Mathematical Models for Hospital Inpatient Flow Management

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Outline

- Part I: Data
- Part II: Model
- Part III: Analytical analysis
- Part IV: Managerial Insights

Overview

- Motivation
  - Inpatient flow management
  - Impact of *early discharge* policy
    - Waiting time for admission to ward
    - Stabilize hourly waiting time performance

- A stochastic network model
  - Allocation delays
  - Overflow policy
  - Endogenous service times

- Predict the time-dependent waiting time
  - A two-time-scale approach

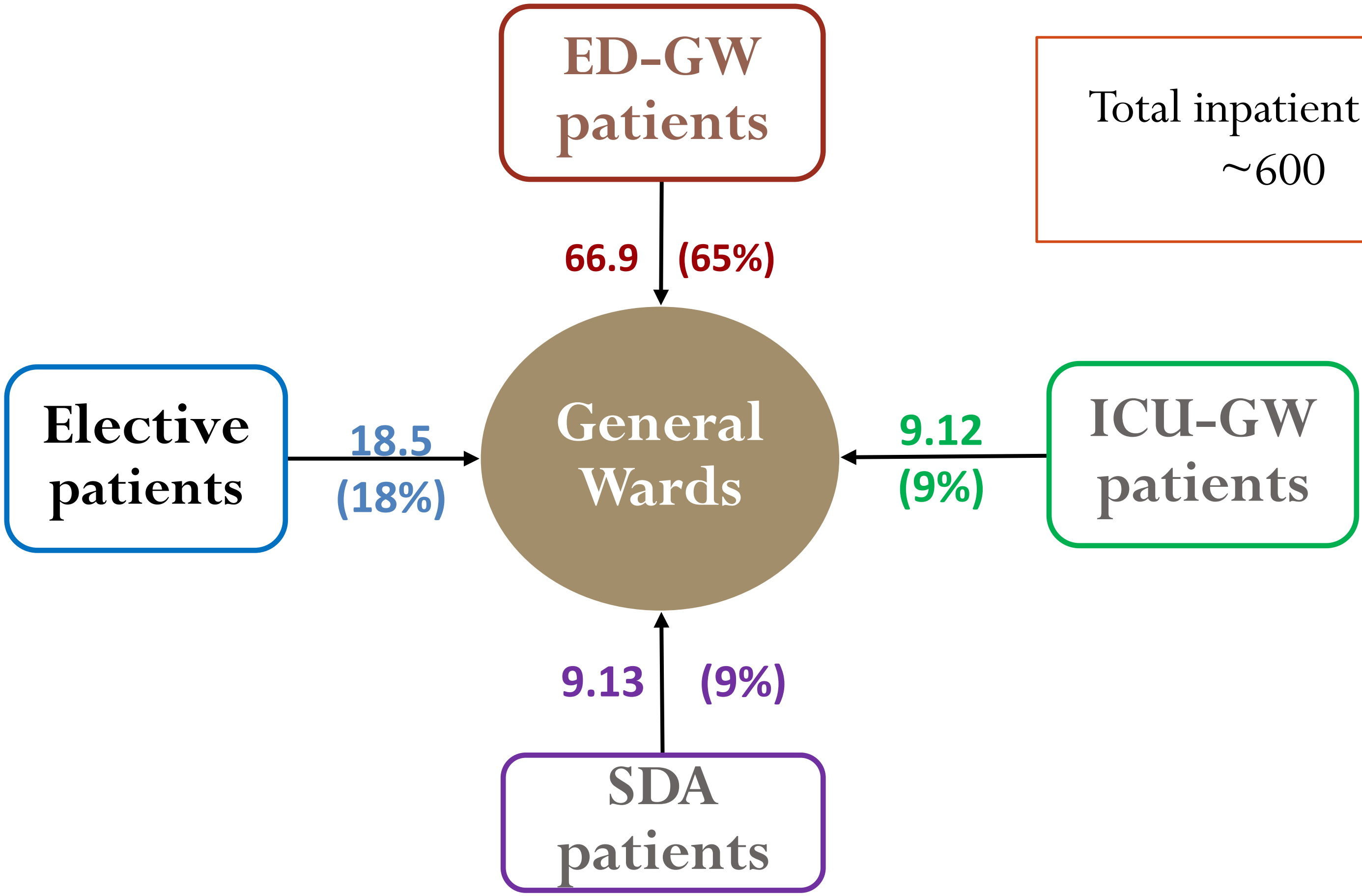
Part I

- Empirical observations
  - Online Supplement for "Hospital Inpatient Operations: Mathematical Models and Managerial Insights" (68 pages)

- Joint work with
  - James ANG and Mabel CHOU (NUS)
  - Ding DING (UIBE, Beijing)
  - Xin JIN and Joe SIM (NUH)

Capacity and source of admission

- Patients from 4 admission sources competing for inpatient beds



Total inpatient beds
~600

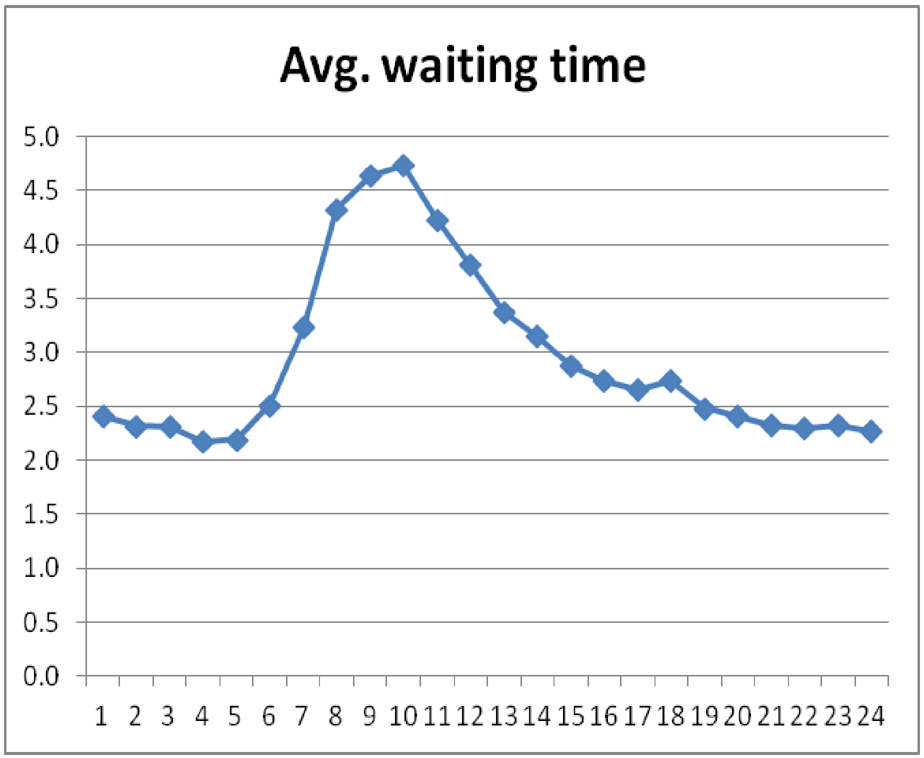
Key performance measures

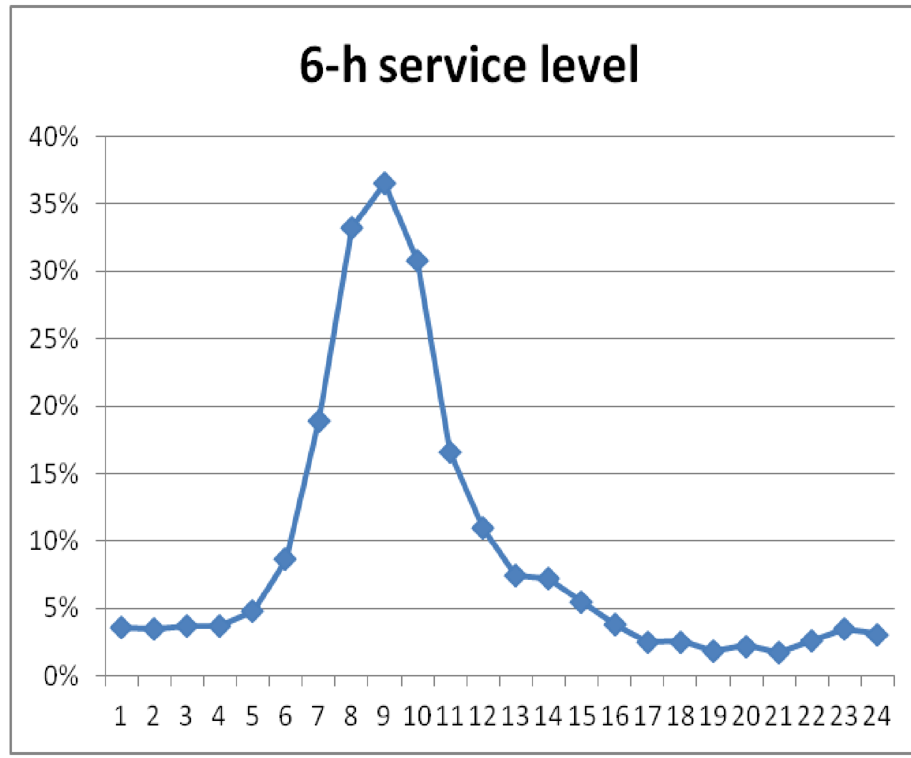
- Waiting time for admission to ward (Jan 08 – Jun 09)
  - Waiting time = admission time – bed request time
  - Average: **2.82** hour
  - **6.52**% of ED-GW patients wait more than 6 hours to get a bed
    - 6-hour service level
    - MOH cares

- Quality- and Efficiency-Driven (QED)
  - Average waiting time = 2.3% (average service time)
  - Average bed utilization = 90%

Time dependency

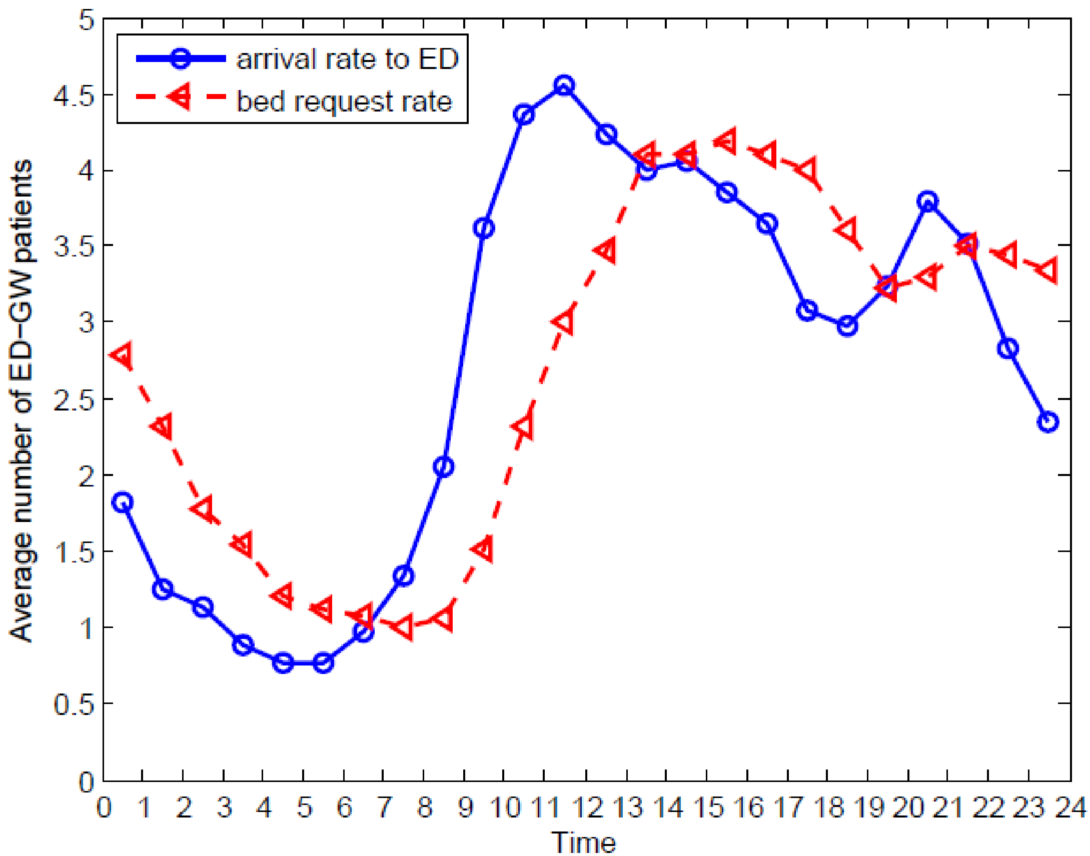
- Waiting time depends on patient's bed request time
  - Can we stabilize?





Time-varying bed request rate

- ED-GW patient's bed request rate (**red** curve) depends on arrival rate to ED (**blue** curve)



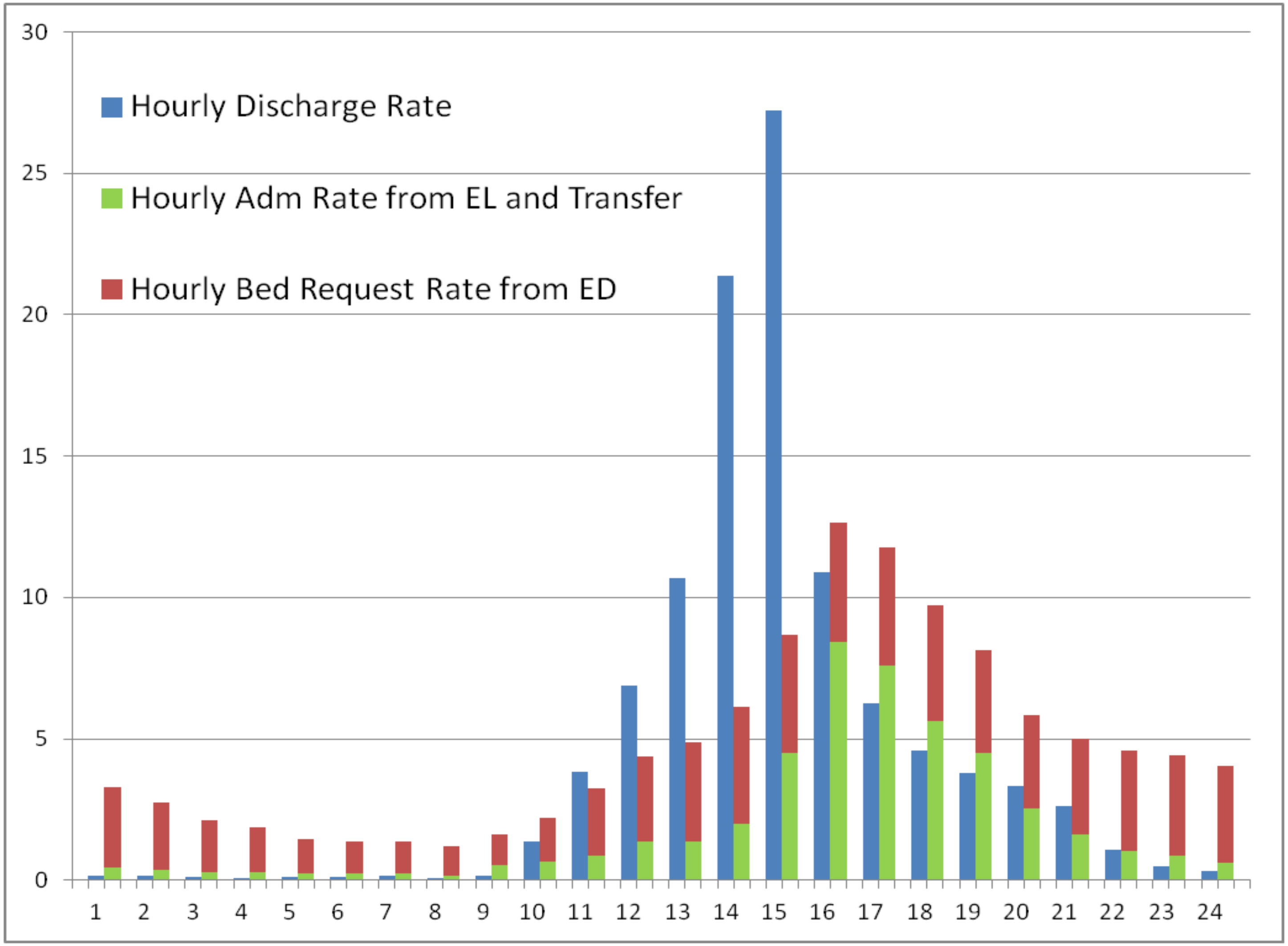
Learning from call center research?

- Zohar Feldman, Avishai Mandelbaum, William A. Massey and Ward Whitt, *Management Sciences*, 2008
  - **Staffing of Time-Varying Queues to Achieve Time-Stable Performance**

- Yunan Liu and Ward Whitt, 2012
  - **Stabilizing customer abandonment in many-server queues with time-varying arrivals**

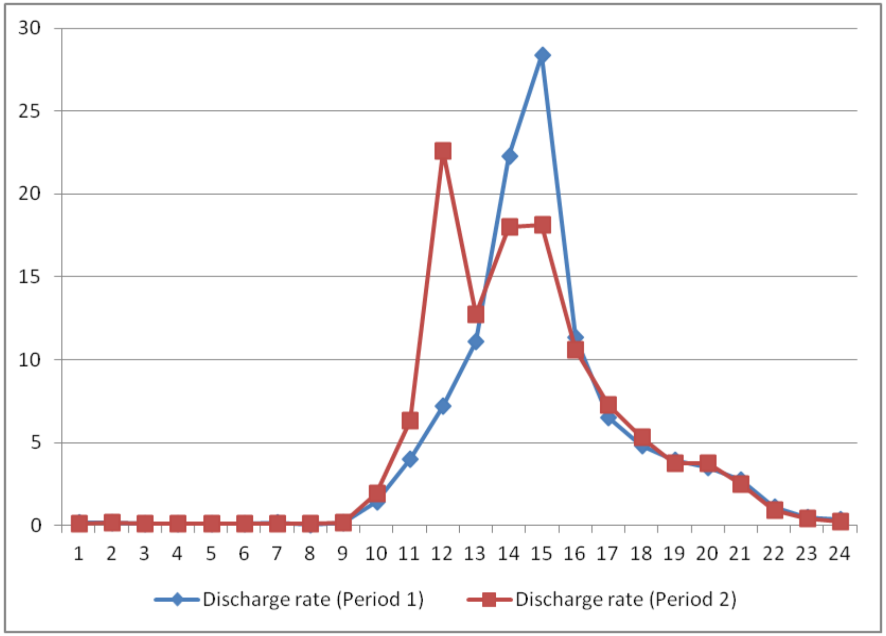
Mismatch between demand and supply of beds

- Jan 08 – Jun 09



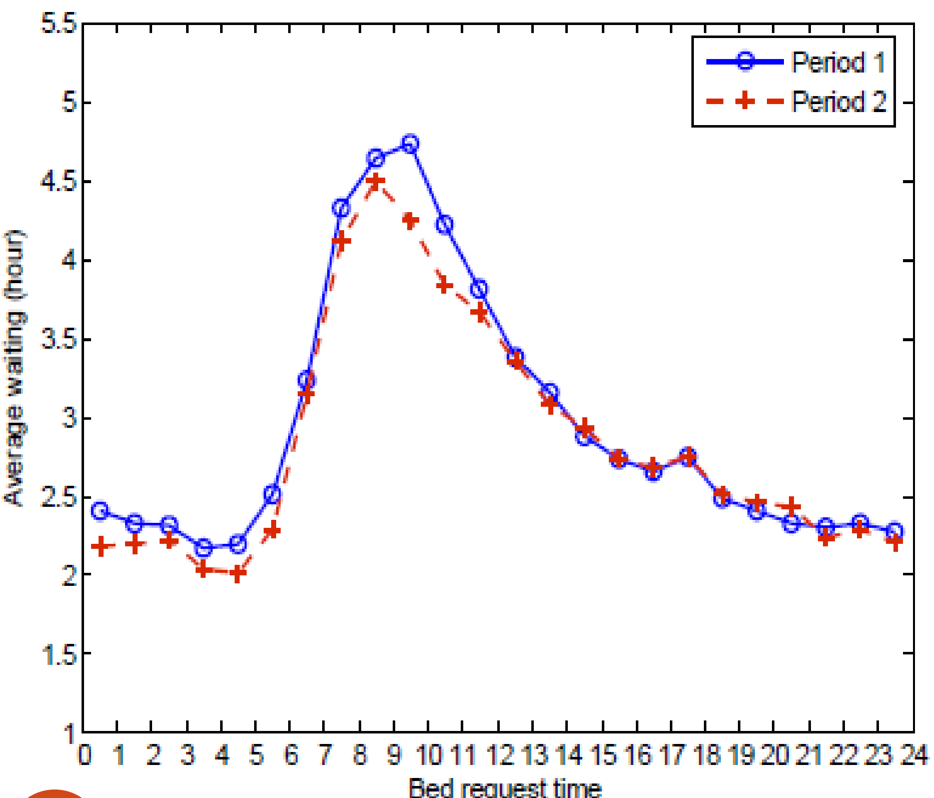
Discharge policy

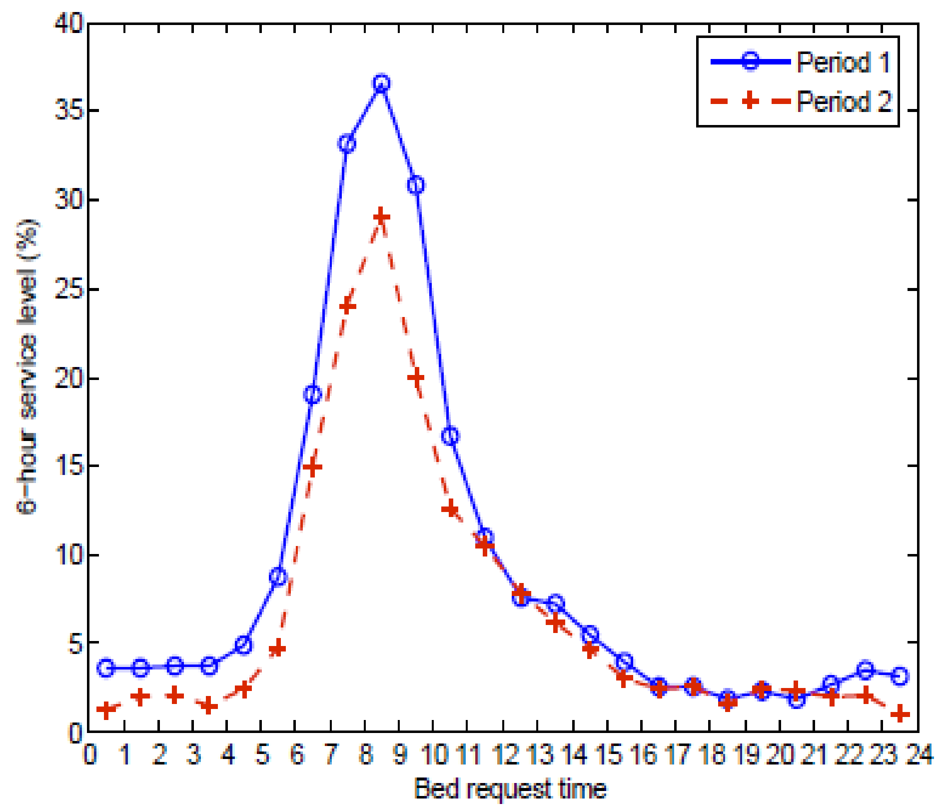
- Discharge timing affects the waiting time
- Early discharge policy
  - Moving the discharge time a few hours earlier in the day
- The hospital implemented early discharge policy since July 2009
  - Study two periods of data
    - Jan 2008 to Jun 2009 (Period 1)
      - 13% before noon
    - Jan 2010 to Dec 2010 (Period 2)
      - 26% before noon



Waiting time for ED-GW patients

	1st period	2nd period
Average waiting time	2.82 h	2.77 h
6-hour service level	6.52%	5.13%





Challenges

- Does the modest improvement come from the early discharge?
  - Changing operating environment
  - Both arrival volume and capacity increases during 2008 to 2010
  - Bed occupancy rate (BOR) reduces in the Period 2
    - Period 1: **90.3%**
    - Period 2: **87.6%**

- More importantly, is there any operational policy that can stabilize the waiting time?

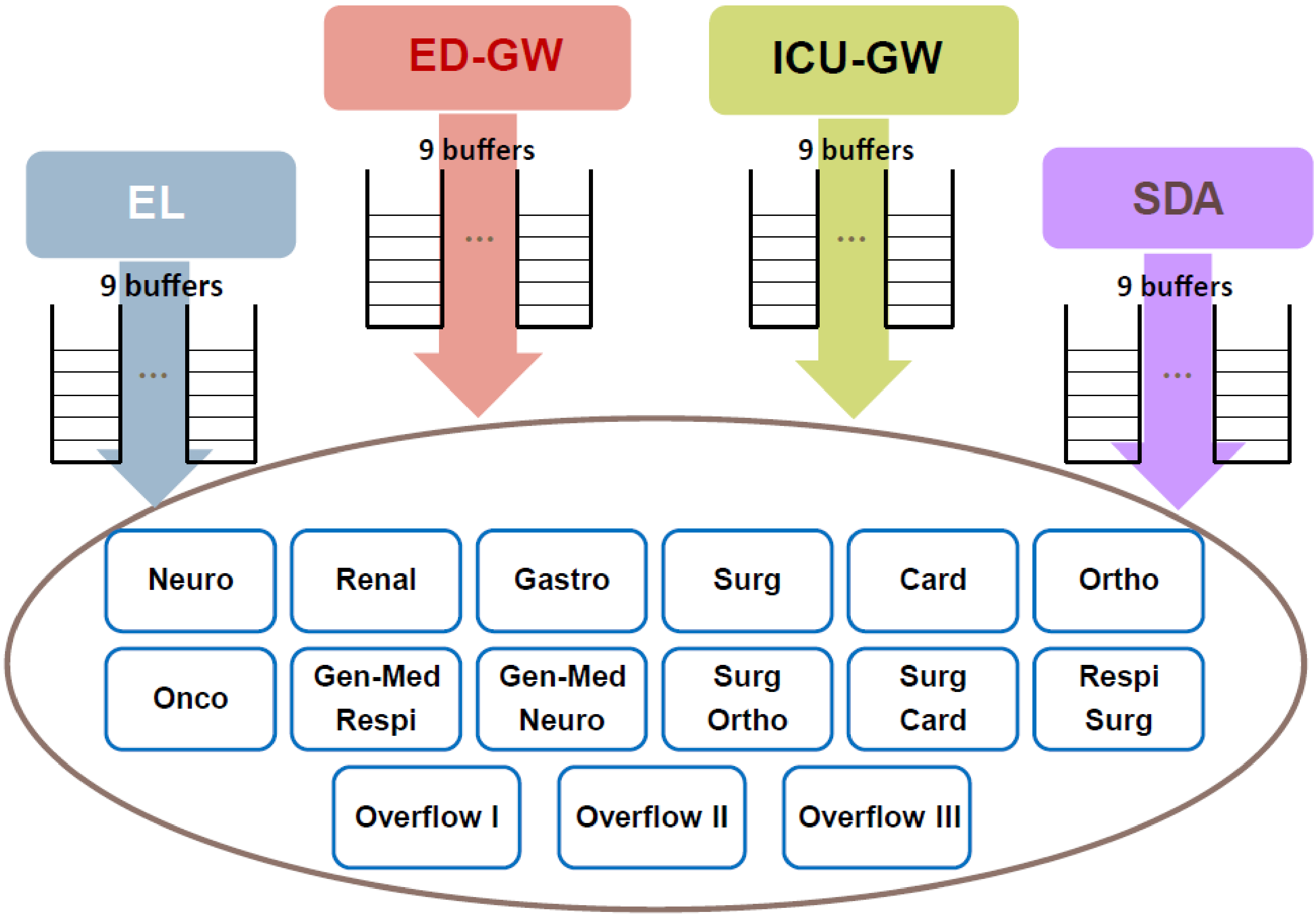
- Need a model to help

Part II: A stochastic model

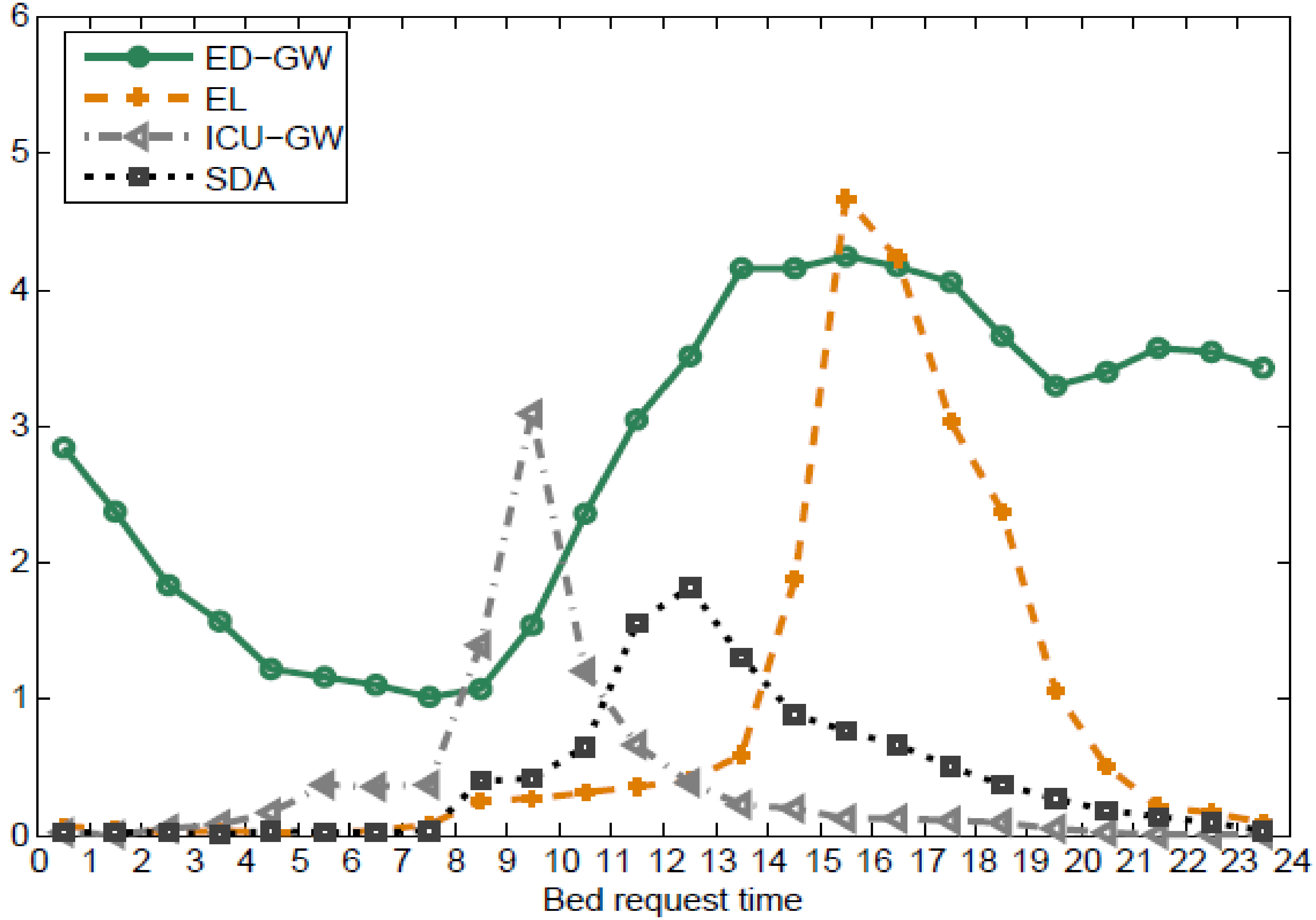
- Model
  - Hospital Inpatient Operations: Mathematical Models and Managerial Insights, submitted

- Joint work with Mabel Chou, Ding Ding, and Joe Sim

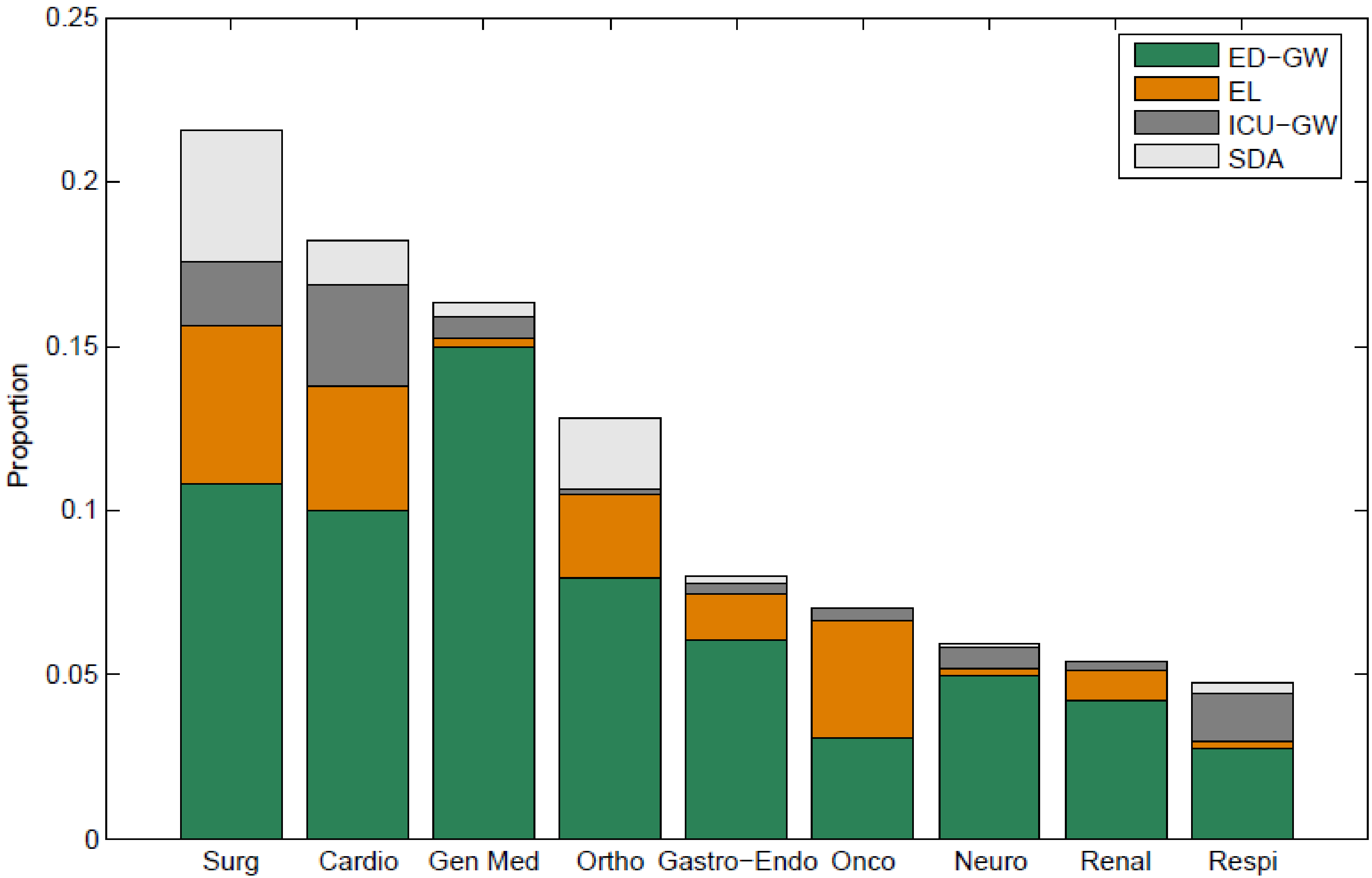
A multiclass, multi-server pool system



Time-varying arrival rates



Specialty distribution



Key modeling components

- Service time model
  - Determined by admission time, **LOS** and discharge distribution
  - An endogenous modeling element
  - No longer i.i.d.

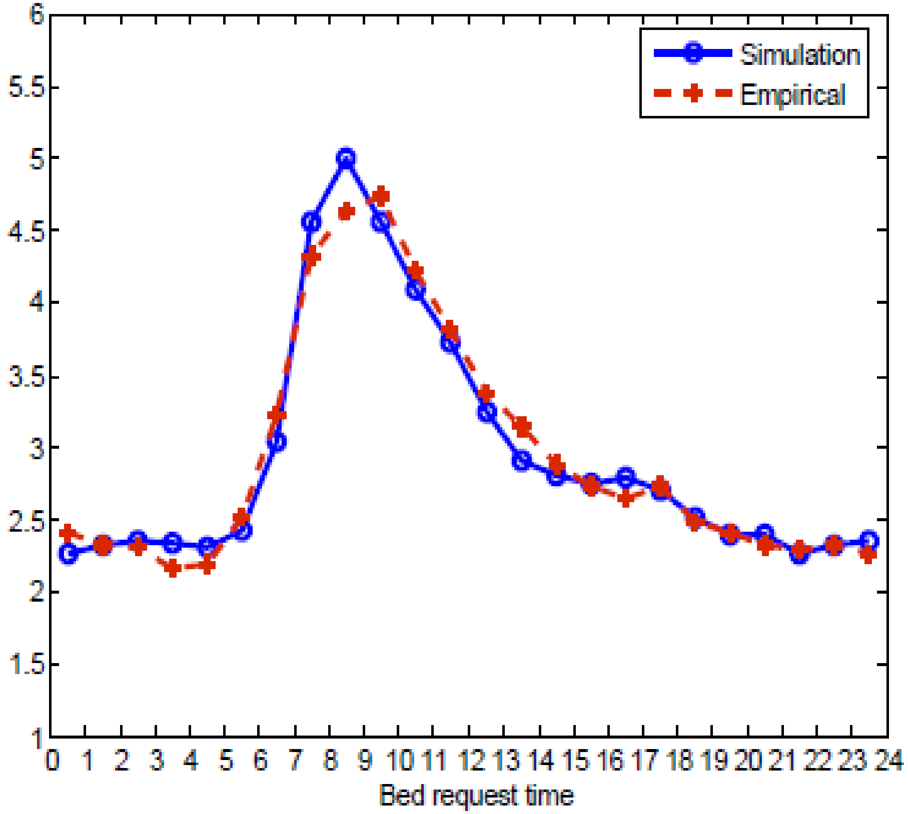
- Allocation delays
  - "Secondary" bottlenecks other than bed availability
    - Yankovic and Green (2011)
    - Armony et al (2011)

- Overflow policy
  - When to overflow a patient
  - Overflow to which server pool

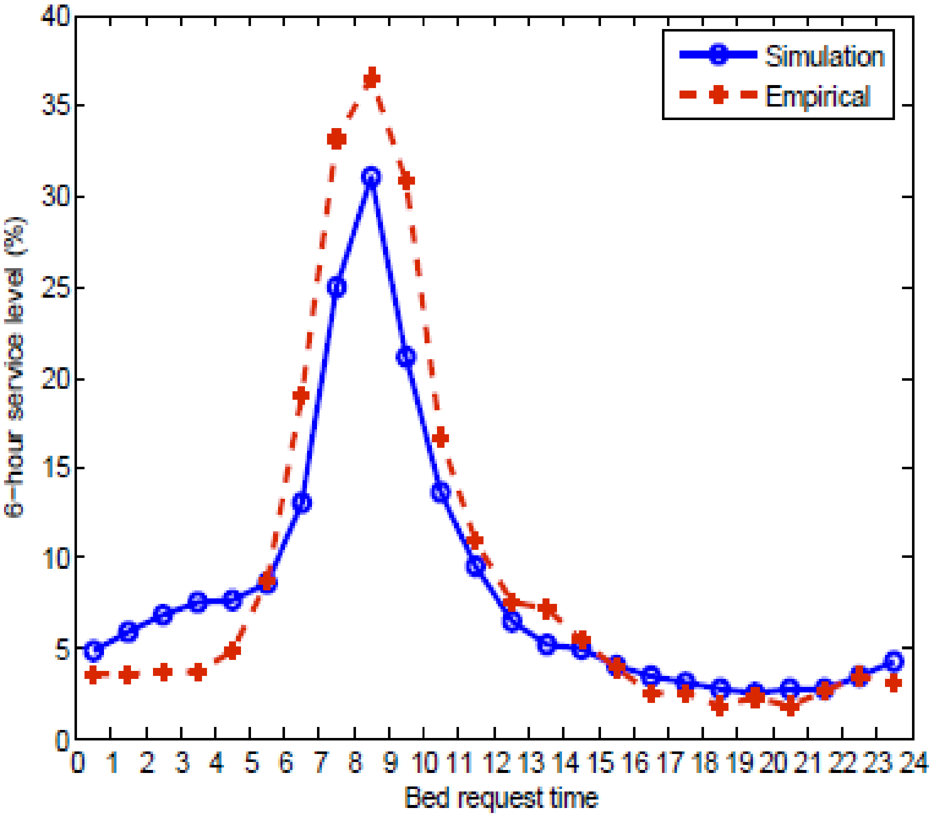
Simulation replicates most performance measures

- Hourly waiting time performances

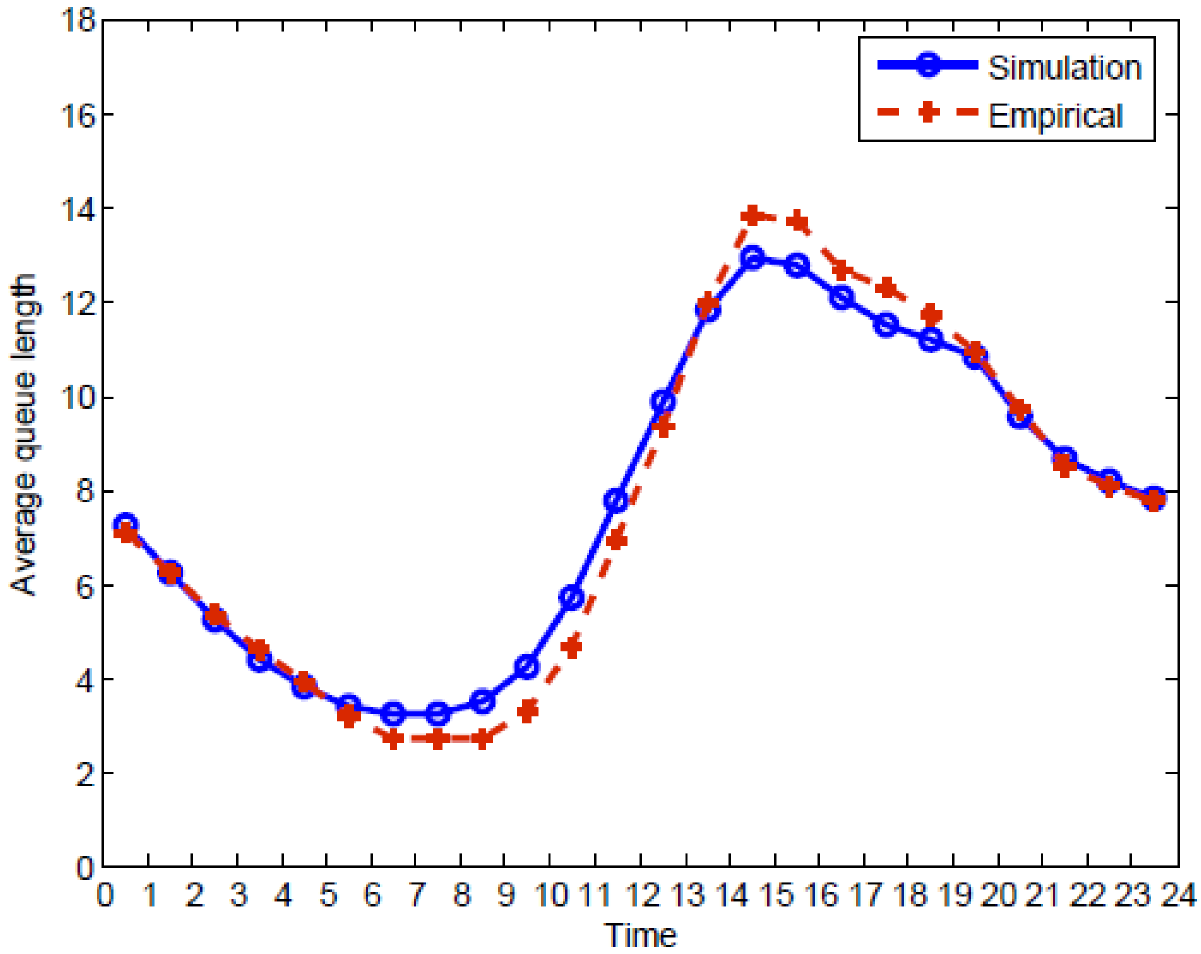
(a) Hourly average waiting time



(b) Hourly 6-hour service level



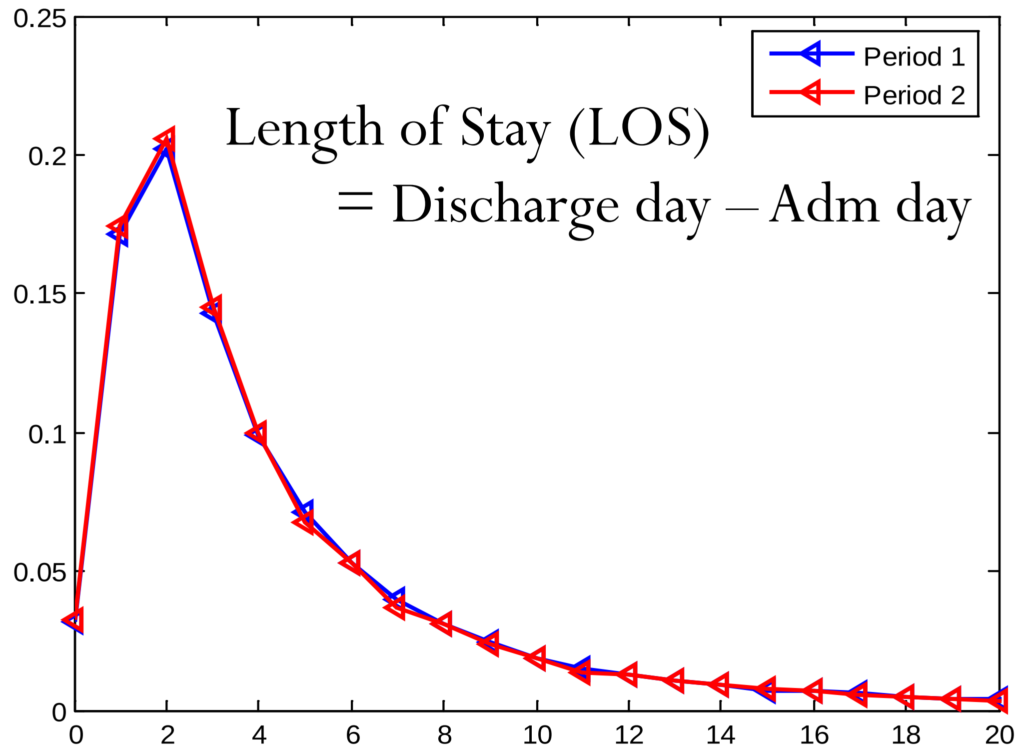
Time-dependent queue length



Service times are endogenous

- Service time model
  - Service time = Discharge time – Admission time
    = **LOS + Dis hour – Adm hour**

- LOS distribution
  - Average is ~ 5 days
  - Depend on *admission source* and *specialty*
  - AM- and PM- dependent for ED-GW patients

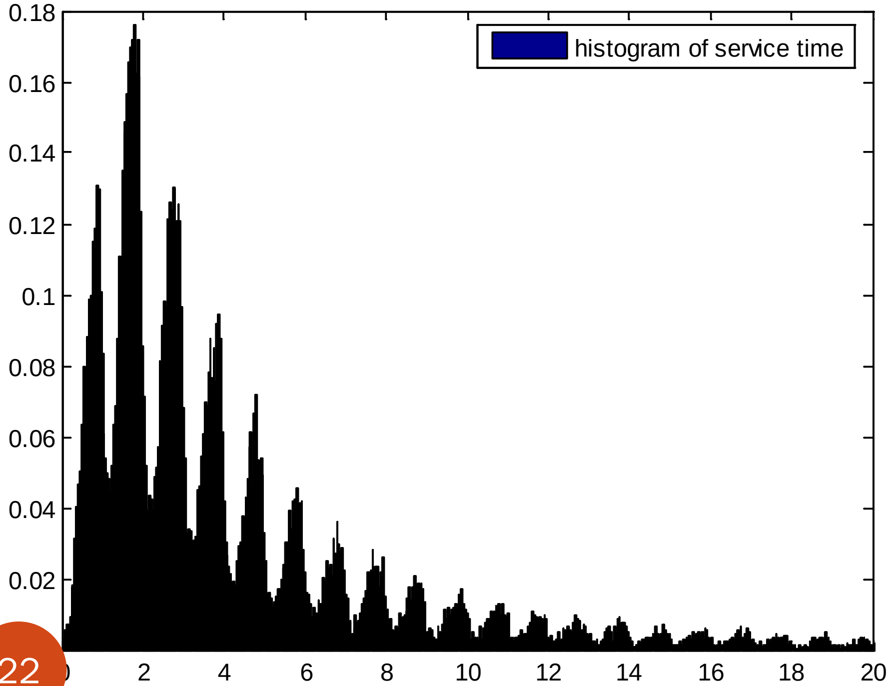


Verify the service time model

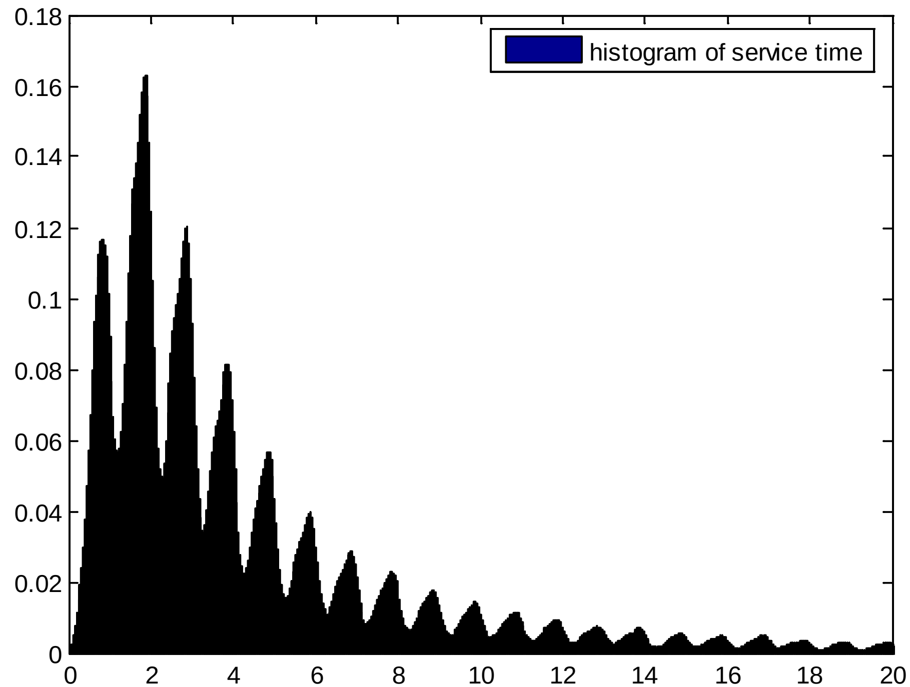
- Service time model
  - Service time = LOS + Discharge hour – Adm hour

Matching empirical

(a) Empirical (Armony et al 2011)

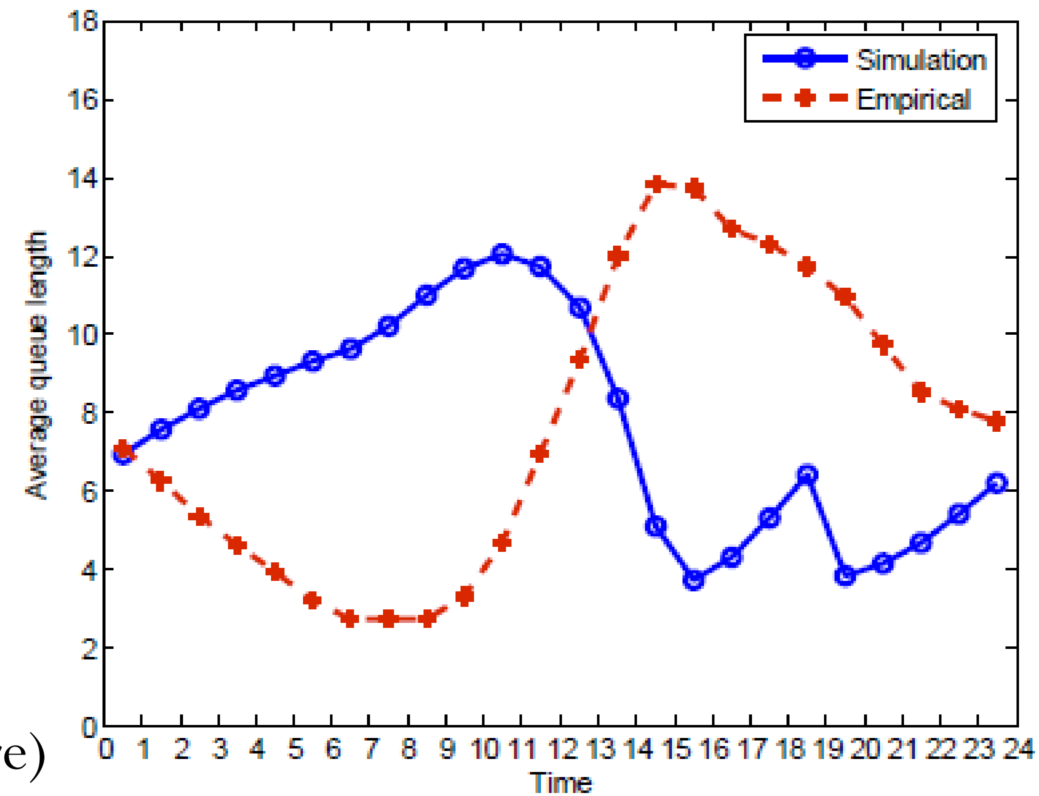


(b) Simulation output



Pre- and post-allocation delays

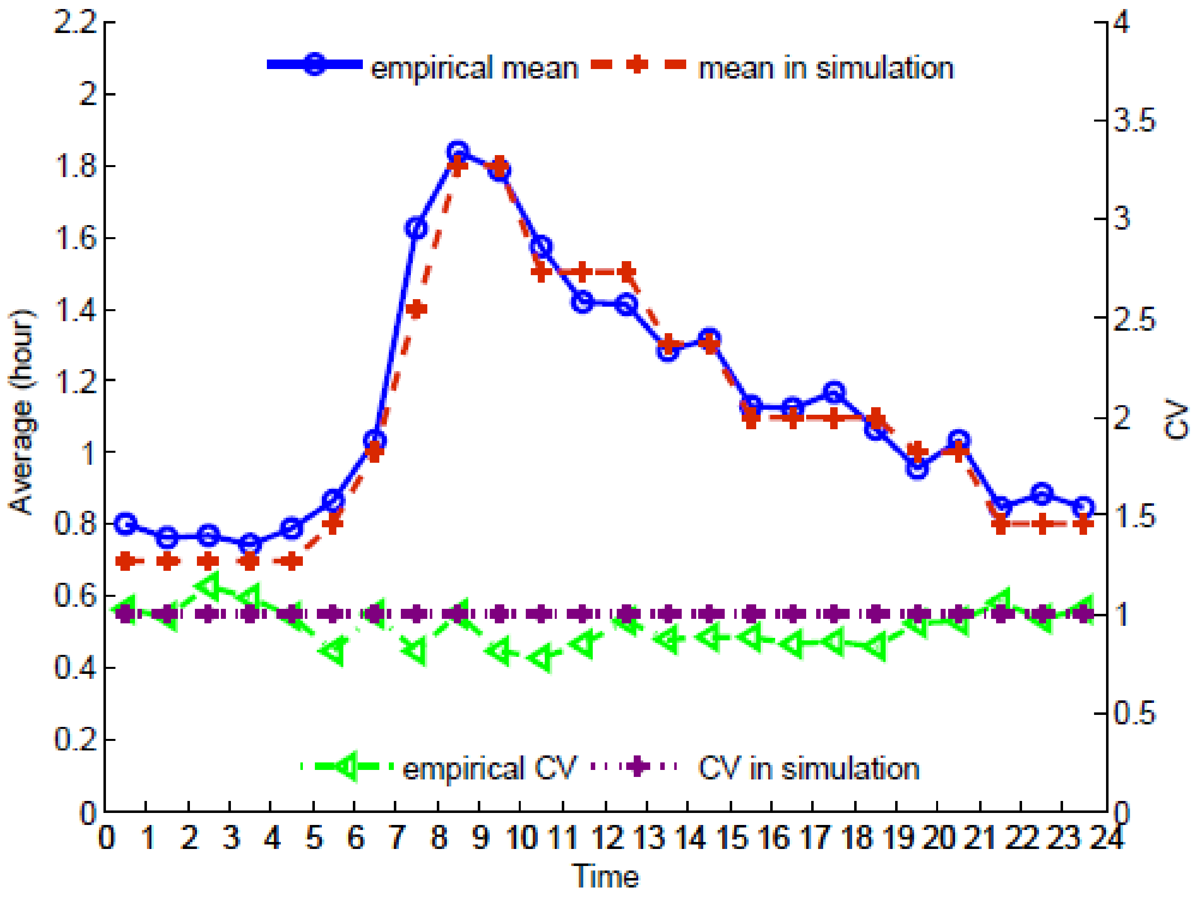
- Patient experiences additional delays upon arrival and when a bed is allocated
  - Pre-allocation delay
    - BMU search/negotiate for beds
  - Post-allocation delay
    - Delays in ED discharge
    - Delays in the transportation
    - Delays in ward admission



- Must model allocation delays
  - If not, hourly queue length does not match (right figure)

Time-dependent allocation delays

- The mean of allocation delay depends on when it is initiated
  - Use log-normal distribution
  - Pre-allocation delay



Overflow policy

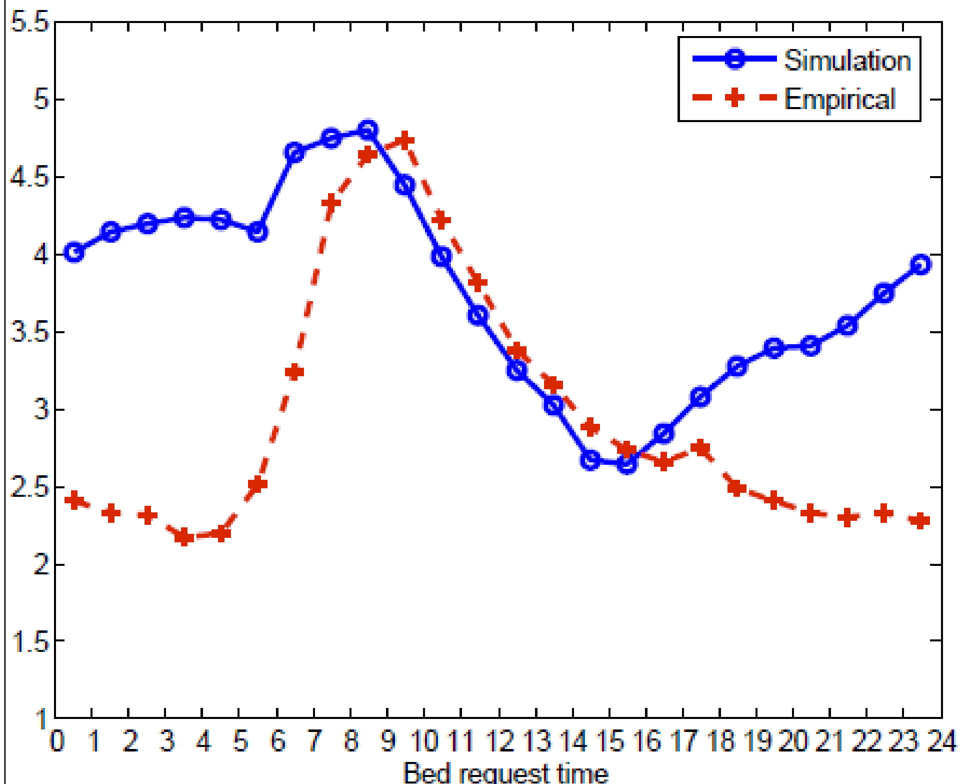
- When a patient's waiting time exceeds certain threshold, the patient can be overflowed to a "wrong" ward
  - Beds are partially flexible
  - Overflow wards have certain priority

Cluster	1st Overflow	2nd Overflow	3rd Overflow
Medicine	Other Med	Surgery/OG	Ortho
Surgery	Other Surg	Ortho /OG	Medicine
Ortho	Other Ortho	Surgery	Medicine

Dynamic overflow policy

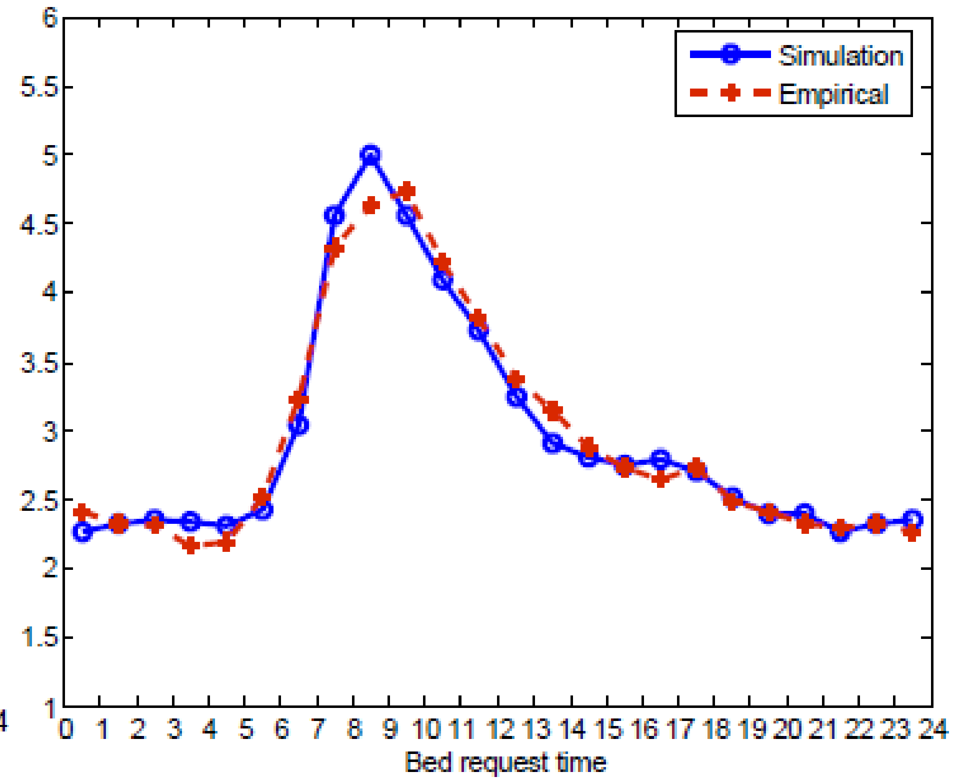
Fixed threshold

- Threshold: 4.0 h



Dynamic threshold

- Threshold: 0.5 h for arrival between 7 pm and 7 am (next day); 5.0 h for others



Part III: Analytical analysis

- Two-time scale method to predict time-dependent performance measures

Two-time scale

- Discrete queue
  - Average LOS and daily arrival rate determine $\{X_k\}$, and thus performances at mid-night (daily level)

- Time-varying performance
  - The arrival rate pattern, discharge timing, and allocation delay distribution determine the hour-of-day behavior

A simplified model

- Single cluster
  - No overflow
- Arrival is periodic Poisson
- LOS follows a Geometric distribution
- Discharge follows a simple discrete distribution

- Service time follows the non-iid model:

$$S = \mathrm{LOS} + h_{\mathrm{dis}} - h_{\mathrm{alloc}}$$

  - Admission time is replaced by allocation time

- Allocation delay
  - Each customer experiences a random delay after allocation time

Predict the time-dependent average queue length

- Decompose the queue length into two parts
  - Queue for beds: patients who are waiting for a bed
  - Alloc-delay queue: patients who are allocated with beds and are experiencing the alloc-delay

Queue for bed (1/2)

- X_k denotes the number of customers at midnight of day k

$$X_{k+1} = X_k - \Phi_k + \Lambda_k$$

  - Discrete queue

- Number of discharges only depends on X_k since
  - LOS is geometric ("coin toss" every day)
  - LOS starts from 1 (i.e., no same-day discharge)

- Number of arrivals follows Poisson distribution
  - Independent of number of discharges

- $\{X_k\}$ is a Markov process
  - Stationary distribution can be solved explicitly
  - Ramakrishnan et al. (2005)

Queue for bed (2/2)

- Using the stationary distribution of $\{X_k\}$
  - The average number of customers in system and the average queue length can be obtained for any time point
  - Average number of customer in system can be solved in a fluid way
    - $E[Y(t)] = E[X_k] + \int_0^t \lambda(t)dt - E[\text{discharge}(0,t)]$
    - Powell et al. (2012)

  - Queue length needs to be obtained from the distribution of number of customers in system at each time point $Y(t)$
    - Conditioning on X_k
      - $Y(t)$ is a convolution between arrival (Poisson r.v.) and discharge (Binomial r.v. depends on the value of X_k) till t

Related work

- E. S. Powell, R. K. Khare, A. K. Venkatesh, B. D. Van Roo, J. G. Adams, and G. Reinhardt, *The Journal of Emergency Medicine*, 2012
  - **The relationship between inpatient discharge timing and emergency department boarding**
  - Affiliations: Department of Emergency Medicine, Northwestern University; Harvard Affiliated Emergency Medicine Residency, Brigham and Women's Hospital–Massachusetts General Hospital, …

Alloc-delay queue

- Each patient experiences a random amount of delay
  - The alloc-delays follow an iid distribution with CDF $F(x)$
  - Patient gets a bed before entering the alloc-delay queue

- Two scenarios
  - Unlucky patient: no bed available upon arrival
    - Waits in the queue for bed first
    - Gets a bed at a discharge time point
  - Lucky patient: gets a bed allocated upon arrival
    - Directly joins the alloc-delay queue

Unlucky patients

- Suppose discharges occur at $t_1,\ t_2,\ t_3,\ t_4$
- The mean number of admissions at each discharge point can be calculated from X_k , arrivals and discharges

- Given the mean number of admissions $Z(t_i)$
  - Mean number of customers in the alloc-delay queue after s hours is $Z(t_i) \cdot (1 - F(s))$

Lucky patients

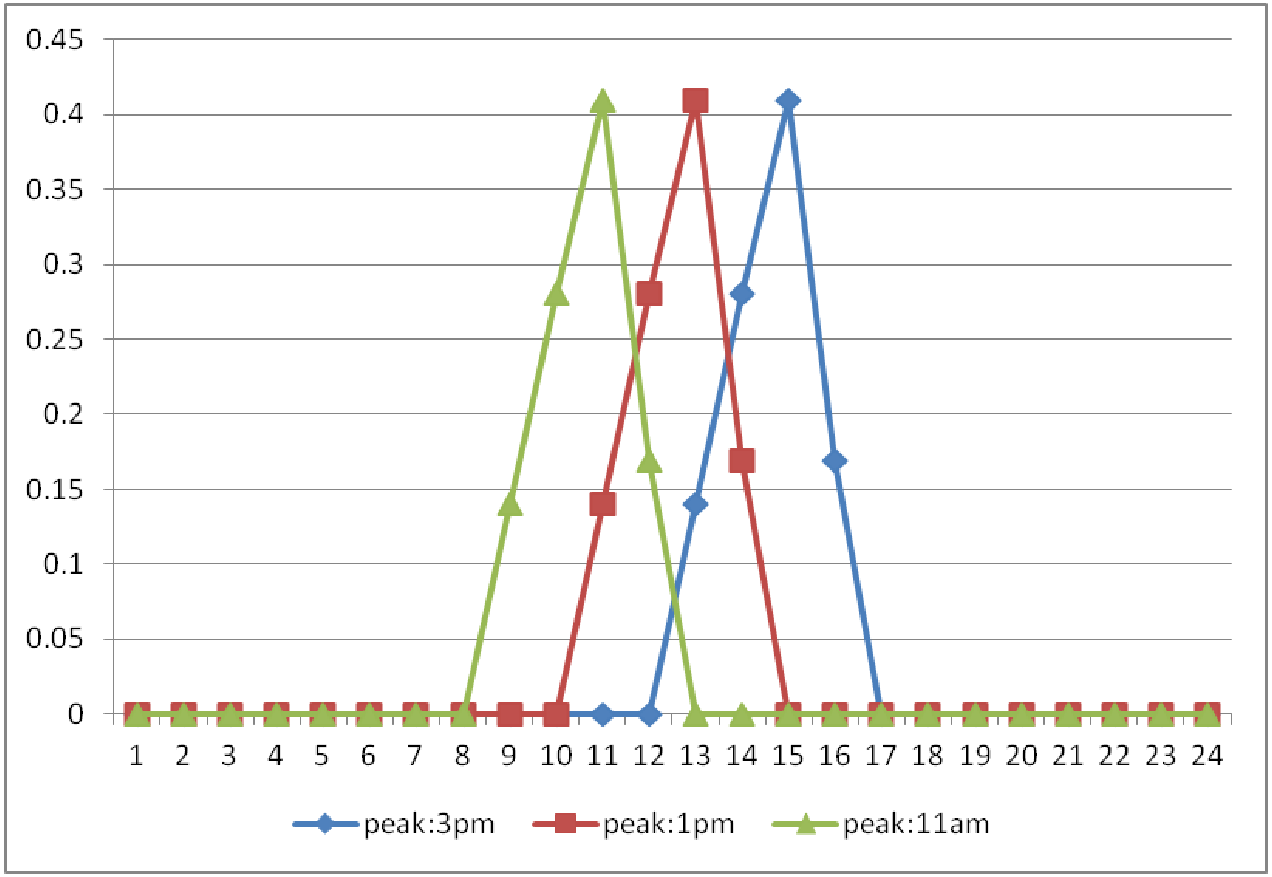
- The *effective* admission process (bed-allocation process) is non-homogeneous Poisson
  - The probability of an arriving patient being lucky or unlucky is independent of the arrival itself
  - The effective admission rate can be calculated from X_k, arrivals and discharges

- Consider the alloc-delay queue as an infinite-server queue
  - Service time is the allocation delay
  - The effective admission process constitutes the arrival
  - Infinite-server queue theory (Eick - 1993):

$$m(t) = E[\lambda(t - F_e)]E[F]$$

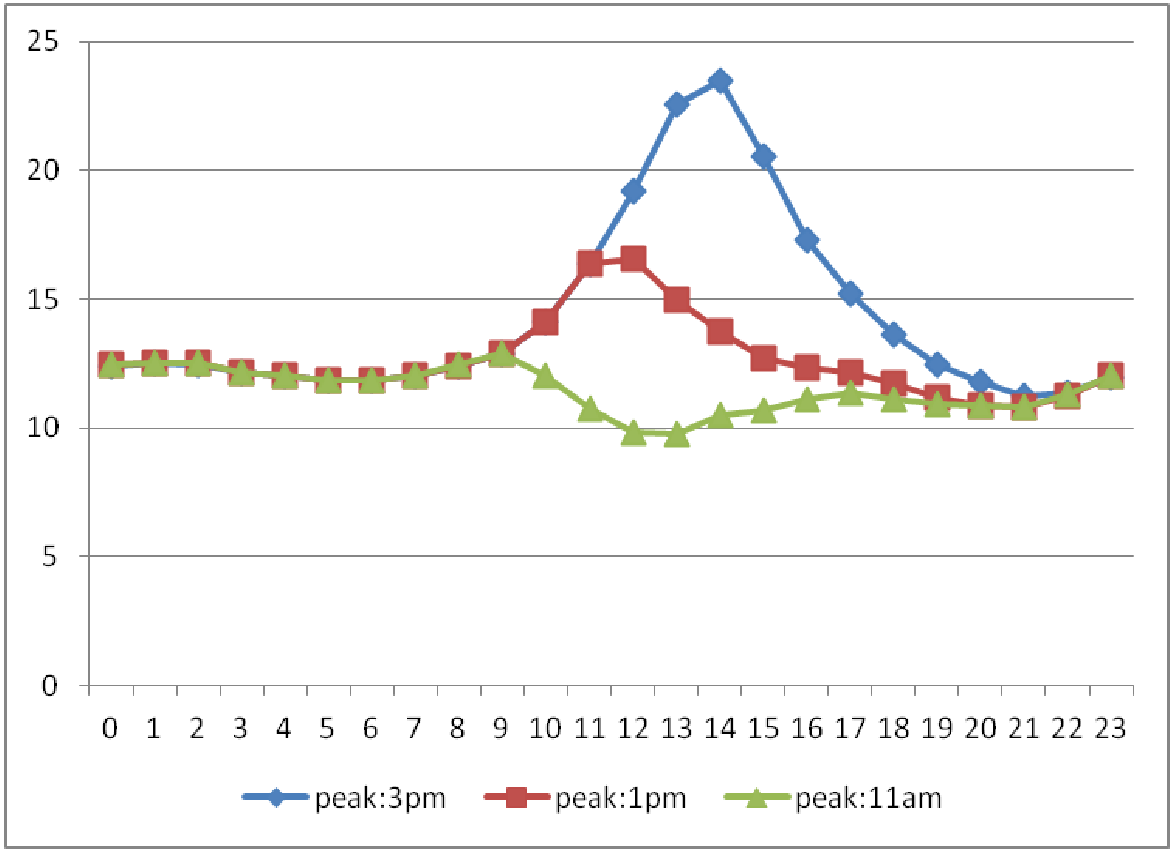
Numerical results

- Alloc delays follow iid exponential distribution with mean 2 hours
- Simple discrete distribution:



Numerical results

- Avg queue length



Insights from the simplified model

- The average number of customers in the system remain the same in scenarios with and without allocation delays

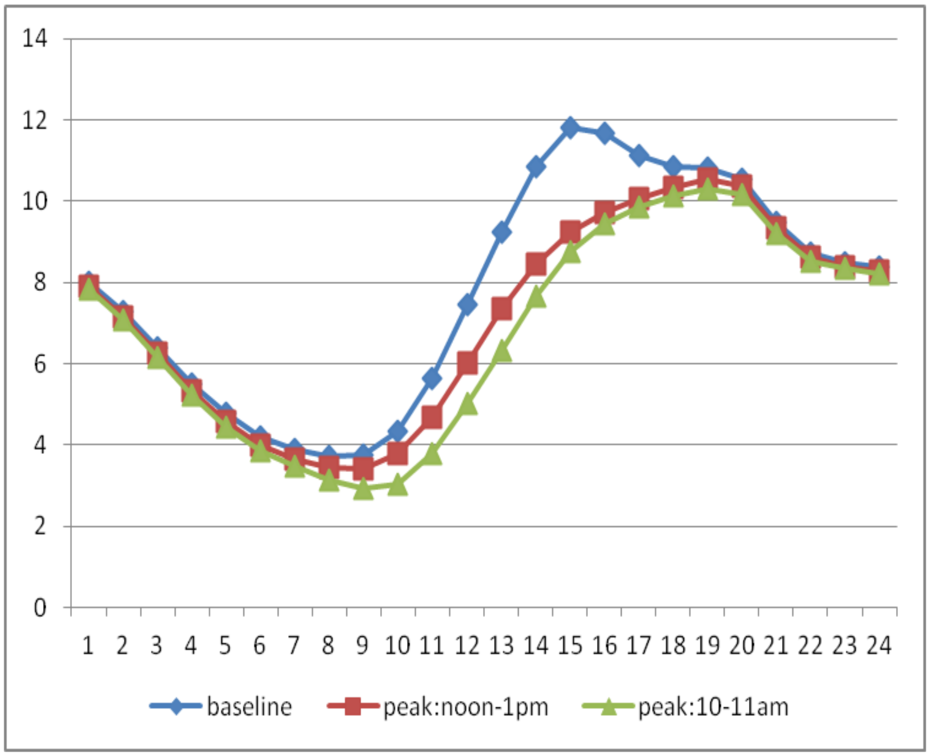
- Challenging to predicting the hourly queue length
  - Necessary to model allocation delays
    - Slower drop in the queue length after 2pm

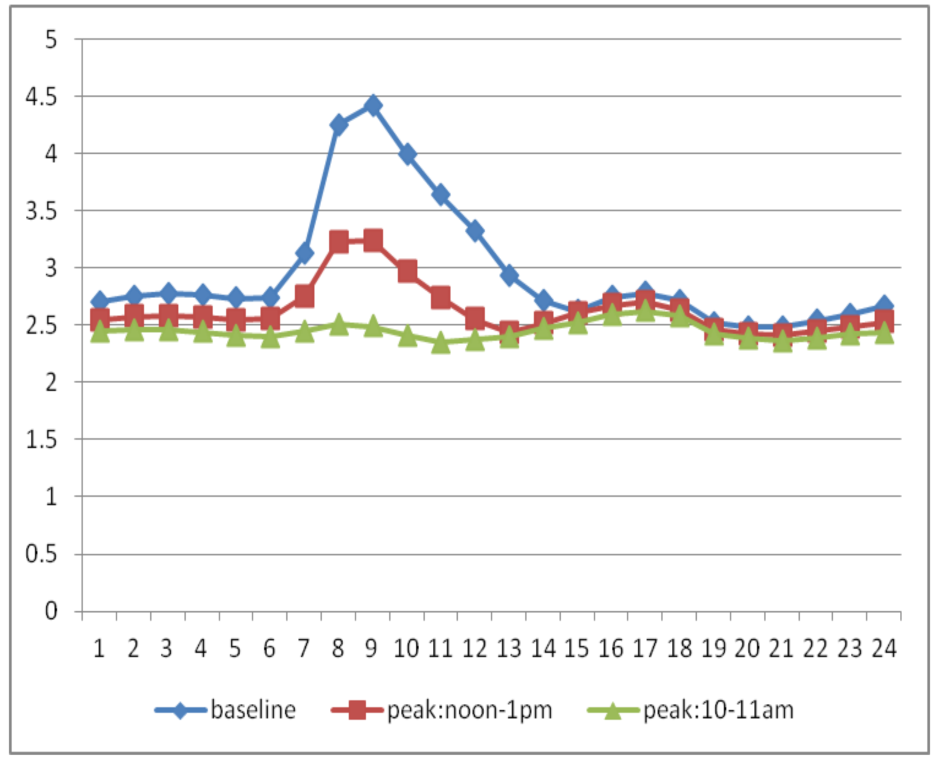
- Early discharge helps stabilize the hourly queue length

Shift the Period 1 discharge curve

- Using constant-mean allocation delay
  - Avg queue length

Avg waiting time





Part IV: Managerial insights

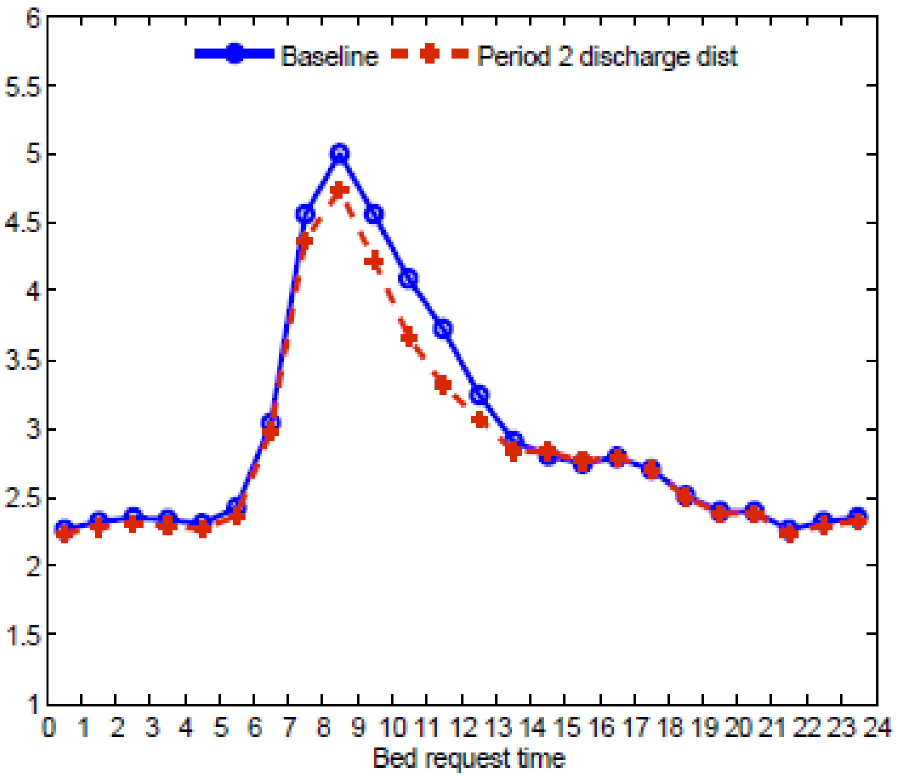
- Whether early discharge policy is beneficial or not

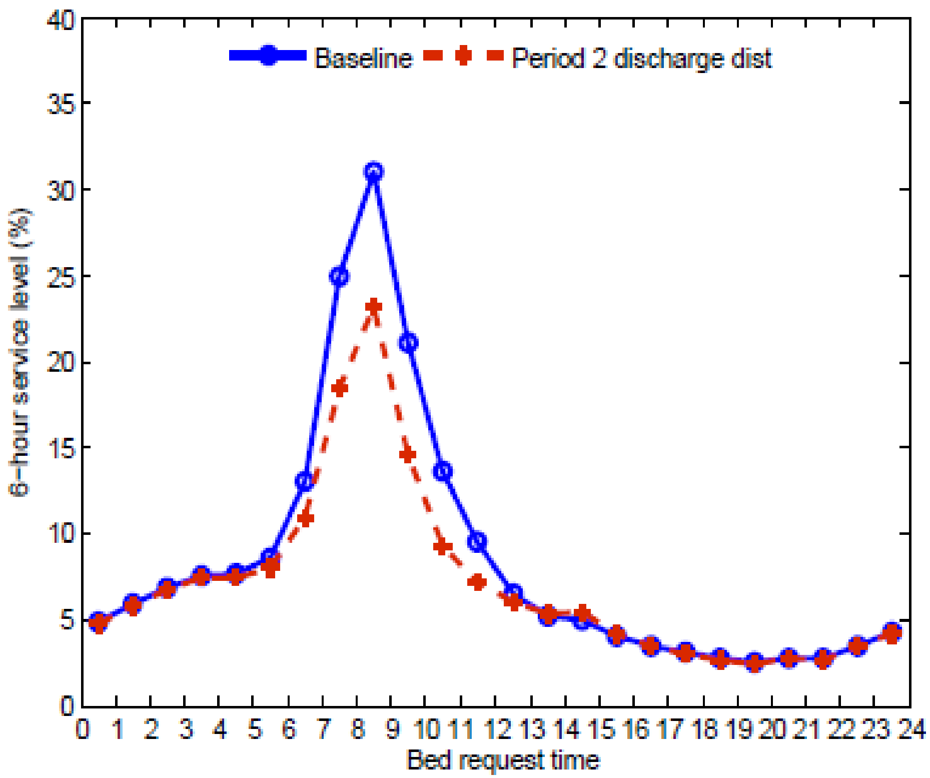
- What-if analysis

Simulation results

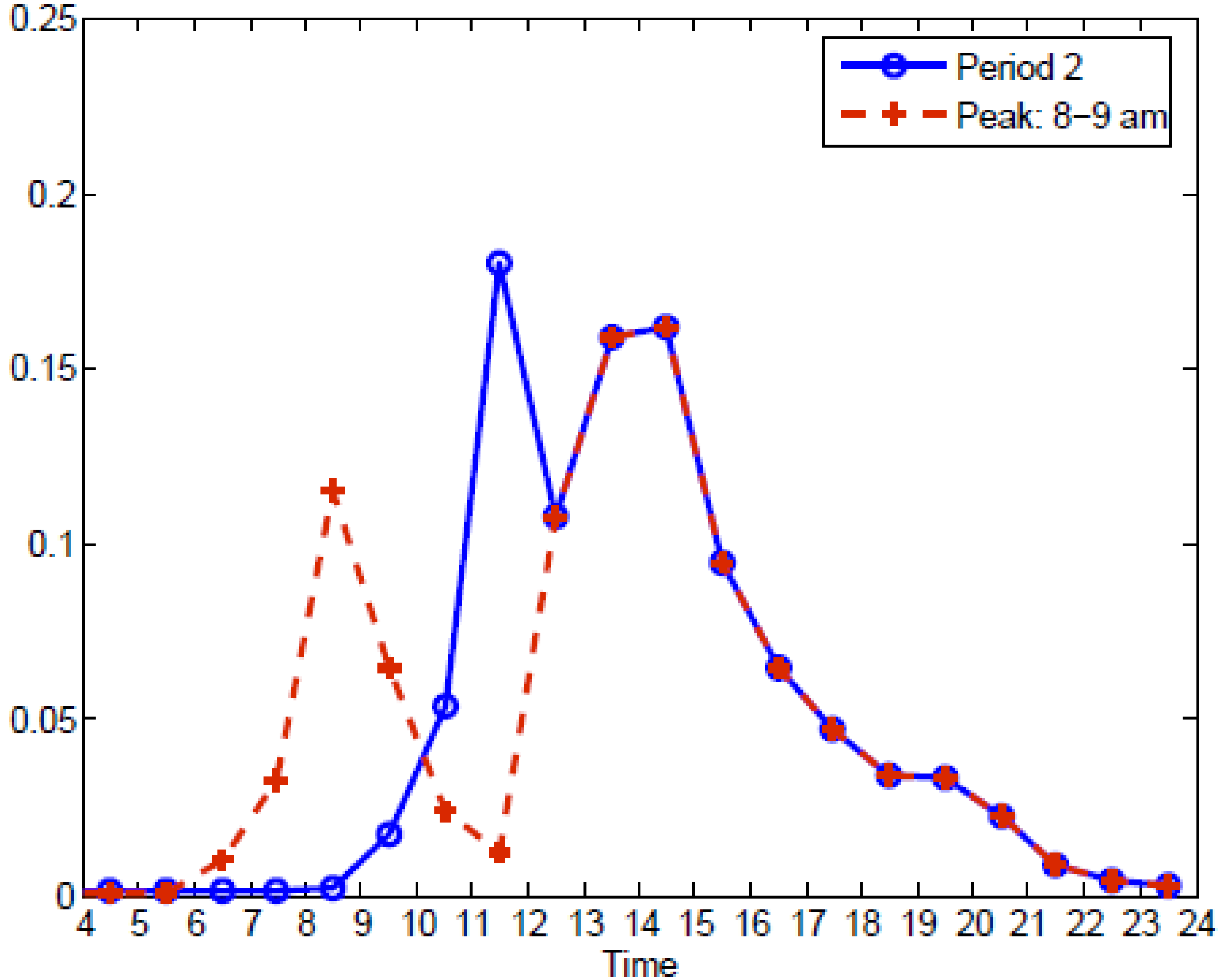
- Simulation shows NUH early discharge policy has little improvement

  (a) hourly avg. waiting time (b) 6-hour service level





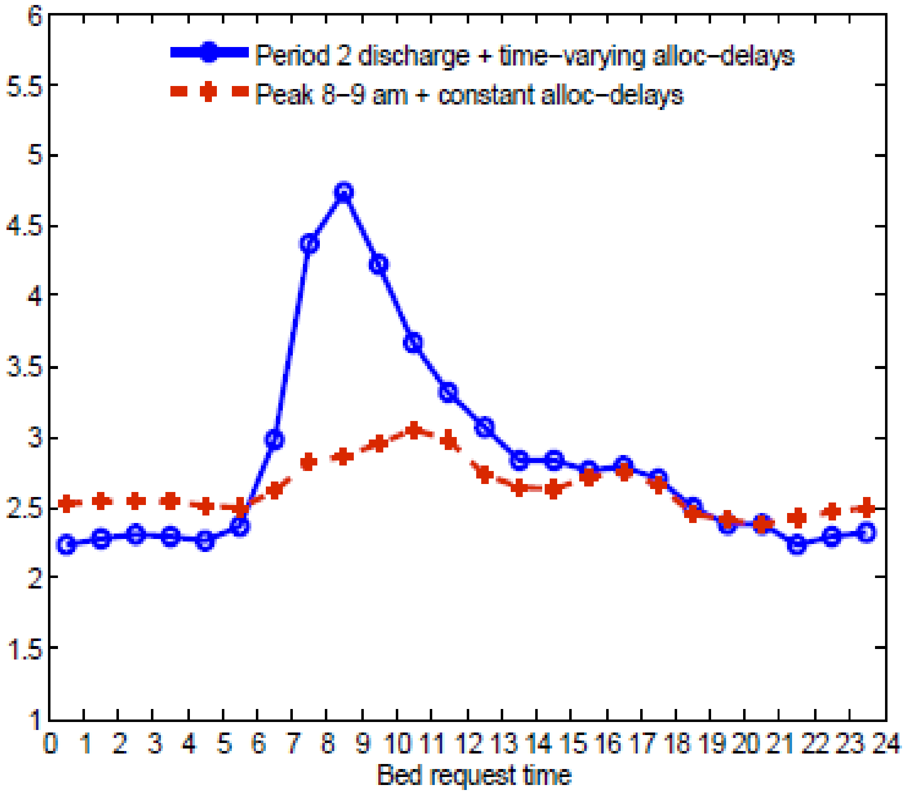
Aggressive early discharge policy

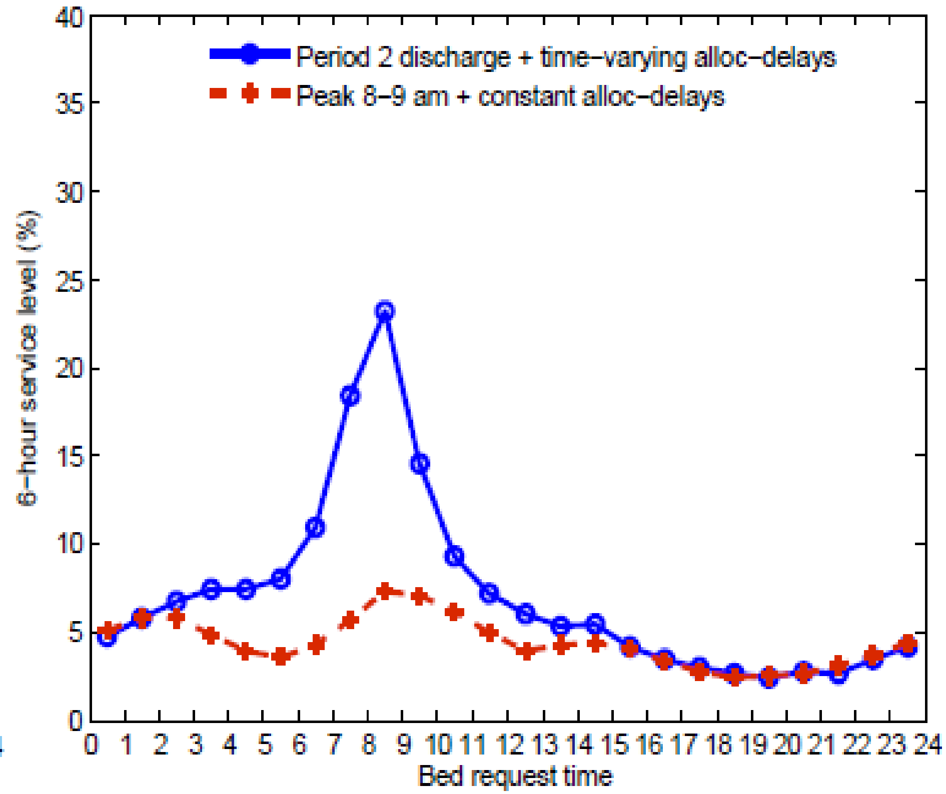


Aggressive early discharge + smooth allocation delay

- Waiting time performances can be stabilized

(a) hourly avg. waiting time (b) 6-hour service level

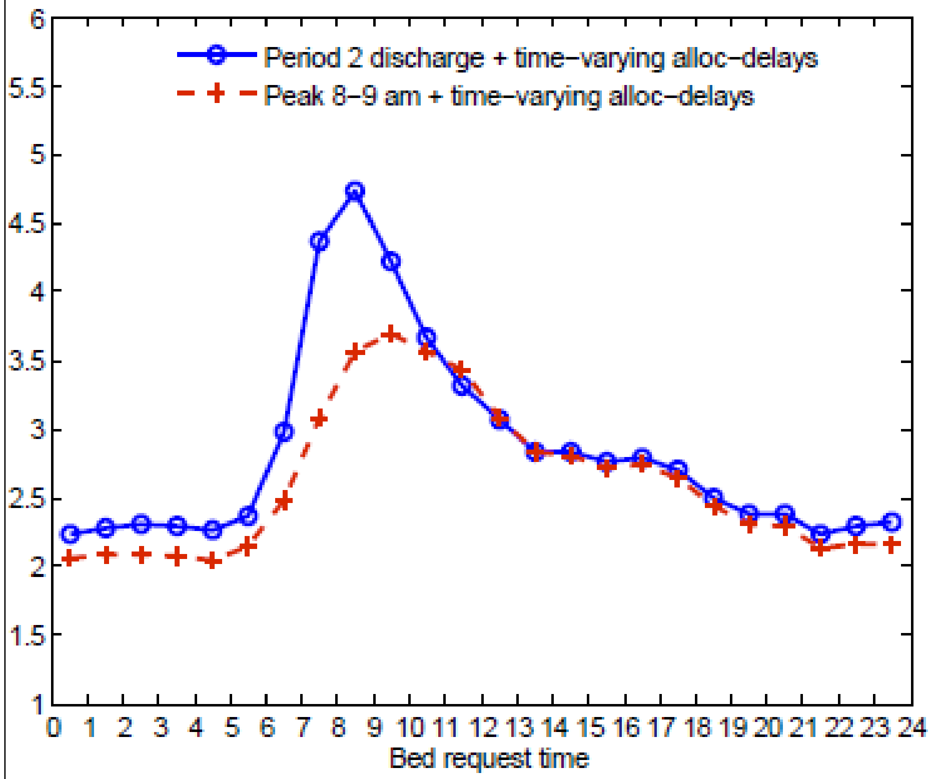




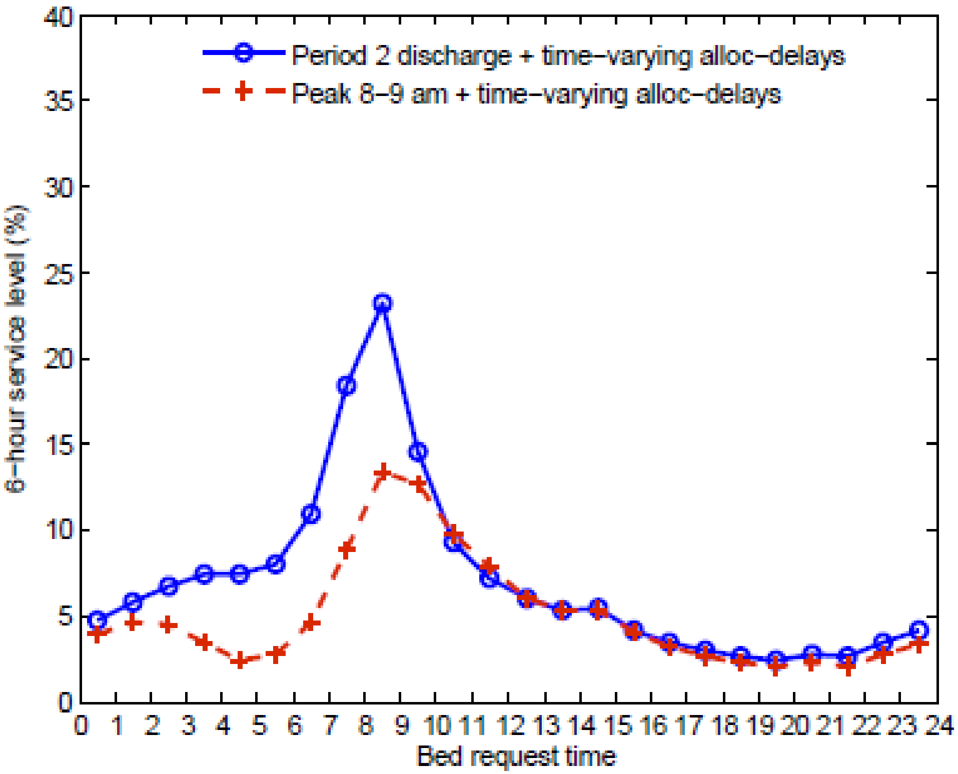
Only use aggressive early discharge

- Cannot be stabilized

(a) hourly avg. waiting time



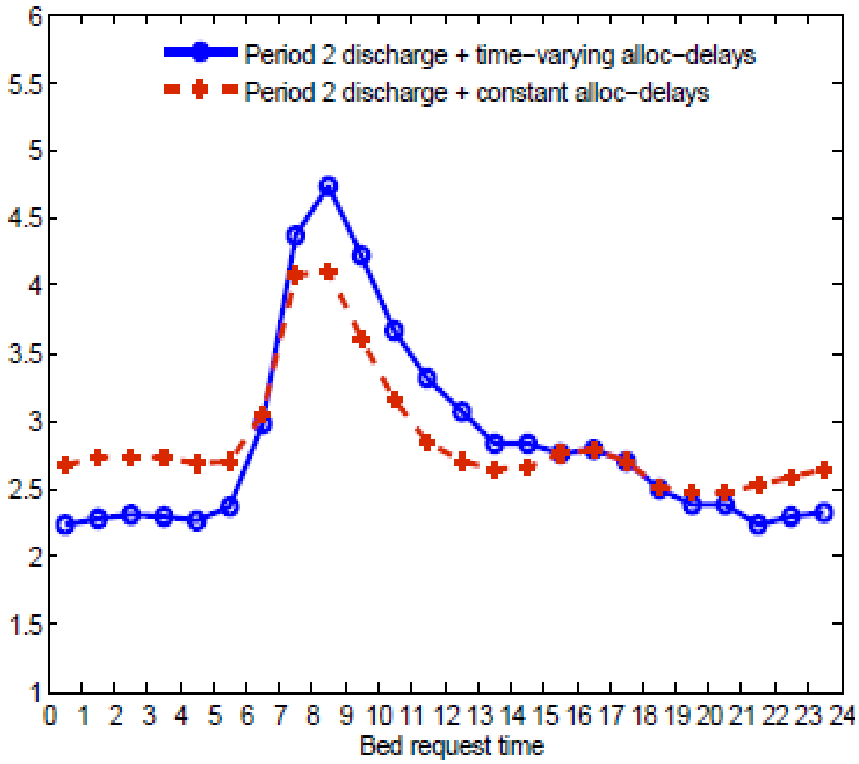
(b) 6-hour service level

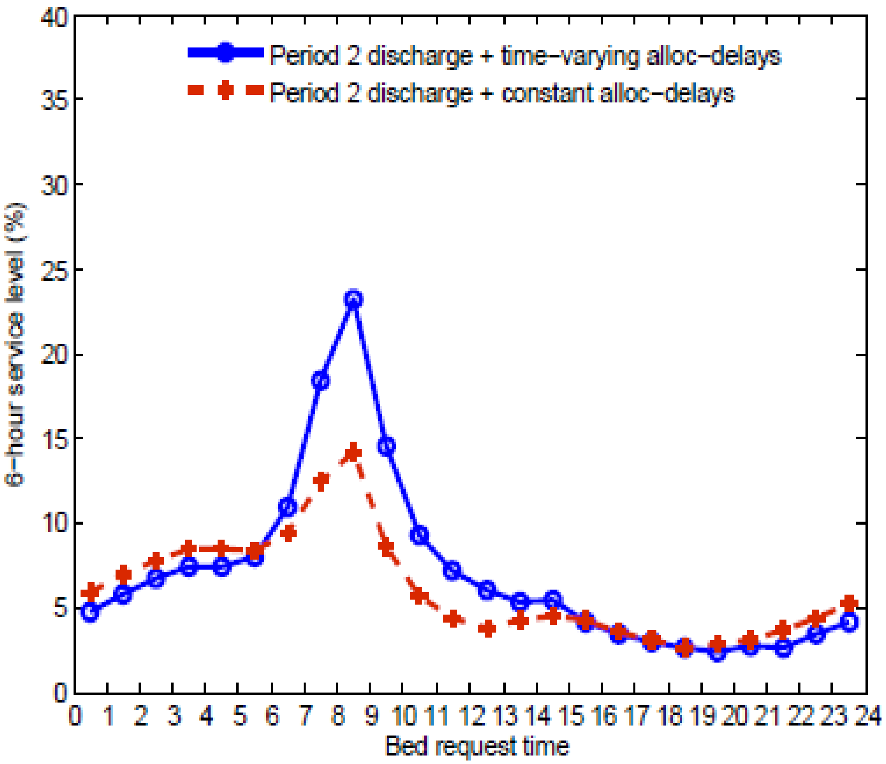


Only smooth the allocation delays

- Assuming allocation delay has a constant mean

(a) hourly avg. waiting time (b) 6-hour service level

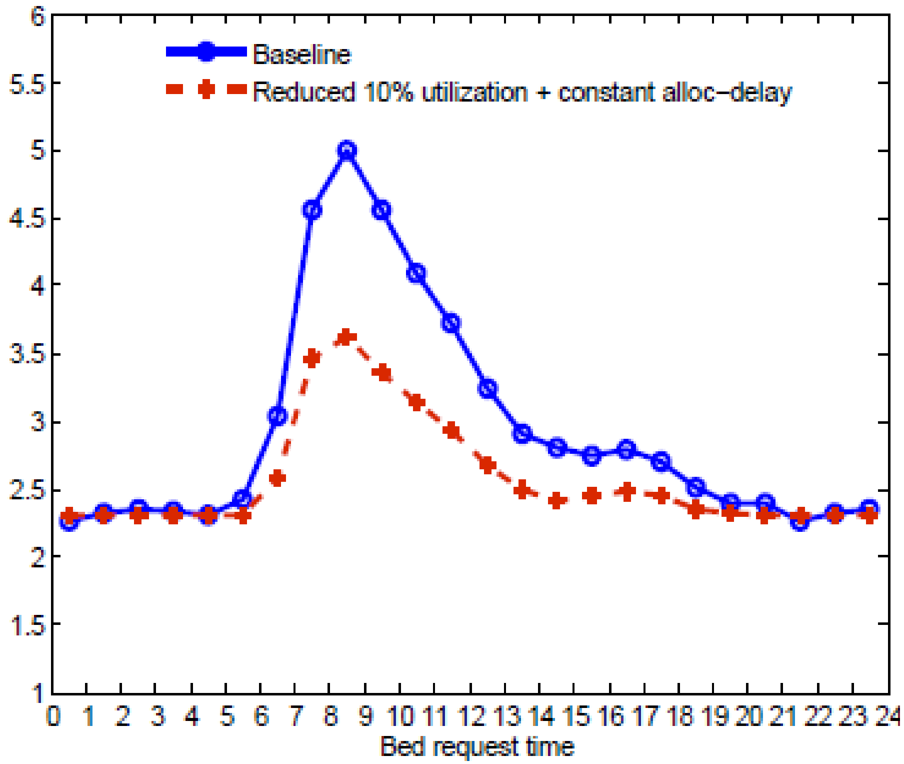




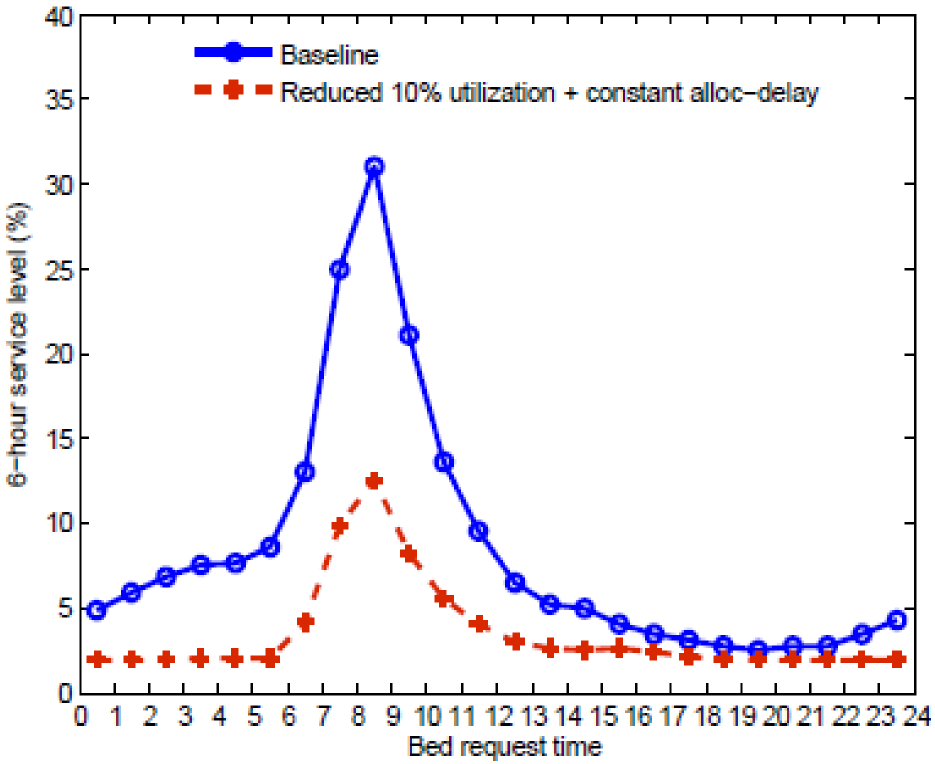
Impact of capacity increase

- 10% reduction in utilization, plus assuming allocation delay has a constant mean

(a) hourly avg. waiting time



(b) 6-hour service level



Summary

- Conduct an empirical study of patient flow of the entire inpatient department
- Build and calibrate a stochastic model to evaluate the impact of discharge distribution on waiting for admission to ward
- Analyze a simplified version of the stochastic model using a two-time scale approach
- Achieve stable waiting time by aggressive early discharge + smooth allocation delay

Questions?

Limitations

- Simulation cannot fully calibrate with the overflow rate
  - Bed class (A, B, C)
  - Gender mismatch
  - Hospital acquired infections
    - Example: a female Surg patient has to be overflowed to a Med ward, since the only available Surg beds are for males

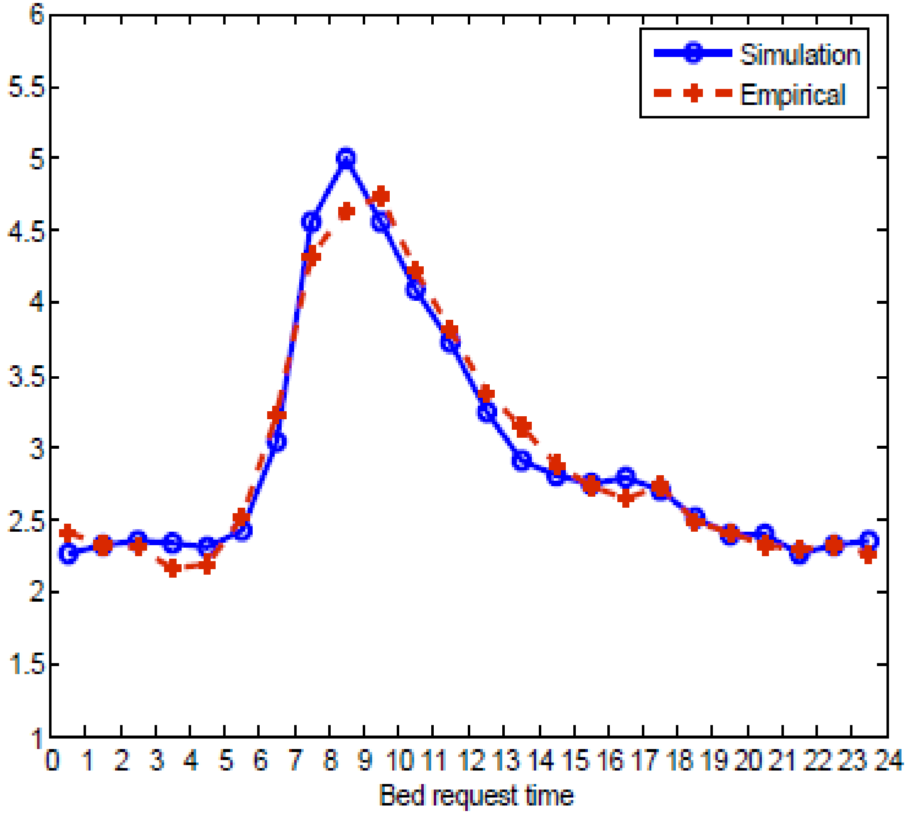
- Day-of-week phenomenon
  - Admission and discharge both depends on the day of week
  - LOS depends on admission day
  - Performances (BOR, waiting time) varies among days

Appendix

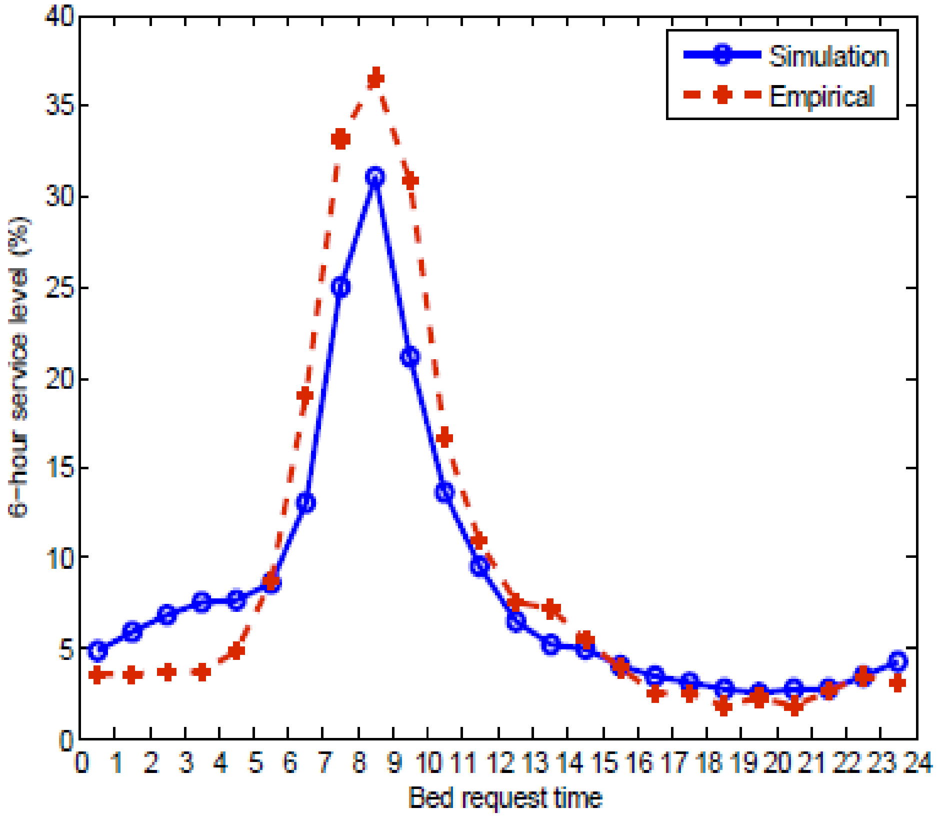
Simulation replicates most performance measures

- Hourly waiting time performances

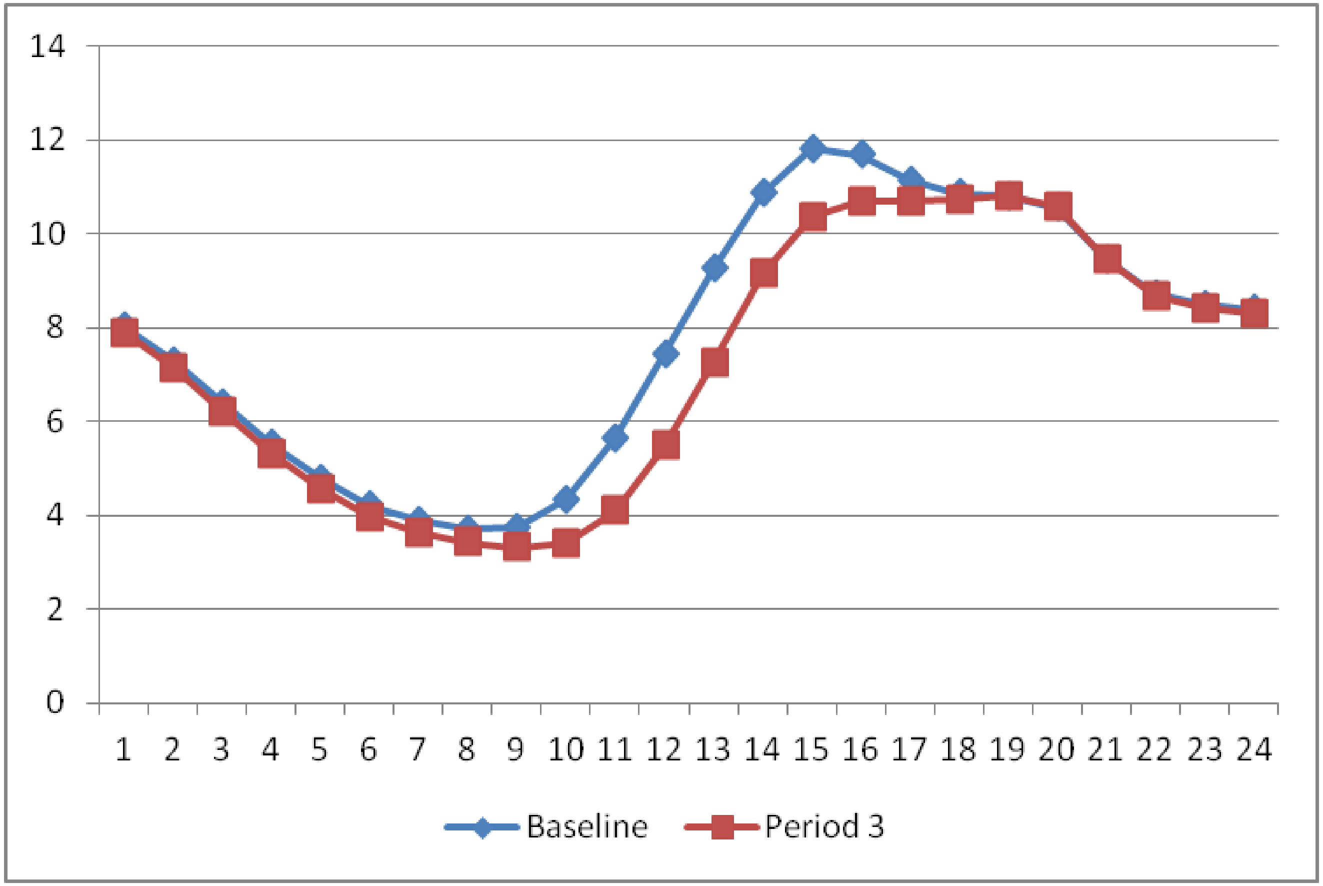
(a) Hourly average waiting time



(b) Hourly 6-hour service level

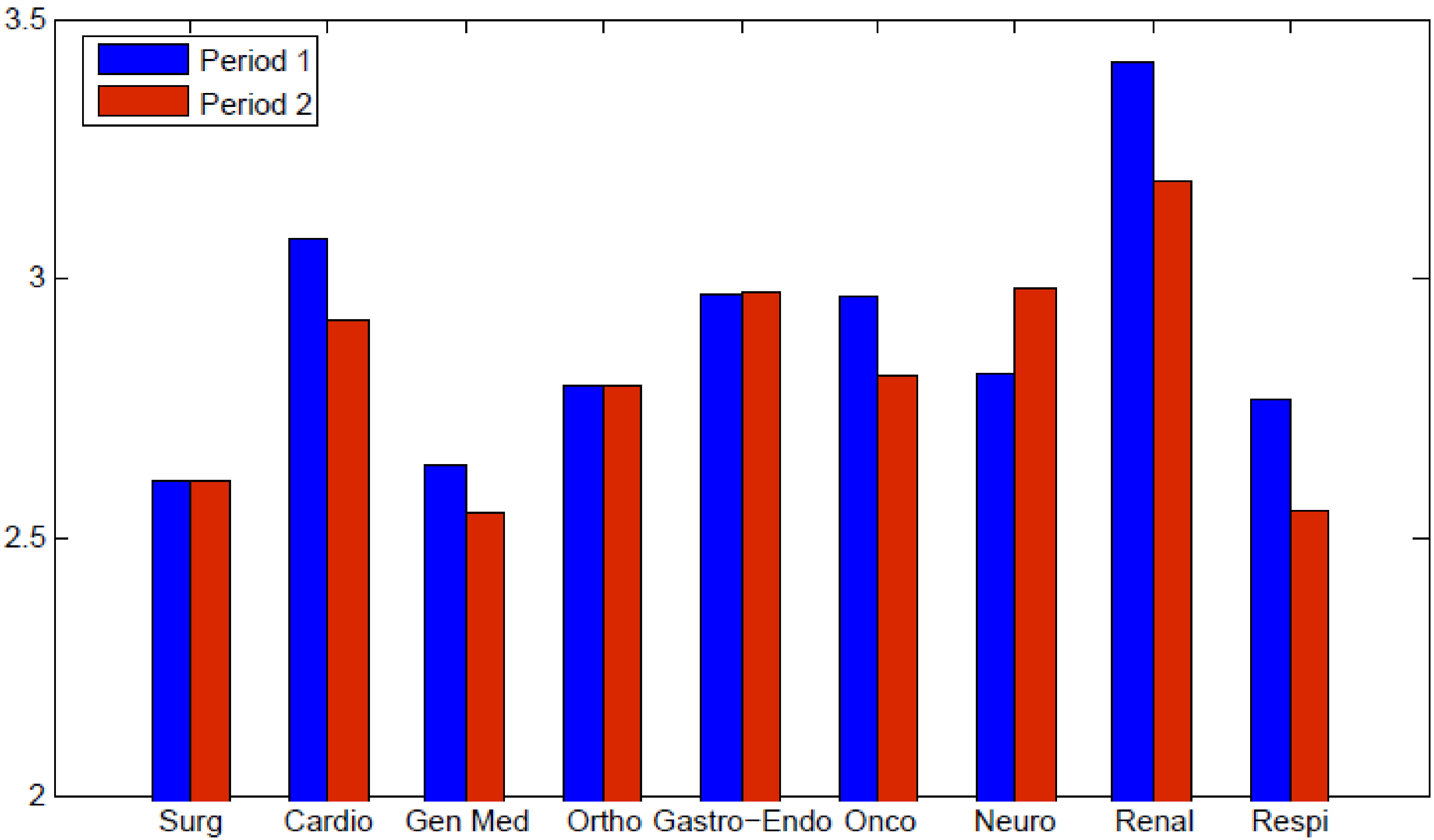


Average queue length (simulation result)



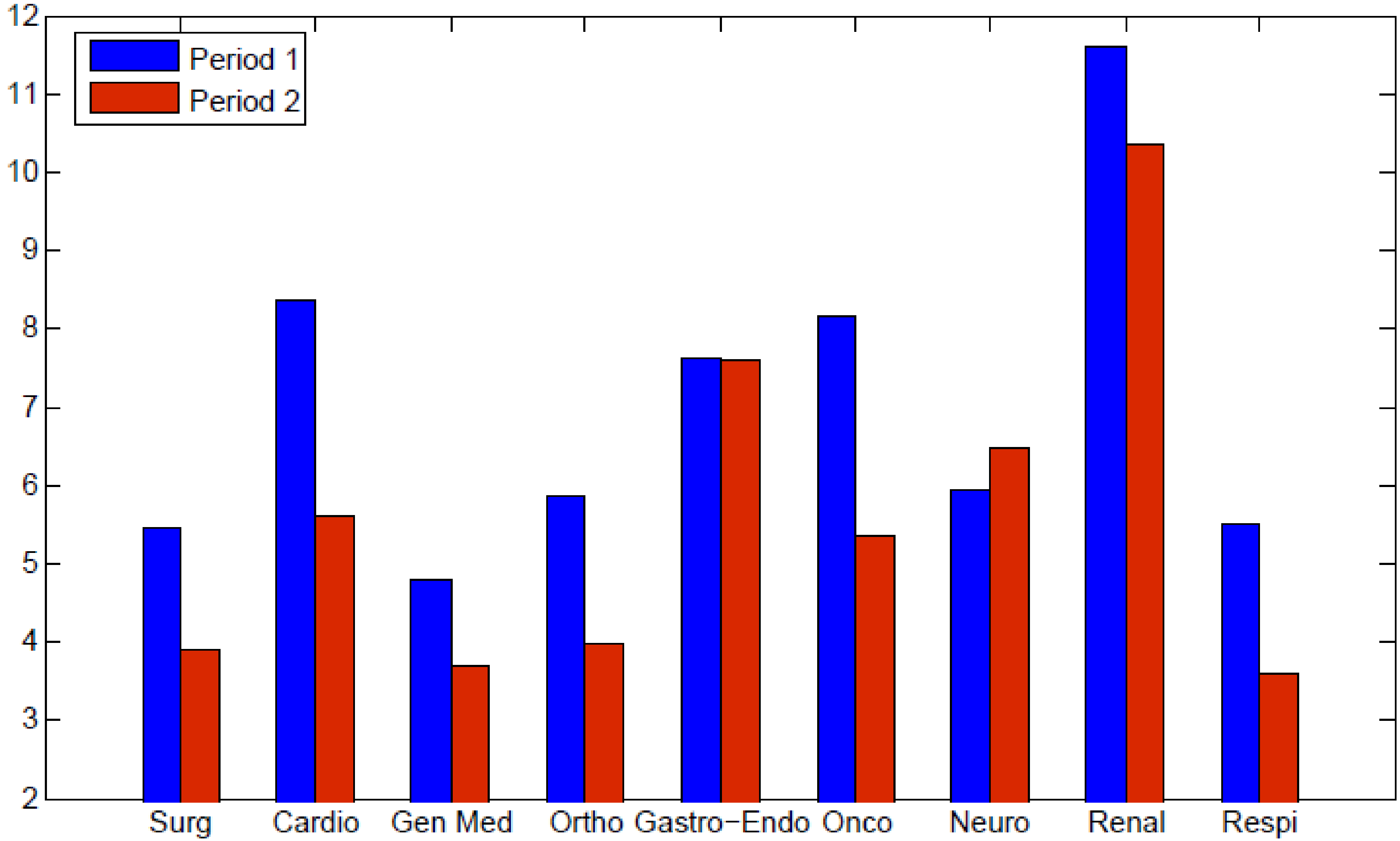
Average waiting time for each specialty

- Renal patients have longest average waiting time



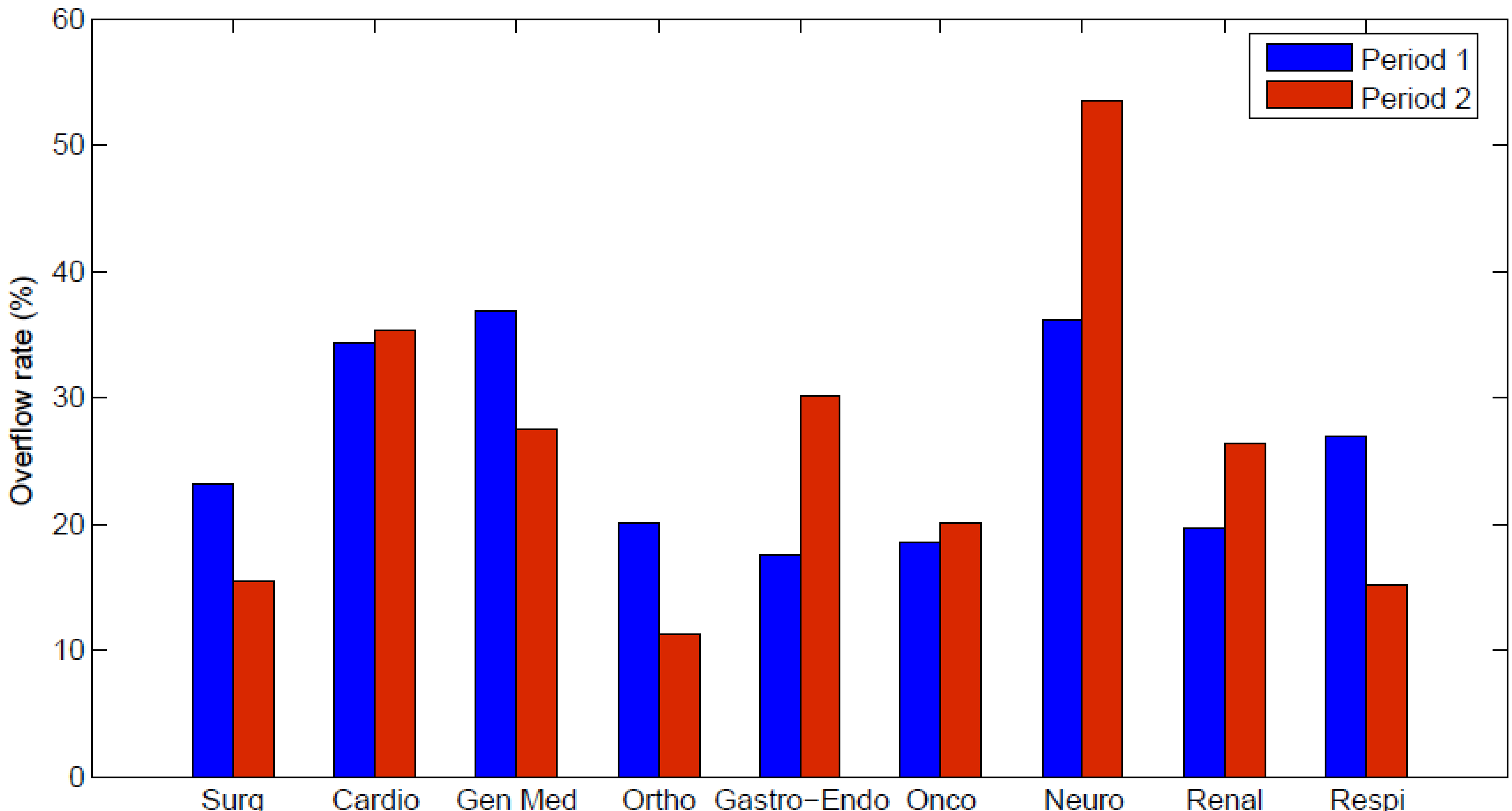
6-hour service level for each specialty

- Cardio and Oncology patients show significant improvement in the 6-hour service level



Overflow rate

- Overall overflow rate reduces in Period 2

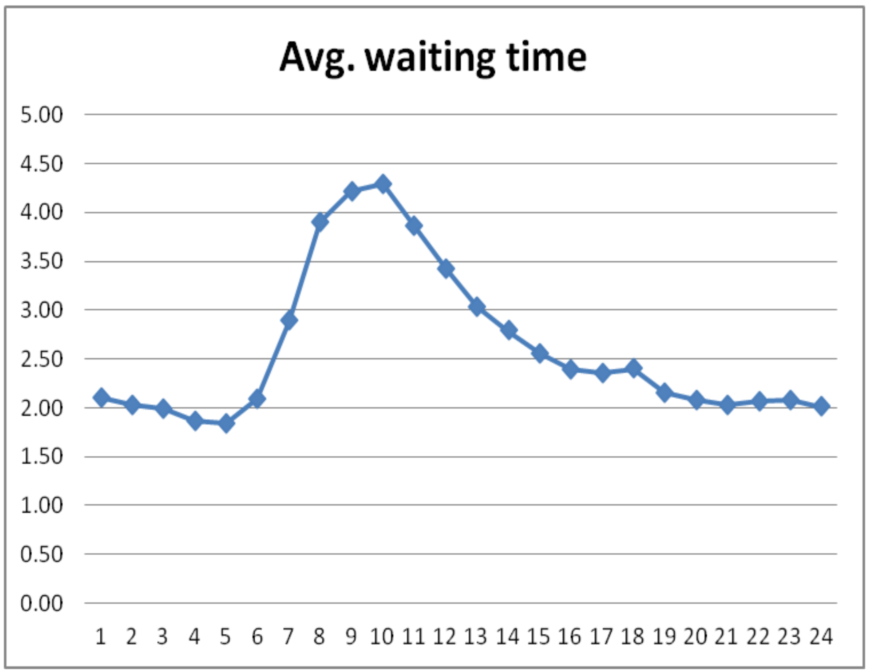


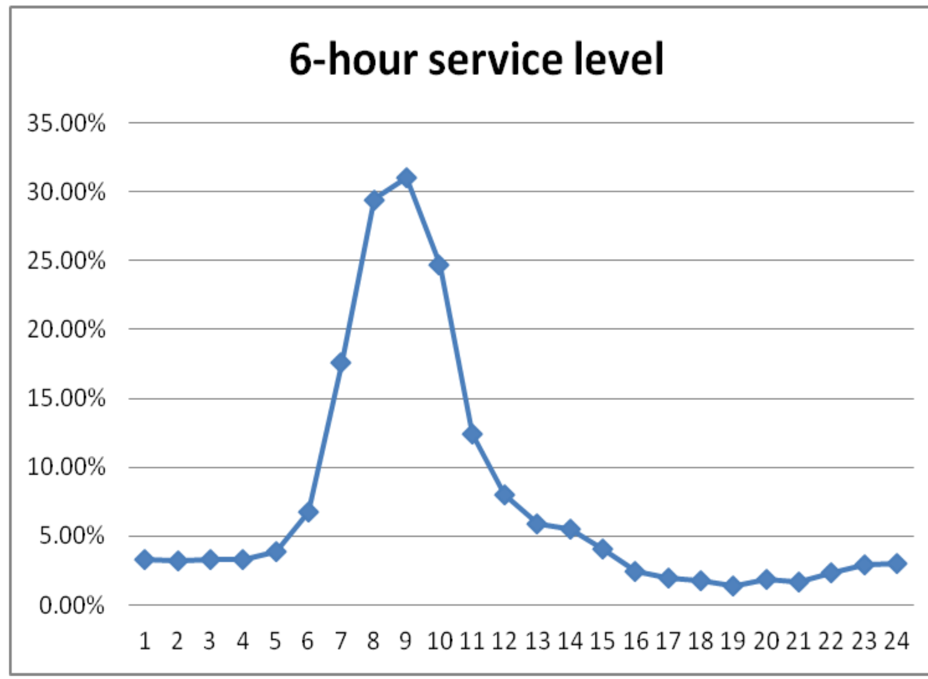
Background

- One of the major hospitals in Singapore
  - Around 1,000 beds in total
- 38 inpatient wards
  - We focus on 21 general wards
  - ICU, ISO, pediatric wards are excluded
  - Wards are dedicated to one specialty or shared by two and more specialties
- Serving around 90,000 patients annually
  - Data from 2008 to 2010

Time dependency

- Waiting time depends on patient's bed request time
  - Use time exit from ED
  - Jan 08 – Jun 09

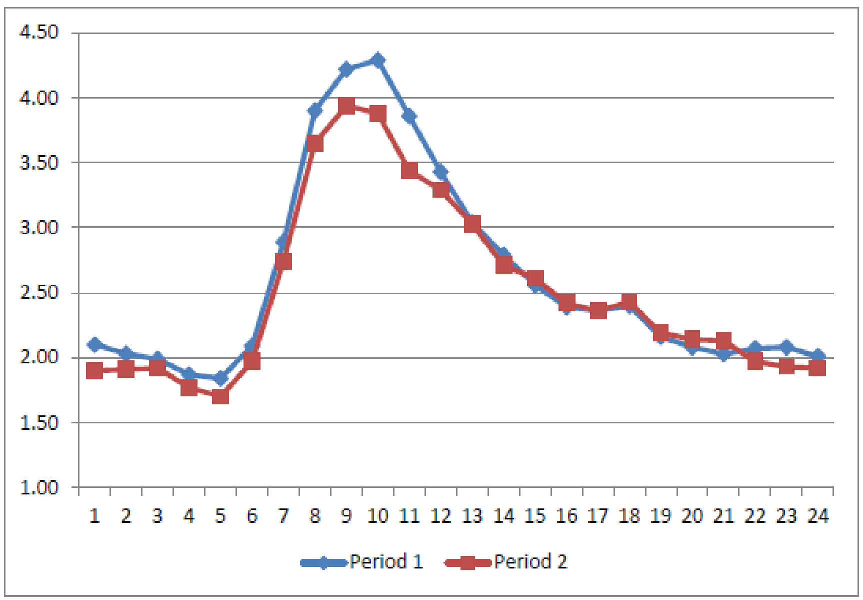


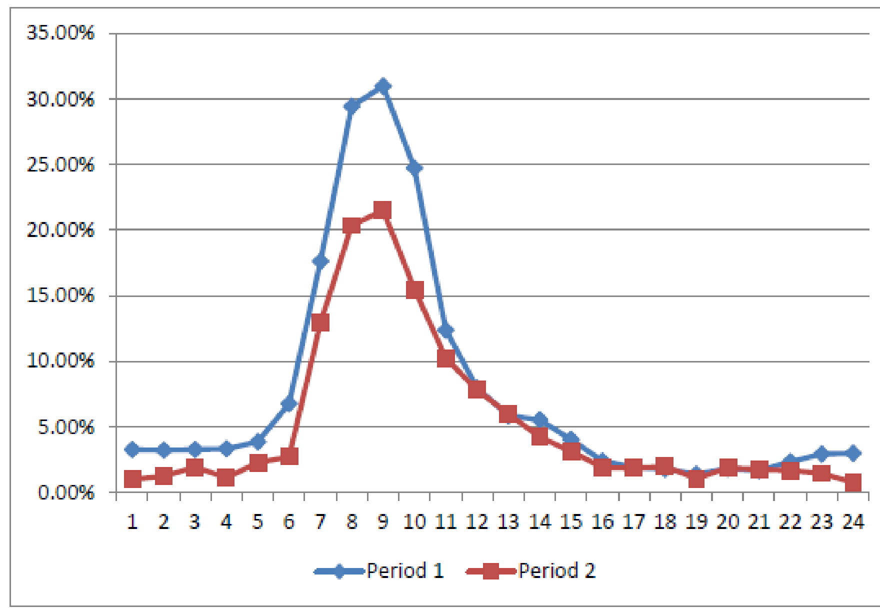


Waiting time for ED patients (using MOH definition)

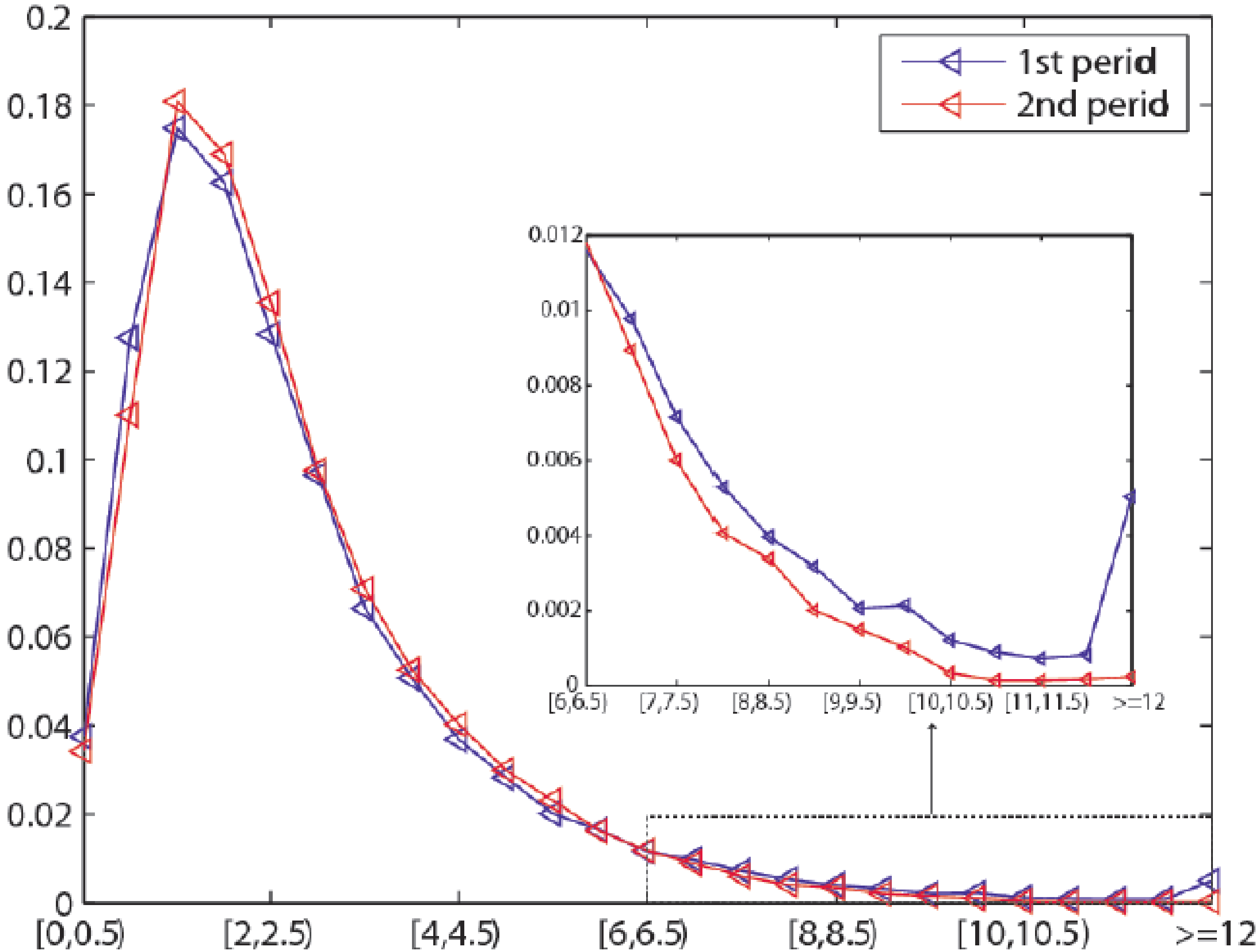
	1st period	2nd period
Average waiting time	2.50 h	2.44 h
6-hour service level	5.24%	3.90%

- (a) hourly avg. waiting time (b) hourly 6-hour service level



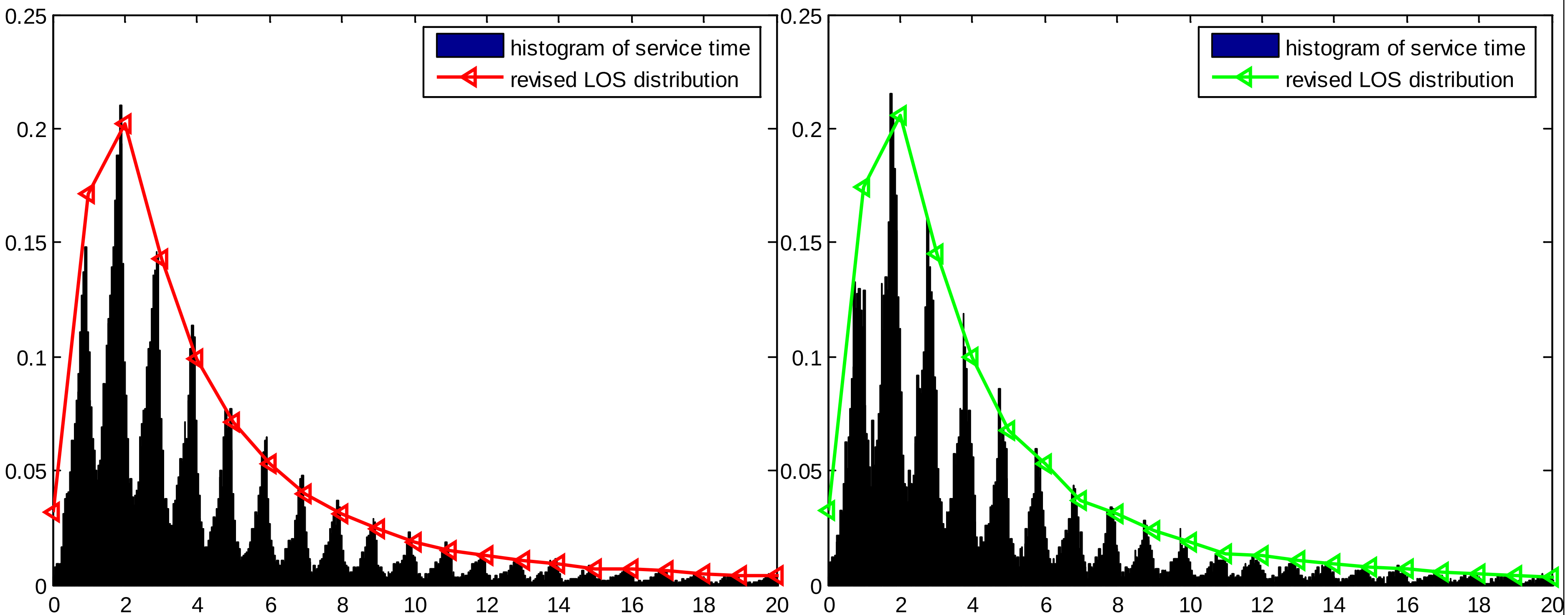


Histogram of waiting time (MOH definition)

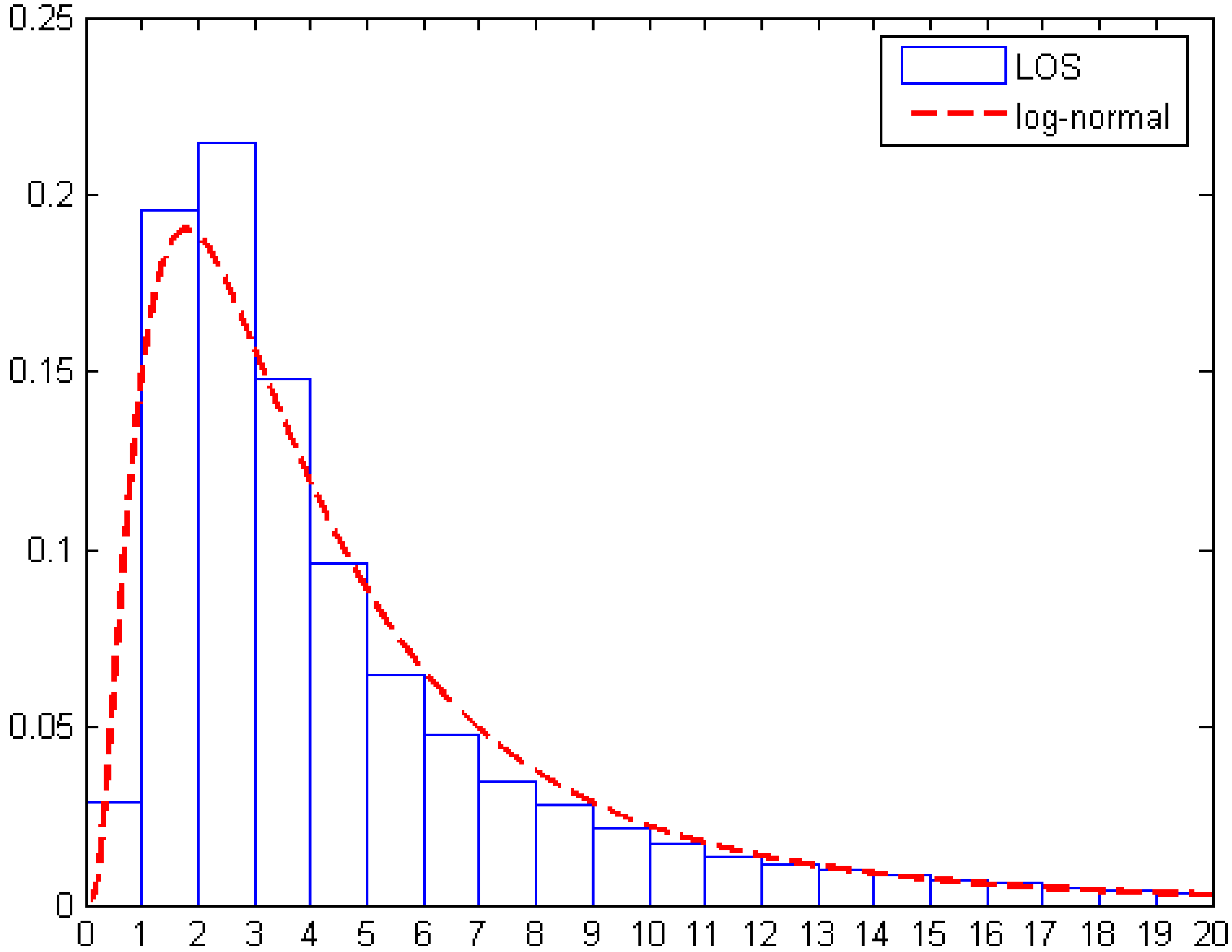


Histogram of service time

- Resolution of 1 hour
  - Period 1 Period 2



Log-normal fit for LOS distribution



Relation between residual, T_{adm}, and T_{dis}

- Residual

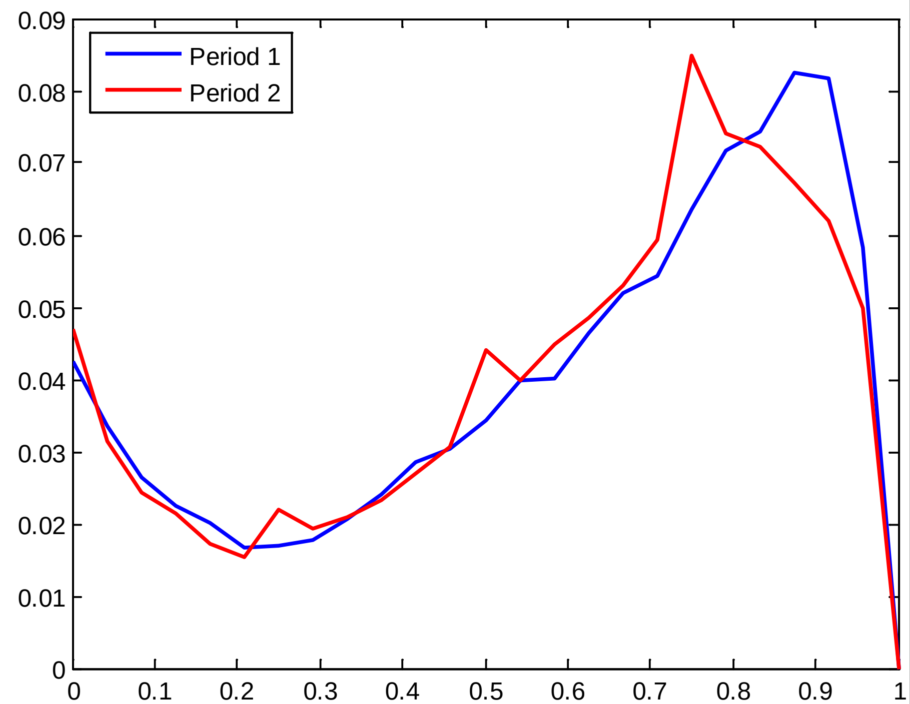
$$\begin{aligned} \mathrm{res}(S) &= S - \lfloor S \rfloor \\ &= T_{\mathrm{dis}} - T_{\mathrm{adm}} - \lfloor (T_{\mathrm{dis}} - T_{\mathrm{adm}}) \rfloor \\ &= (T_{\mathrm{dis}} - \lfloor T_{\mathrm{dis}} \rfloor - (T_{\mathrm{adm}} - \lfloor T_{\mathrm{adm}} \rfloor)) \bmod 1, \end{aligned}$$

where for two real numbers x and $y \neq 0$, $x \bmod y = x - \lfloor x/y \rfloor \cdot y$.

Alternative service time model (1/2)

- $S = T_{dis} - T_{adm}$
  - S denote service time (in unit of day)
  - T_{adm} denote the admission time, T_{dis} denote the discharge time

- Residual = S – floor(S)
  - histogram (right fig)

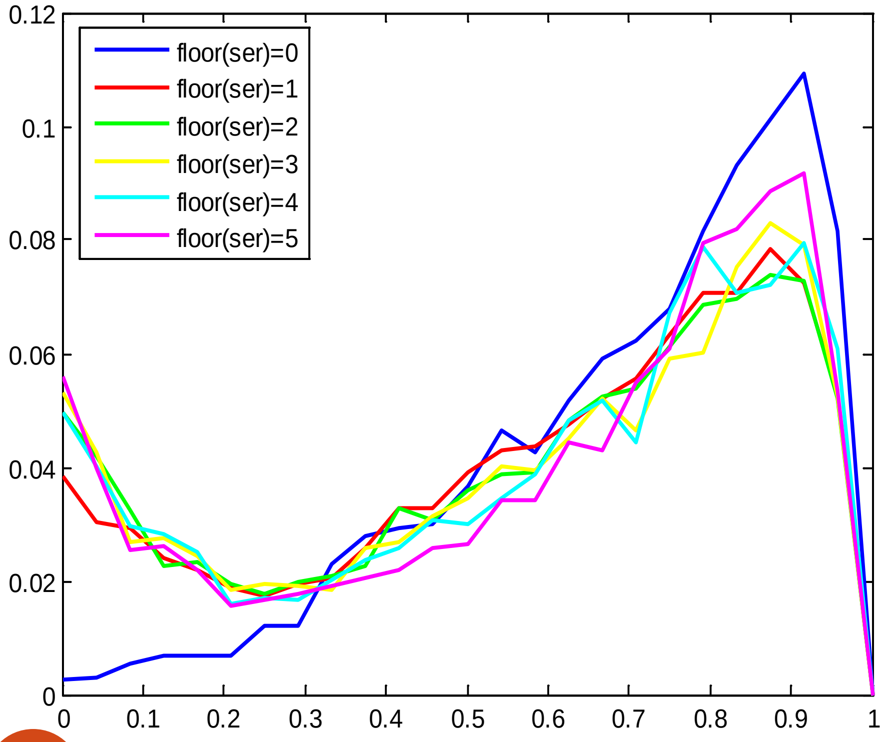


- In the alternative model
  - Generate the integer part **floor(S)** from empirical distribution
  - *Independently* generate the **residual** from another empirical distribution

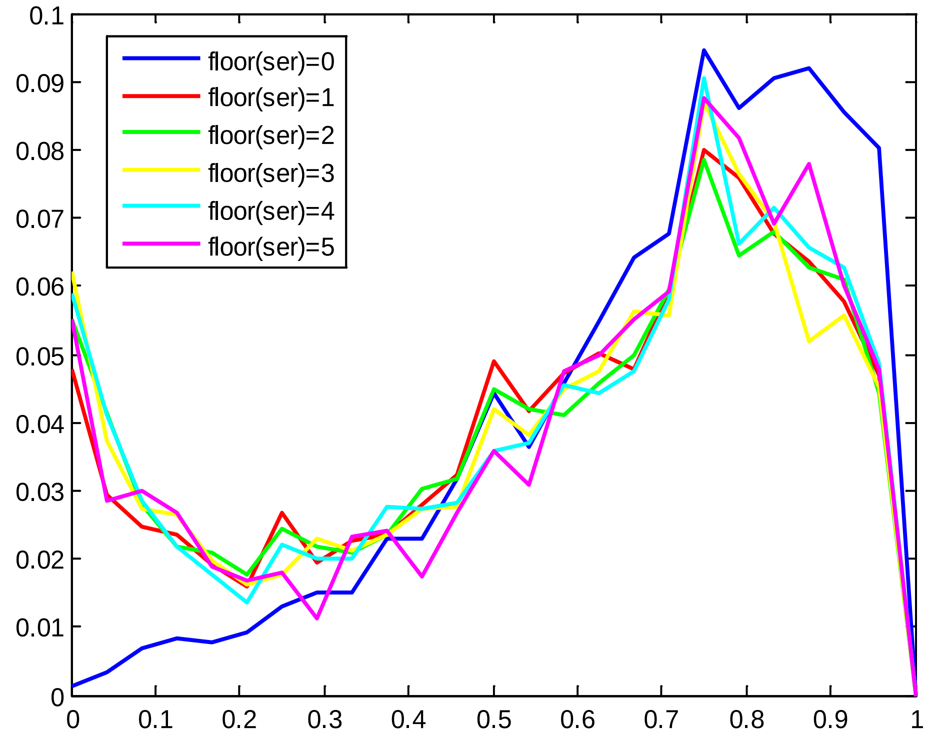
Alternative service time model (2/2)

- Histogram of residual conditioning on each integer value
  - The conditional distribution are close, except when floor(S) = 0

Period 1

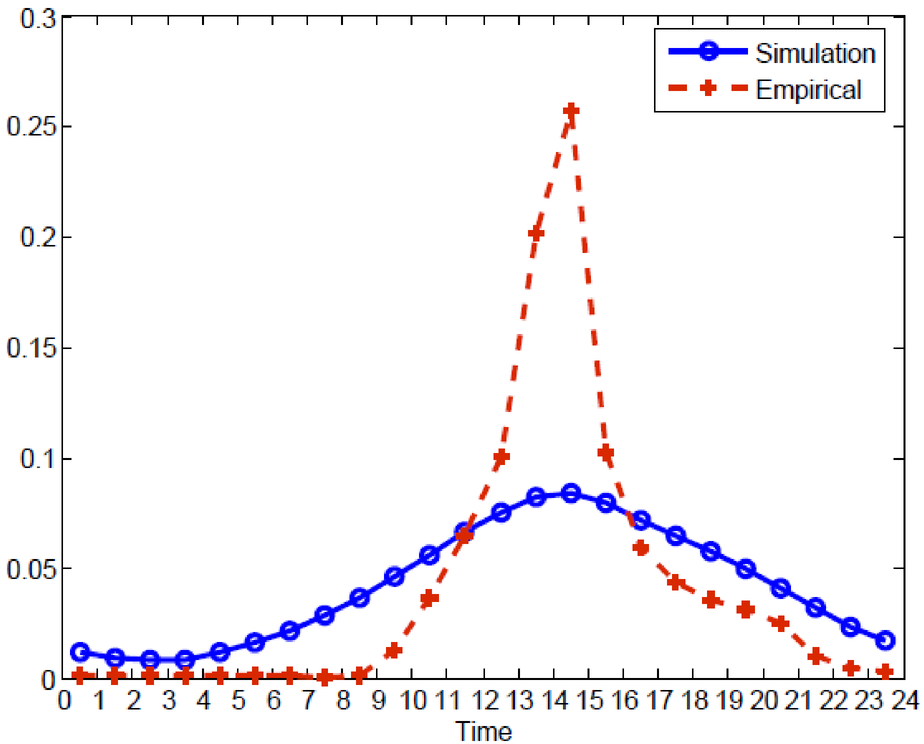


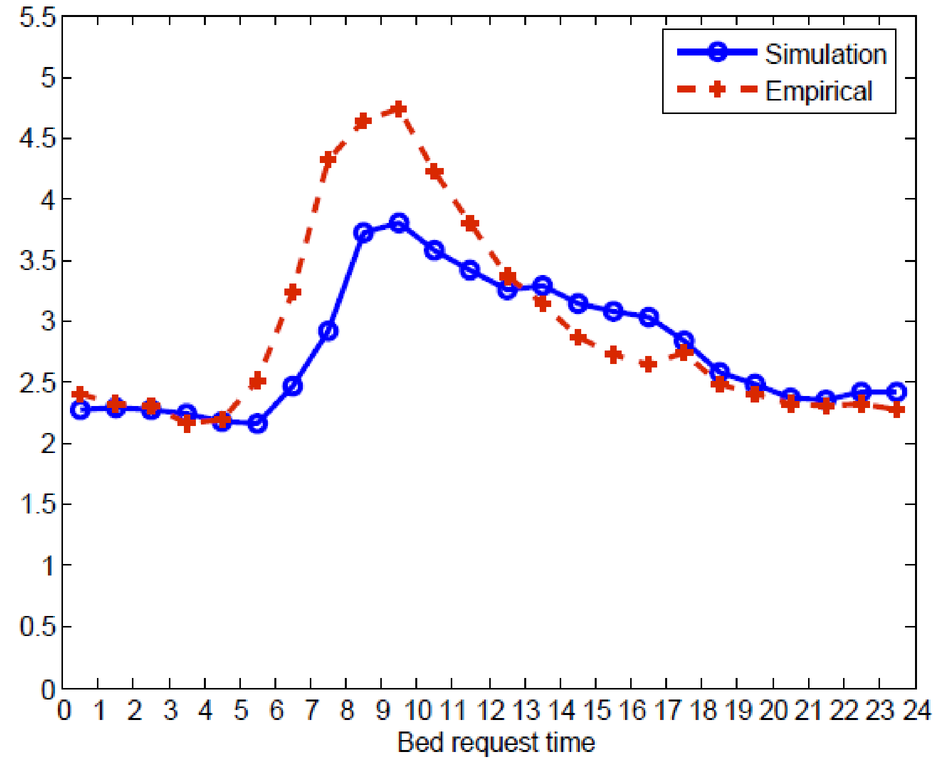
Period 2



Alternative service time model

- If directly generating service time
  - Discharge distribution does not match
  - Avg. waiting time does not match





Stochastic network models

- Multiclass, multi-server pools with some flexible pools
  - 30 ~ 60 servers in each pool
  - 15 server pools
- Typical BOR is 86% ~ 93%
- Periodic arrival processes
- Long service times = several arrival periods
  - Average LOS = 5 days
- Waiting time is a small fraction of service time
  - Average waiting time = 2.5 hours = 1/48 average LOS
- Must overflow in a fraction of the service time

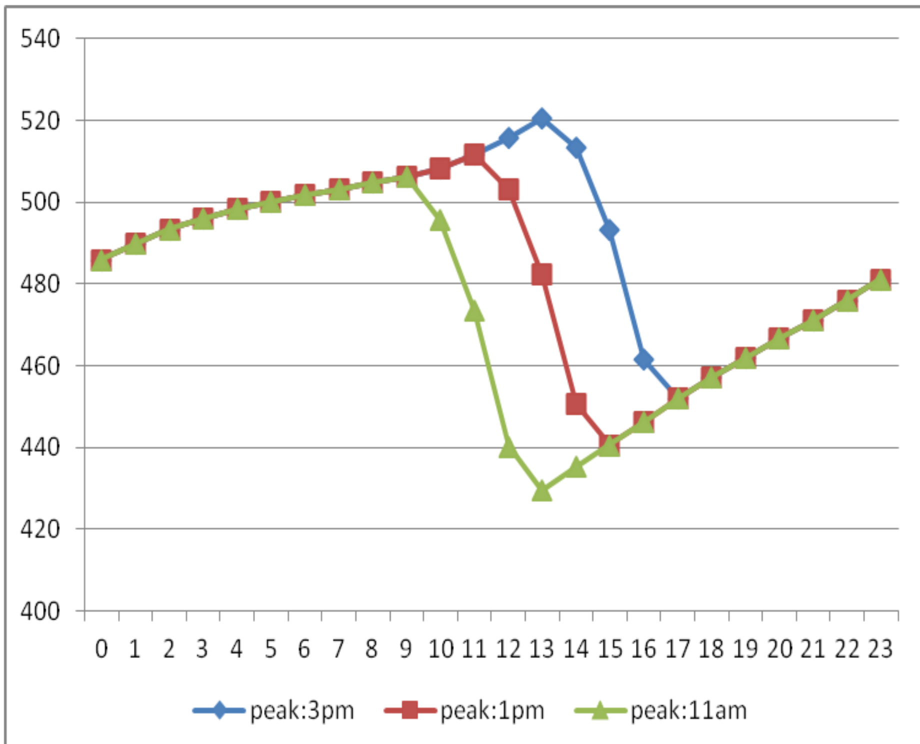
Simulation model

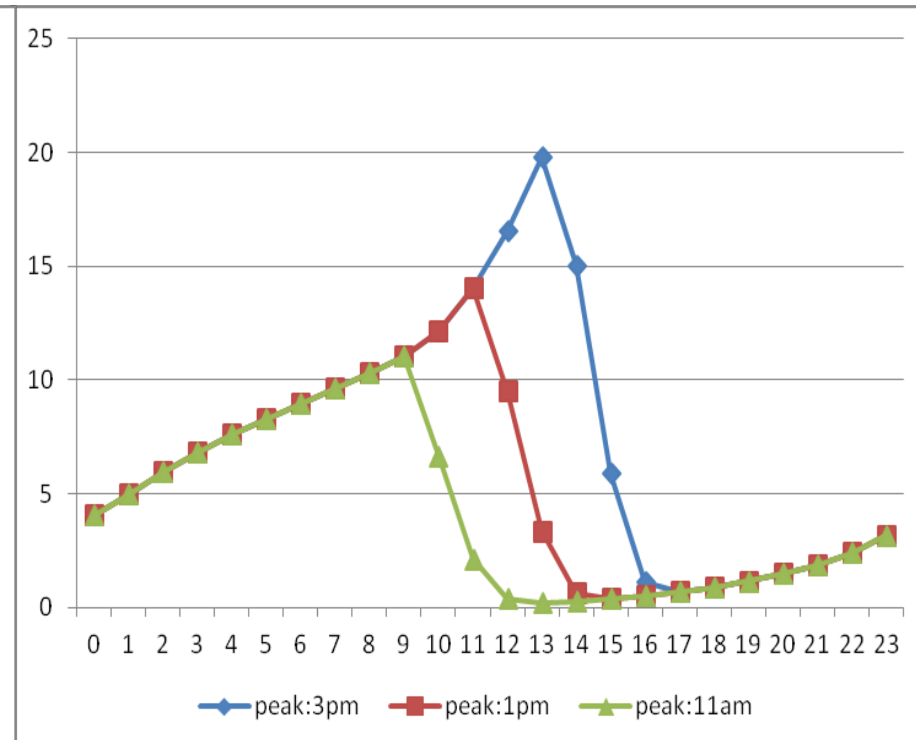
- Using 9 cluster of patients and 15 server pools
  - **Utilization (Sim): 90.5%; (empirical): 88.0%**
  - We did not catch gender/ bed class /sub-specialty mismatch in simulation

- 4 types of arrivals for each cluster
  - ED-GW
  - EL
  - ICU-GW
  - SDA
    - Use empirical arrival rate and service time for each type of patients

Analytical results: no allocation delay

- Compare with simulation results
  - Number of customer in system   Avg queue length





A stochastic model

- Multi-class, multi-server pool system
  - Each server pool is either dedicated to one class of customer or flexible to serve two and more classes of customers
- Periodic arrival
  - 4 types of arrival (ED-GW, Elective, ICU-GW, SDA) for each specialty
- A novel service time model
- And other key components

AM PM patients (ED-GW patients)

- The admission time affects LOS
  - AM patients: average LOS = 4.24 days
  - PM patients: average LOS = 5.31 days