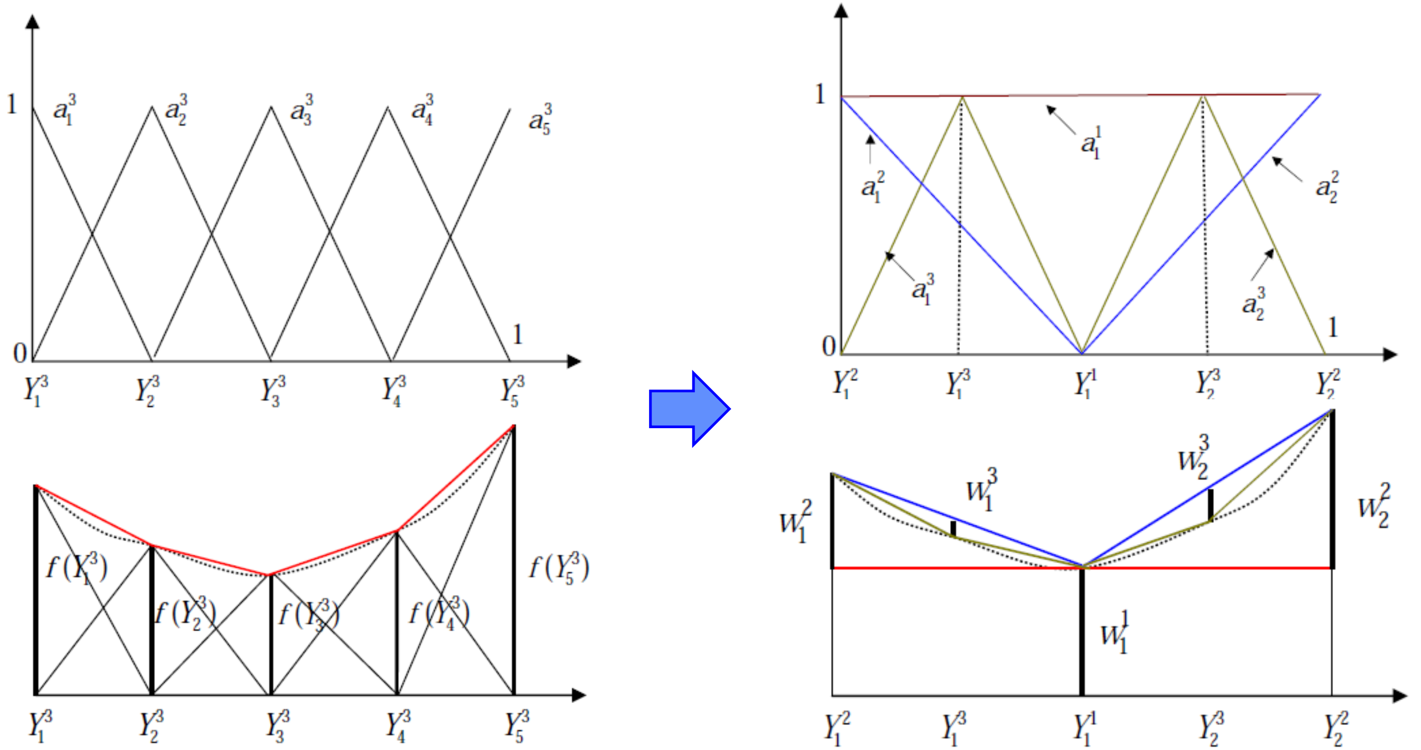
Gradient-Enhanced Interpolation: Next Steps

- Add hierarchical basis for use alongside nodal basis → local refinement (element splitting, point-wise GSG), improved precision in incremental effects, hybrid metric estimation



Hierarchical linear splines; from Xiang Ma, Ph.D. dissertation, Cornell Univ., 2010

- Extend support for infinite and semi-infinite domains

- Global Hermite interpolation polynomials: stable 1D generators + integration weights

Build on efficient/robust/scalable UQ core

Drivers

- Efficient/robust/*scalable* core → adaptive methods, adjoint enhancement
- Complex random env. → mixed UQ, model form/multifidelity, RF/SP, multiphysics/multiscale

Stochastic sensitivity analysis

- Aleatory or combined expansions including nonprobabilistic dimensions *s* → sensitivities of moments w.r.t. design and/or epistemic parameters

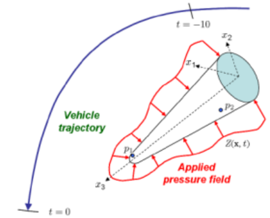
$$R(\xi, s) = \sum_{j=0}^{P} \alpha_j(s)\Psi_j(\xi)$$

$$R(\xi, s) = \sum_{j=0}^{P} \alpha_j \Psi_j(\xi, s)$$

Design and Model Calibration Under Uncertainty

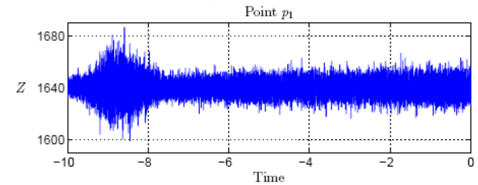
Mixed Aleatory-Epistemic UQ

- SOP, IVP, and DSTE approaches that are more accurate and efficient than traditional nested sampling



Bayesian Inference:

- Collaborations w/ LANL (GPM), UT (Queso), Purdue/MIT (gPC)

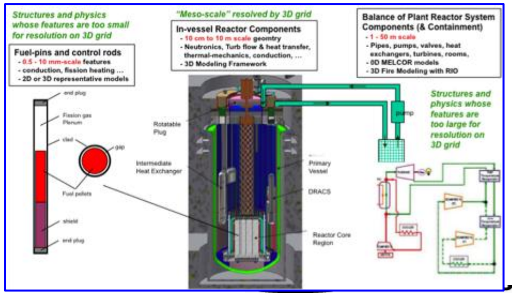


Model form:

- Multifidelity UQ (hierarchy), Bayesian model averaging (ensemble)

Multiphysics (multiscale) UQ:

- Invert UQ & multiphysics loops → transfer UQ stats among codes



Random Fields / Stochastic Processes (Encore, PECOS)

Stochastic Sensitivity Analysis

- **PCE/SC have convenient analytic features**
  - Expansions readily differentiated w.r.t. ξ
  - Analytic moment expressions
- **Augment w/ nonprobabilistic dimensions *s***
  - Design, epistemic uncertain

$$R = \sum_{j=0}^{P} \alpha_j \Psi_j(\boldsymbol{\xi}) \qquad R = \sum_{j=1}^{N_p} r_j \boldsymbol{L}_j(\boldsymbol{\xi})$$

$$\begin{cases} \mu_R = \alpha_0 \\ \sigma_R^2 = \sum_{j=1}^{P} \alpha_j^2 \langle \Psi_j^2 \rangle \end{cases} \qquad \begin{cases} \mu_R = \sum_{j=1}^{N_p} r_j w_j \\ \sigma_R^2 = \sum_{j=1}^{N_p} r_j^2 w_j - \mu_R^2 \end{cases}$$

- **Approach 1: PCE/SC over prob. vars for each set of nonprob. vars**

$$R(\boldsymbol{\xi}, \boldsymbol{s}) = \sum_{j=0}^{P} \alpha_j(\boldsymbol{s}) \Psi_j(\boldsymbol{\xi})$$

$$R(\boldsymbol{\xi}, \boldsymbol{s}) = \sum_{j=1}^{N_p} r_j(\boldsymbol{s}) \boldsymbol{L}_j(\boldsymbol{\xi})$$

⇨ Moment sensitivity = expectation of response sensitivity

$$\begin{cases} \frac{d\mu_R}{ds} = \langle \frac{dR}{ds} \rangle \\ \frac{d\sigma_R^2}{ds} = 2\sum_{j=1}^{P} \alpha_j \langle \frac{dR}{ds}, \Psi_j \rangle \end{cases} \qquad \begin{cases} \frac{d\mu_R}{ds} = \sum_{j=1}^{N_p} w_j \frac{dr_j}{ds} \\ \frac{d\sigma_R^2}{ds} = \sum_{j=1}^{N_p} 2 w_j (r_j - \mu_R) \frac{dr_j}{ds} \end{cases}$$

→Additional data requirements (*dR/ds*), but no additional dimensions

- **Approach 2: PCE/SC over all vars**

$$R(\boldsymbol{\xi}, \boldsymbol{s}) = \sum_{j=0}^{P} \alpha_j \Psi_j(\boldsymbol{\xi}, \boldsymbol{s}) \qquad R(\boldsymbol{\xi}, \boldsymbol{s}) = \sum_{j=1}^{N_p} r_j \boldsymbol{L}_j(\boldsymbol{\xi}, \boldsymbol{s})$$

Moment sensitivity = expectations over ξ + differentiation of remaining polynomial in *s*

⇨

$$\mu_R(\boldsymbol{s}) = \sum_{j=0}^{P} \alpha_j \langle \Psi_j(\boldsymbol{\xi}, \boldsymbol{s}) \rangle_{\boldsymbol{\xi}} \qquad \mu_R(\boldsymbol{s}) = \sum_{j=1}^{N_p} r_j \langle \boldsymbol{L}_j(\boldsymbol{\xi}, \boldsymbol{s}) \rangle_{\boldsymbol{\xi}}$$

→ Additional dimensions, but no additional data requirements

Optimization Under Uncertainty

Standard NLP

$$\begin{aligned} \min \quad & f(d) \\ \text{s.t.} \quad & g_l \le g(d) \le g_u \\ & h(d) = h_t \\ & d_l \le d \le d_u \end{aligned}$$

optimize, accounting for uncertainty metrics

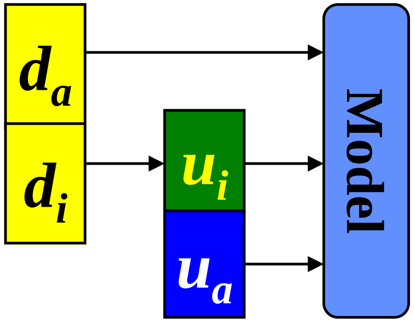
(using any UQ method)

Add resp stats $\boldsymbol{s_u}$ ***(μ, σ, z/β/p)***

$$\begin{aligned} \min \quad & f(d) + W s_u(d) \\ \text{s.t.} \quad & g_l \le g(d) \le g_u \\ & h(d) = h_t \\ & d_l \le d \le d_u \\ & a_l \le A_i\, s_u(d) \le a_u \\ & A_e\, s_u(d) = a_t \end{aligned}$$

Input design parameterization

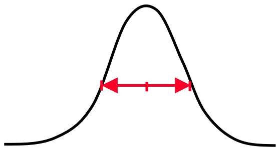
- **Design vars may augment uncertain vars in simulation**
- **Inserted design vars: an optimization design var may be a parameter of an uncertain dist, e.g., the mean of a normal**



Control response statistics to design for…

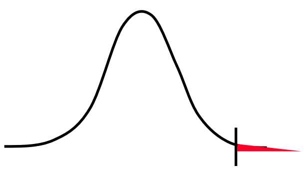
…robustness:

min/constrain moments
μ, σ^2, or $z(\beta)$ range



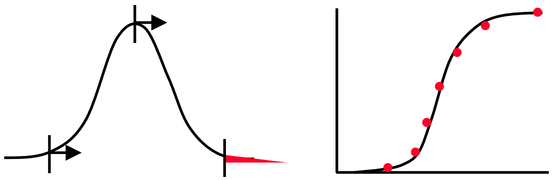
…reliability:

min/max/constrain p/β
(tail stats, failure)



…combined/other:

Pareto, inversion/model calibration under uncertainty



Aleatory

Optimization Under Uncertainty

Standard NLP

$$\begin{aligned} \min \quad & f(d) \\ \text{s.t.} \quad & g_l \leq g(d) \leq g_u \\ & h(d) = h_t \\ & d_l \leq d \leq d_u \end{aligned}$$

optimize, accounting for uncertainty metrics

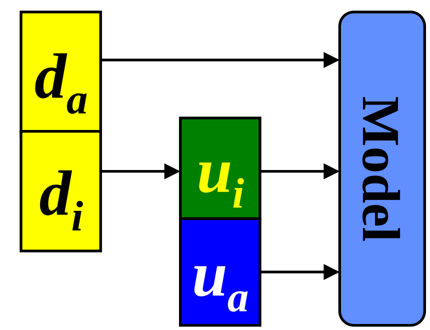
(using any UQ method)

Add resp stats s_u *(μ, σ, z/β/p)*

$$\begin{aligned} \min \quad & f(d) + W s_u(d) \\ \text{s.t.} \quad & g_l \leq g(d) \leq g_u \\ & h(d) = h_t \\ & d_l \leq d \leq d_u \\ & a_l \leq A_i\, s_u(d) \leq a_u \\ & A_e\, s_u(d) = a_t \end{aligned}$$

Input design parameterization

- **Design vars may augment uncertain vars in simulation**
- **Inserted design vars: an optimization design var may be a parameter of an uncertain dist, e.g., the mean of a normal**



Control response statistics to design for…

…robustness:
min/constrain moments
μ, σ^2, or $z(\beta)$ range

…reliability:
min/max/constrain p/β
(tail stats, failure)

…combined/other:
Pareto, inversion/model calibration under uncertainty

Epistemic/Mixed

PCE-based and SC-based OUU

Fully analytic Bi-level:

- **Analytic moment/reliability sensitivities *(avoid numerical derivs. at design level)***
- **Uncertain or Combined expansions**

Reliability:

$$\begin{cases} \text{minimize} & f \\ \text{subject to} & \beta \geq \bar{\beta} \end{cases}$$

(β initially based on moment proj)

Robustness:

$$\begin{cases} \text{minimize} & f \\ \text{subject to} & \sigma^2 \leq \bar{\sigma}^2 \end{cases}$$

Sequential/Surrogate-based:

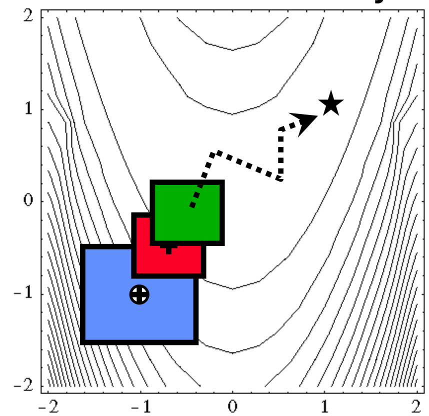
- **Break nesting: iterate between opt & UQ w/ (surrogate) linkage**
- **Uncertain expansions**

$$\begin{cases} \text{minimize} & f(\mathbf{s}_c) + \nabla_s f(\mathbf{s}_c)^T(\mathbf{s} - \mathbf{s}_c) \\ \text{subject to} & \beta(\mathbf{s}_c) + \nabla_s \beta(\mathbf{s}_c)^T(\mathbf{s} - \mathbf{s}_c) \geq \bar{\beta} \\ & \parallel \mathbf{s} - \mathbf{s}_c \parallel_\infty \leq \Delta^k \end{cases}$$

$$\begin{cases} \text{minimize} & f(\mathbf{s}_c) + \nabla_s f(\mathbf{s}_c)^T(\mathbf{s} - \mathbf{s}_c) \\ \text{subject to} & \sigma^2(\mathbf{s}_c) + \nabla_s \sigma^2(\mathbf{s}_c)^T(\mathbf{s} - \mathbf{s}_c) \leq \bar{\sigma}^2 \\ & \parallel \mathbf{s} - \mathbf{s}_c \parallel_\infty \leq \Delta^k \end{cases}$$

- 1^{st}-order
- Also QN 2^{nd}-order

TR-SBO with local data fit & multifidelity



Multifidelity (focused on UQ fidelity):

- **Optimize corrected LF UQ model over TR**
  - **LF = Combined expansion (over *s*), MVFOSM**
  - **HF = Uncertain expansion (at single design pt)**
- **Additive corrections enforce LF/HF consistency**
  - **1^{st} order & QN 2^{nd}-order**

$$\hat{\beta_{hi}}(\mathbf{s}) = \beta_{lo}(\mathbf{s}) + \alpha_\beta(\mathbf{s})$$
$$\hat{\sigma_{hi}}^2(\mathbf{s}) = \sigma^2_{lo}(\mathbf{s}) + \alpha_{\sigma^2}(\mathbf{s})$$

$$\begin{cases} \text{minimize} & f(\mathbf{s}) \\ \text{subject to} & \hat{\sigma_{hi}}^2(\mathbf{s}) \leq \bar{\sigma}^2 \\ & \parallel \mathbf{s} - \mathbf{s}_c \parallel_\infty \leq \Delta^k \end{cases}$$

$$\begin{cases} \text{minimize} & f(\mathbf{s}) \\ \text{subject to} & \hat{\beta_{hi}}(\mathbf{s}) \geq \bar{\beta} \\ & \parallel \mathbf{s} - \mathbf{s}_c \parallel_\infty \leq \Delta^k \end{cases}$$

Unilevel approach (sim residuals @ collocation pts)

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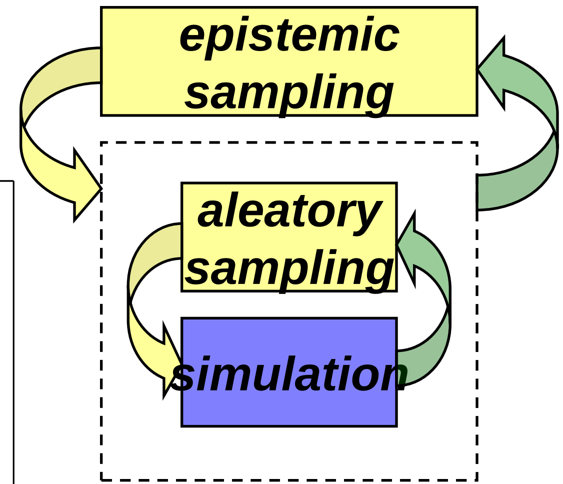
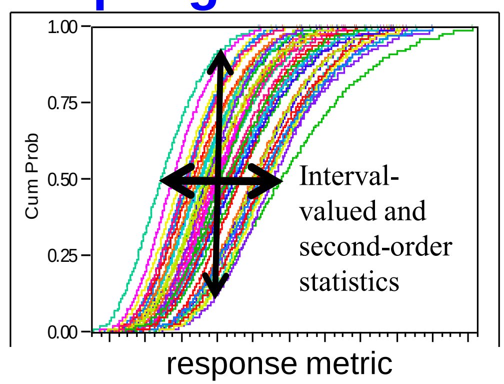
Mixed Aleatory-Epistemic UQ: IVP, DSTE, and SOP

Epistemic uncertainty (aka: subjective, reducible, lack of knowledge uncertainty): insufficient info to specify objective probability distributions



Traditional approach: nested sampling

- Expensive sims → under-resolved sampling (especially @ outer loop)
- Under-prediction of credible outcomes



Algorithmic approaches

- Interval-valued probability (IVP), *aka* PBA
- Dempster-Shafer theory of evidence (DSTE)
- Second-order probability (SOP), *aka* PoF

Increasing epistemic structure (stronger assumptions)

Address accuracy and efficiency

- Inner loop: stochastic exp. that are epistemic-aware (aleatory, combined)
- Outer loop:
  - IVP, DSTE: opt-based interval estimation, global (EGO) or local (NLP)
  - SOP: nested stochastic exp. (nested expectation is only post-processing in special cases)

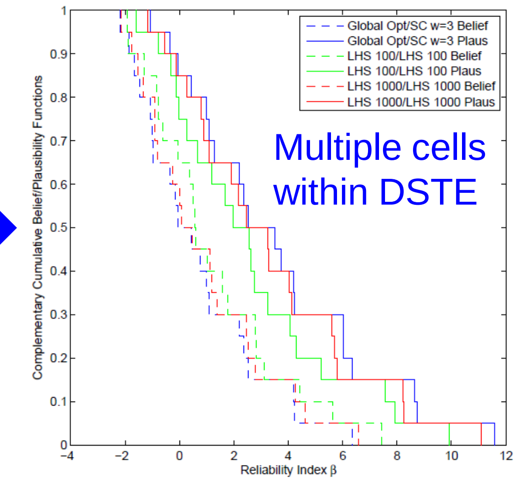
$$\begin{array}{ll} \text{minimize} & M(s) \\ \text{subject to} & s_L \leq s \leq s_U \end{array}$$

$$\begin{array}{ll} \text{maximize} & M(s) \\ \text{subject to} & s_L \leq s \leq s_U \end{array}$$

Mixed Aleatory-Epistemic UQ:
IVP, SOP, and DSTE based on Stochastic Expansions

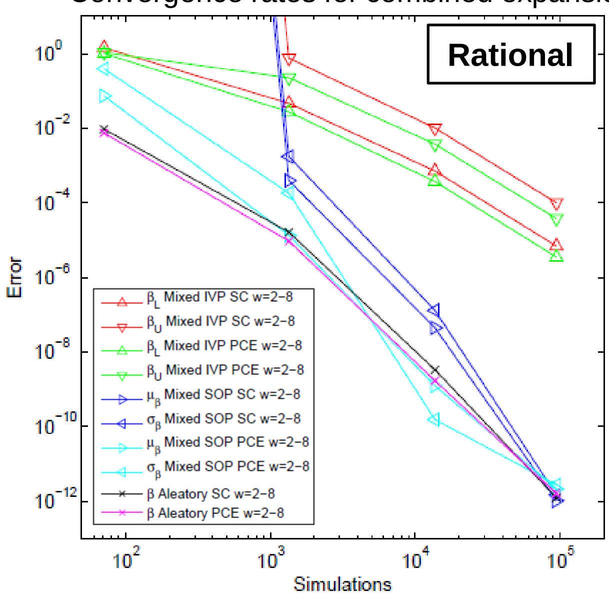
Interv Est Approach	UQ Approach	Expansion Variables	Evaluations (Fn, Grad)	Area	β
IVP SC SSG Aleatory: β interval converged to 5-6 digits by 300-400 evals					
EGO	SC SSG w = 1	Aleatory	(84/91, 0/0)	[75.0002, 374.999]	[-2.26264, 11.8623]
EGO	SC SSG w = 2	Aleatory	(372/403, 0/0)	[75.0002, 374.999]	[-2.18735, 11.5900]
EGO	SC SSG w = 3	Aleatory	(1260/1365, 0/0)	[75.0002, 374.999]	[-2.18732, 11.5900]
EGO	SC SSG w = 4	Aleatory	(3564/3861, 0/0)	[75.0002, 374.999]	[-2.18732, 11.5900]
NPSOL	SC SSG w = 1	Aleatory	(21/77, 21/77)	[75.0000, 375.000]	[-2.26264, 11.8623]
NPSOL	SC SSG w = 2	Aleatory	(93/341, 93/341)	[75.0000, 375.000]	[-2.18735, 11.5901]
NPSOL	SC SSG w = 3	Aleatory	(315/1155, 315/1155)	[75.0000, 375.000]	[-2.18732, 11.5900]
NPSOL	SC SSG w = 4	Aleatory	(891/3267, 891/3267)	[75.0000, 375.000]	[-2.18732, 11.5900]
IVP nested LHS sampling: converged to 2-3 digits by 10^8 evals					
LHS 100	LHS 100	N/A	$(10^4/10^4, 0/0)$	[80.5075, 338.607]	[-2.14505, 8.64891]
LHS 1000	LHS 1000	N/A	$(10^6/10^6, 0/0)$	[76.5939, 368.225]	[-2.19883, 11.2353]
LHS 10^4	LHS 10^4	N/A	$(10^8/10^8, 0/0)$	[76.4755, 373.935]	[-2.16323, 11.5593]

Fully converged area interval = [75., 375.], β interval = [−2.18732, 11.5900]



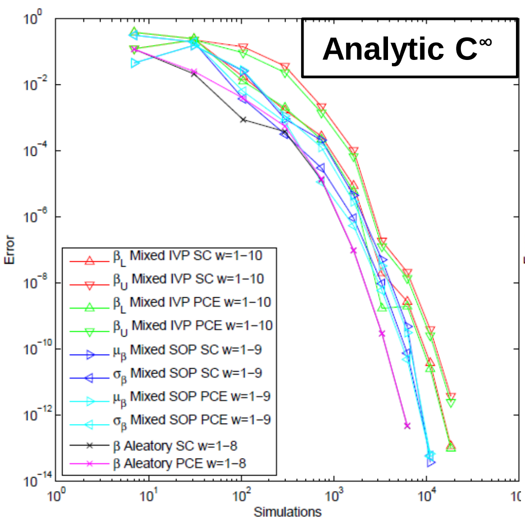
Multiple cells within DSTE

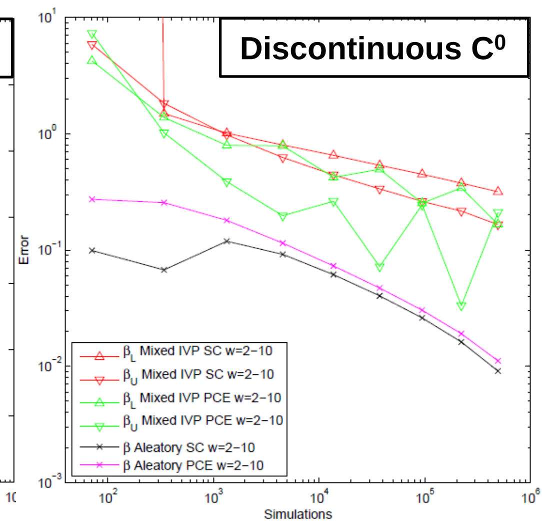
Convergence rates for combined expansions



L^∞ metrics: IVP mixed, DSTE mixed

L^2 metrics: Aleatory, SOP mixed





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Concluding Remarks

R&D in Adaptive UQ Methods → scalability

- Stochastic expansions: PCE, SC
  - Dimension-adaptive refinement (p- and h-)
    - Uniform p-refinement with *isotropic* grids
    - Adaptive p-refinement with *anisotropic* grids (VBD/Sobol', Spectral decay → anisotropy)
    - Goal-oriented adaptive p-refinement with *generalized* sparse grids (statistical QOI: μ, σ^2, β, β^*, p_{fail})
  - Adjoint enhancement
  - Hierarchical basis formulations → region-adaptive h-refinement

R&D in UQ Complexity → mixed uncertainties, multiphysics/multiscale

- Stochastic sensitivity analysis → enables OUU/MCUU and mixed UQ
- Design under uncertainty: bilevel, sequential, multifidelity approaches
- Mixed UQ with IVP/SOP/DSTE → greater accuracy/efficiency than nested sampling
  - Inner loop: stochastic expansions (aleatory or combined)
  - Outer loop: opt-based interval est.; global with data reuse (robust) or local with SSA (scalable)
- Multi-* → Multi-physics/Multi-scale/Multifidelity UQ
- Random fields and stochastic processes
- Nondeterministic calibration (MLE, ME, Bayesian inference) → Model Form UQ

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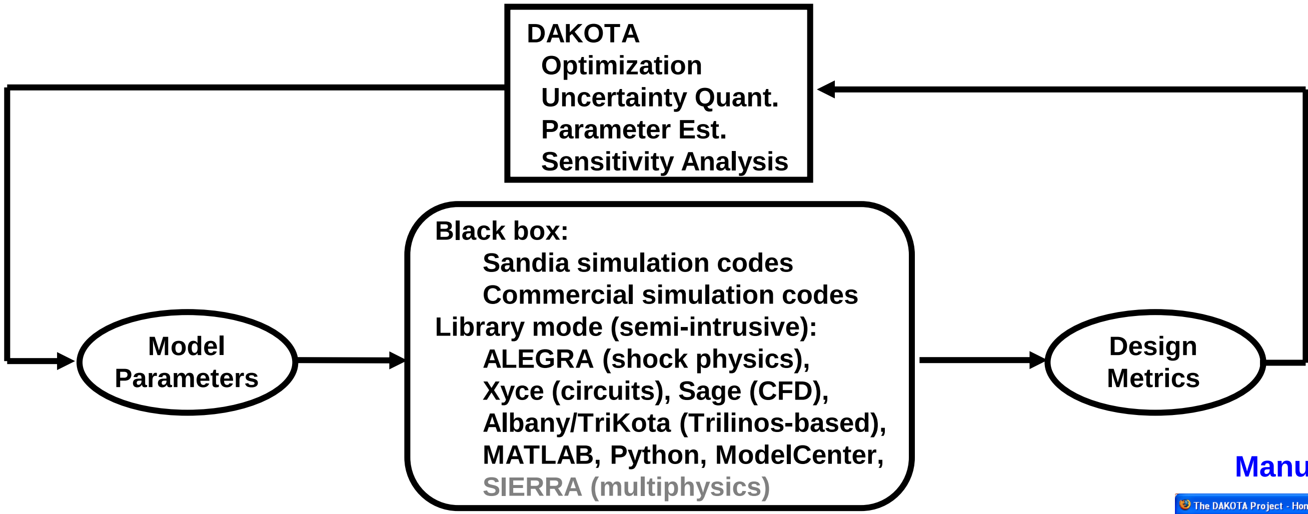
DAKOTA Software



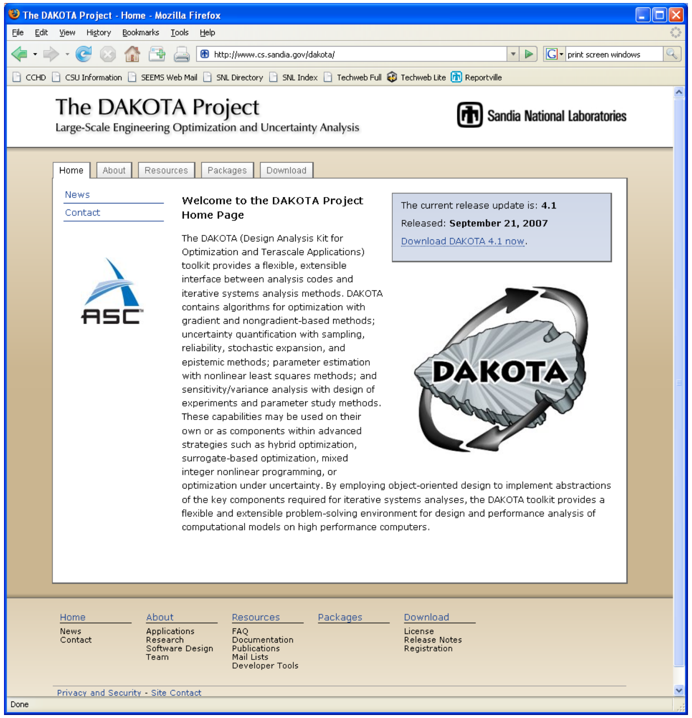
Iterative systems analysis

Multilevel parallel computing

Simulation management

http://dakota.sandia.gov

Manuals, Publications, Training matls. online



Releases: **Major/Interim, Stable/VOTD; *5.1 released 12/10***

Modern SQE: **Linux/Unix, Mac, Windows; Nightly builds/testing; subversion, TRAC, autotools/Cmake**

GNU LGPL: **free downloads worldwide (>7000 total ext. registrations, ~3500 distributions last yr.)**

Community development: **open checkouts now available**

Community support: **dakota-users, dakota-help**