Stochastic Quadratures and Financial Applications

Jaya Bishwal

UNC Charlotte

http://www.math.uncc.edu/$\sim$jpbishwa

Continuous Time Model

Brownian Motion

A continuous time continuous state stochastic process W_t with the following properties:

i) Process starts at 0: $W_0 = 0$.

ii) Independent increments : $W_{t_2} - W_{t_1}$ and $W_{t_4} - W_{t_3}$ are independent for $0 \le t_1 \le t_2 \le t_3 \le t_4$.

iii) $W_{t_{i+1}} - W_{t_i}$ has normal distribution with mean zero and variance $t_{i+1} - t_i$.

Parametric Stochastic Differential Equation (SDE)

$$dX_t = U(\mu, t, X_t)\, dt + V(\sigma, t, X_t)\, dW_t, 0 \le t \le T$$

$\{X_t\}$ is called a diffusion process. U is called the drift coefficient, V is called the volatility coefficient. μ and σ are unknown parameters in the model.

Itô Integral

Nonparametric diffusion:

$$dX_t \; = \; a(t, X_t) \; dt \; + \; b(t, X_t) \; dW_t, \quad t \in [0, T]$$

Consider partition of $[0, T]$

$$\pi_n := \{0 = t_0 < t_1 < \ldots < t_n = T\}$$

$$t_k = kh, \;\; k = 0, 1, \ldots, n, \quad h \rightarrow \; 0 \;\; as \;\; n \rightarrow \infty$$

Itô integral :

$$\int_0^T f(t, X_t) dW_t = \lim_{h \to 0} \sum_{i=1}^{n} f(t_{i-1}, X_{t_{i-1}})(W_{t_i} - W_{t_{i-1}}).$$

Fisk Integral

McKean integral:

$$\int_0^T f(t, X_t)dW_t = \lim_{h \to 0} \sum_{i=1}^{n} f(t_i, X_{t_i})(W_{t_i} - W_{t_{i-1}}).$$

Fisk integral:

$$\oint_0^T f(t, X_t)dW_t = \lim_{h \to 0} S_{1,n}$$

where

$$S_{1,n} := \sum_{i=1}^{n} \frac{f(t_{i-1}, X_{t_{i-1}}) + f(t_i, X_{t_i})}{2}(W_{t_i} - W_{t_{i-1}}).$$

Itô, McKean and Fisk Unified

For $\beta \in [0, 1]$, define

$$S_{A,n} := \sum_{i=1}^{n} [\beta f(t_{i-1}, X_{t_{i-1}}) + (1-\beta) f(t_i, X_{t_i})](W_{t_i} - W_{t_{i-1}}).$$

$\beta = 1$ gives Itô's scheme
$\beta = 0$ gives McKean's scheme
$\beta = \frac{1}{2}$ gives Fisk's scheme.

Stratonovich Integral

Stratonovich integral:

$$\oint_0^T f(t, X_t)dW_t = \lim_{h\to 0} S_{2,n}$$

where

$$S_{2,n} := \sum_{i=1}^{n} f\left(\frac{t_{i-1}+t_i}{2}, \frac{X_{t_{i-1}}+X_{t_i}}{2}\right)(W_{t_i} - W_{t_{i-1}}).$$

In analogy with ordinary numerical integration,
Itô's scheme is stochastic rectangular rule
Fisk's scheme is stochastic trapezoidal rule
Stratonovich's scheme is stochastic midpoint rule.

Chain Rule and Change Rule

Chain Rule for Itô calculus:

$$df(X_t) = f_x(X_t)dX_t + \frac{1}{2}f_{xx}(X_t)dt.$$

Chain Rule for Stratonovich Calculus:

$$d^o f(X_t) = f_x(X_t)dX_t$$

$$\int_a^b W_t dW_t = \frac{W_b^2 - W_a^2}{2} - \frac{b-a}{2}.$$

$$\oint_a^b W_t dW_t = \frac{W_b^2 - W_a^2}{2}$$

$$\oint_0^T f(t, X_t)dX_t = \int_0^T f(t, X_t)dX_t + \frac{1}{2}\int_0 f_x(t, X_t)dt \quad a.s.$$

Rates of Convergence

Results

$$E\left|\sum_{i=1}^{n} f(t_{i-1}, X_{t_{i-1}})(W_{t_i} - W_{t_{i-1}}) - \int_0^T f(t, X_t)dW_t\right|^2 \leq \frac{C}{n}$$

$$E\left|\sum_{i=1}^{n} \frac{f(t_{i-1}, X_{t_{i-1}}) + f(t_i, X_{t_i})}{2}(W_{t_i} - W_{t_{i-1}}) - \oint_0^T f(t, X_t)dW_t\right|^2 \leq \frac{C}{n^2}$$

Illustration

$$
\begin{aligned}
S_{1,n} = S_{2,n} &= \sum_{i=1}^{n} \frac{X_{t_{i-1}} + X_{t_i}}{2} (X_{t_i} - X_{t_{i-1}}) \\
&= \frac{1}{2} \sum_{i=1}^{n} \left(X_{t_i}^2 - X_{t_{i-1}}^2 \right) \\
&= \frac{1}{2} \left(X_T^2 - X_0^2 \right) \\
&= \int_0^T X_t dX_t - \frac{1}{2} \int_0^T X_t^2 dt = \oint_0^T X_t dX_t
\end{aligned}
$$

$$
\sum_{i=1}^{n} \frac{X_{t_{i-1}} + X_{t_i}}{2} (X_{t_i} - X_{t_{i-1}}) - \oint_0^T X_t dX_t = 0.
$$

Generalized Simpson's Rule

Convex combination of $S_{1,n}$ and $S_{2,n}$.
For $0 \leq \alpha \leq 1$ define

$$S_{B,n} := \sum_{i=1}^{n} \left[\alpha \left(\frac{f(t_{i-1}, X_{t_{i-1}}) + f(t_i, X_{t_i})}{2} \right) \right.$$
$$\left. + (1-\alpha) f\left(\frac{t_{i-1} + t_i}{2}, \frac{X_{t_{i-1}} + X_{t_i}}{2} \right) \right] (W_{t_i} - W_{t_{i-1}}),$$

$\alpha = 1$ gives $S_{1,n}$ (Fisk)
$\alpha = 0$ gives $S_{2,n}$ (Stratonovich)

Stochastic Simpson's Rule

$\alpha = \frac{1}{3}$ gives

$$S_{5,n} := \frac{1}{6} \sum_{i=1}^{n} \Big[f(t_{i-1}, X_{t_{i-1}}) + 4f(\frac{t_{i-1}+t_i}{2}, \frac{X_{t_{i-1}}+X_{t_i}}{2}) + f(t_i, X_{t_i}) \Big] (W_{t_i} - W_{t_{i-1}})$$

In analogy with ordinary numerical integration, it is the Stochastic Simpson's rule.

Generalized Stochastic Integral

$$
\begin{aligned}
B_T &:= \oint_0^T f(t, X_t) dW_t \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} p_j \; f\left((1 - s_j)\, t_{i-1} + s_j\, t_i,\right. \\
&\qquad \left.(1 - s_j) X_{t_{i-1}} + s_j X_{t_i}\right) \; \left(W_{t_i} - W_{t_{i-1}}\right)
\end{aligned}
$$

$p_j, \;\; j \in \{1, 2, \cdots, m\}$ is a probability mass function of a discrete random variable S on $0 \le s_1 < s_2 < \cdots < s_m \le 1$ with

$$P(S = s_j) = p_j, \; j \in \{1, 2, \cdots, m\}.$$

Moments

Denote the k-th moment of the random variable S as

$$\mu_k := \sum_{j=1}^{m} s_j^k p_j, \; k = 1, 2, \cdots .$$

The new integral and the Itô integral are connected as follows:

$$B_T = I_T + \mu_1 \int_0^T f_x(t, X_t)dt$$

where $I_T = \int_0^T f(t, X_t)dW_t$ is the Itô integral.
When $\mu_1 = 0$, the new integral is the Itô integral.
When $\mu_1 = \frac{1}{2}$, the new integral is the Fisk-Stratonovich integral.

Order of Approximation

The order of mean square approximation error (rate of convergence) in the new integral is $n^{-\nu}$ where

$$\nu := \inf\left\{k : \mu_k \neq \frac{1}{1+k},\ \mu_j = \frac{1}{1+j}, j = 0, 1, \cdots, k-1\right\}.$$

Given a positive integer m, how does one construct a probability mass function $p_j,\ \ j \in \{1, 2, \cdots, m\}$ on $0 \leq s_1 < s_2 < \cdots < s_m \leq 1$ so that

$$\sum_{j=1}^{m} s_j^r p_j = \frac{1}{r+1},\ r \in \{0, \cdots, m-2\} \qquad (1)$$

$$\sum_{j=1}^{m} s_j^{m-1} p_j \neq \frac{1}{m}? \qquad (2)$$

First Order Schemes

$\nu = 1$: Mass 1 at the point $s = 0$ gives Itô scheme for which $\mu_1 = 0, \; \mu_1 \neq \frac{1}{2}$.

$\nu = 1$: Mass 1 at the point $s = 1$ gives the McKean scheme for which $\mu_1 = 1, \; \mu_1 \neq \frac{1}{2}$.

Second Order Schemes

$\nu = 2$: Masses $\frac{1}{2}, \frac{1}{2}$ at the respective points $0, 1$ produces the Fisk scheme $S_{1,n}$ for which $\mu_1 = \frac{1}{2},\ \mu_2 = \frac{1}{4}$.

$\nu = 2$: Mass 1 at the point $\frac{1}{2}$ produce the Stratonovich scheme $S_{2,n}$ for which $\mu_1 = \frac{1}{2},\ \mu_2 = \frac{1}{2}$.

Third Order Schemes

$\nu = 3$: Masses $\frac{1}{4}, \frac{3}{4}$ at the respective points $0, \frac{2}{3}$ produce the asymmetric scheme

$$S_{3,n} :=$$

$$\frac{1}{4}\sum_{i=1}^{n}\left[f(t_{i-1}, X_{t_{i-1}}) + 3f(\tfrac{t_{i-1}+2t_i}{3}, \tfrac{X_{t_{i-1}}+2X_{t_i}}{3})\right](W_{t_i} - W_{t_{i-1}})$$

for which $\mu_1 = \frac{1}{2},\ \mu_2 = \frac{1}{3},\ \mu_3 = \frac{2}{9}$.

$\nu = 3$: Masses $\frac{3}{4}, \frac{1}{4}$ at the respective points $\frac{1}{3}, 1$ produce asymmetric scheme

$$S_{4,n} :=$$

$$\frac{1}{4}\sum_{i=1}^{n}\left[3f(\tfrac{2t_{i-1}+t_i}{3}, \tfrac{2X_{t_{i-1}}+X_{t_i}}{3}) + f(t_i, X_{t_i})\right](W_{t_i} - W_{t_{i-1}})$$

for which $\mu_1 = \frac{1}{2},\ \mu_2 = \frac{1}{3},\ \mu_3 = \frac{10}{36}$.

Fourth Order Schemes

$\nu = 4$: Masses $\frac{1}{6}, \frac{2}{3}, \frac{1}{6}$ at the respective points $0, \frac{1}{2}, 1$ produce the Simpson's scheme $S_{5,n}$ for which $\mu_1 = \frac{1}{2},\ \mu_2 = \frac{1}{3},\ \mu_3 = \frac{1}{4},\ \mu_4 = \frac{5}{24}$.

$\nu = 4$: Masses $\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$ at the respective points $0, \frac{1}{3}, \frac{2}{3}, 1$ produce the symmetric scheme

$$S_{6,n} := \frac{1}{8} \sum_{i=1}^{n} \Big[f(t_{i-1}, X_{t_{i-1}}) + 3f(\tfrac{2t_{i-1}+t_i}{3}, \tfrac{2X_{t_{i-1}}+X_{t_i}}{3}) + 3f(\tfrac{t_{i-1}+2t_i}{3}, \tfrac{X_{t_{i-1}}+2X_{t_i}}{3}) + f(t_i, X_{t_i}) \Big] (W_{t_i} - W_{t_{i-1}})$$

for which $\mu_1 = \frac{1}{2},\ \mu_2 = \frac{1}{3},\ \mu_3 = \frac{1}{4},\ \mu_4 = \frac{11}{54}$.

Fifth Order Scheme

$\nu = 5$: Masses $\frac{1471}{24192}, \frac{6925}{24192}, \frac{1475}{12096}, \frac{2725}{12096}, \frac{5675}{24192}, \frac{1721}{24192}$. at the respective points $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$ produce the asymmetric scheme

$$S_{7,n} :=$$

$$\frac{1}{24192} \sum_{i=1}^{n} \Big[1471 f(t_{i-1}, X_{t_{i-1}}) + 6925 f\big(\tfrac{t_{i-1}+t_i}{5}, \tfrac{X_{t_{i-1}}+X_{t_i}}{5}\big)$$

$$+2950 f\big(\tfrac{2t_{i-1}+2t_i}{5}, \tfrac{2X_{t_{i-1}}+2X_{t_i}}{5}\big) + 5450 f\big(\tfrac{3t_{i-1}+3t_i}{5}, \tfrac{3X_{t_{i-1}}+3X_{t_i}}{5}\big)$$

$$+5675 f\big(\tfrac{4t_{i-1}+4t_i}{5}, \tfrac{4X_{t_{i-1}}+4X_{t_i}}{5}\big) + 1721 f(t_i, X_{t_i}) \Big] (W_{t_i} - W_{t_{i-1}})$$

for which $\mu_1 = \frac{1}{2}, \mu_2 = \frac{1}{3}, \mu_3 = \frac{1}{4},\ \mu_4 = \frac{1}{5},\ \mu_5 = \frac{841}{5040}$

Sixth Order Scheme

$\nu = 6$: Masses $\frac{7}{90}, \frac{16}{45}, \frac{2}{15}, \frac{16}{45}, \frac{7}{90}$ at the respective points $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ produce the symmetric scheme

$$S_{8,n} := \frac{1}{90}\sum_{i=1}^{n}\Big[7f(t_{i-1}, X_{t_{i-1}}) + 32f\big(\tfrac{3t_{i-1}+t_i}{4}, \tfrac{3X_{t_{i-1}}+X_{t_i}}{4}\big)$$
$$+12f\big(\tfrac{t_{i-1}+t_i}{2}, \tfrac{X_{t_{i-1}}+X_{t_i}}{2}\big) + 32f\big(\tfrac{t_{i-1}+3t_i}{4}, \tfrac{X_{t_{i-1}}+3X_{t_i}}{4}\big)$$
$$+7f(t_i, X_{t_i})\Big](W_{t_i} - W_{t_{i-1}})$$

for which

$\mu_1 = \frac{1}{2}, \mu_2 = \frac{1}{3}, \mu_3 = \frac{1}{4},\ \mu_4 = \frac{1}{5},\ \mu_5 = \frac{1}{6},\ \mu_6 = \frac{110}{768}$.

Sixth Order Scheme

$\nu = 6$: Masses $\frac{19}{288}, \frac{75}{288}, \frac{50}{288}, \frac{50}{288}, \frac{75}{288}, \frac{19}{288}$ at the respective points $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$ produce symmetric scheme

$$S_{9,n} := \frac{1}{288} \sum_{i=1}^{n} \Big[19 f(t_{i-1}, X_{t_{i-1}}) + 75 f(\tfrac{4t_{i-1}+t_i}{5}, \tfrac{4X_{t_{i-1}}+X_{t_i}}{5}) \\ + 50 f(\tfrac{3t_{i-1}+2t_i}{5}, \tfrac{3X_{t_{i-1}}+2X_{t_i}}{5}) + 50 f(\tfrac{2t_{i-1}+3t_i}{5}, \tfrac{2X_{t_{i-1}}+3X_{t_i}}{5}) \\ + 75 f(\tfrac{t_{i-1}+4t_i}{5}, \tfrac{X_{t_{i-1}}+4X_{t_i}}{5}) + 19 f(t_i, X_{t_i}) \Big] (W_{t_i} - W_{t_{i-1}})$$

for which

$\mu_1 = \frac{1}{2}, \mu_2 = \frac{1}{3}, \mu_3 = \frac{1}{4}, \mu_4 = \frac{1}{5}, \mu_5 = \frac{1}{6}, \mu_6 = \frac{3219}{22500}$.

Option Pricing Models

Stock Price Model
Black-Scholes Model

$$dX_t = \mu X_t dt + \sigma X_t dW_t.$$

μ is the long run mean, σ is the volatility
Interest Rate Model
Vasicek Model

$$dX_t = \alpha(\beta - X_t)dt + \sigma dW_t.$$

α is the speed of mean reversion, $\alpha\beta$ is the level of mean reversion, σ is the volatility.

Financial Statistics

Ornstein-Uhlenbeck Process (Special Case of Vasicek)

$$dX_t = \theta X_t dt + dW_t,\ t \geq 0, X_0 = 0$$

Girsanov Likelihood based on data $\{X_t, 0 \leq t \leq T\}$

$$L_T = \theta \int_0^T X_t dX_t - \frac{\theta^2}{2} \int_0^T X_t^2 dt$$

and its maximizer is the maximum likelihood estimator

$$\theta_T = \frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt.}$$

Likelihood Discretization

Discretize the likelihood L_T. $L_{n,T,1}$ is obtained by an Itô approximation of the stochastic integral and rectangular rule approximation of the ordinary integral in L_T.

$$L_{n,T,1}(\theta) = \theta \sum_{i=1}^{n} X_{t_{i-1}}(X_{t_i} - X_{t_{i-1}}) - \frac{\theta^2}{2} \sum_{i=1}^{n} X_{t_{i-1}}^2 \Delta t_i.$$

Its maximizer is Approximate maximum likelihood estimator

$$AMLE1 = arg \max_{\theta} L_{n,T,1}(\theta)$$

$$\theta_{n,T,1} = \frac{\sum_{i=1}^{n} X_{t_{i-1}}(X_{t_i} - X_{t_{i-1}})}{\sum_{i=1}^{n} X_{t_{i-1}}^2 \Delta t_i}.$$

Transformed Likelihood

Transform the Itô integral to the Stratonovich integral in L_T and then apply FS type approximation for the Stratonovich integral and rectangular rule approximation for the ordinary in L_T, then we obtain the approximate likelihood $L_{n,T,2}$.

$$\oint_0^T f(t, X_t) dX_t = \int_0^T f(t, X_t) dX_t + \frac{1}{2} \int_0 f_x(t, X_t) dt$$

$$L_T = \theta \left(\oint_0^T X_t dX_t - \frac{1}{2} \int_0^T dt \right) - \frac{\theta^2}{2} \int_0^T X_t^2 dt$$

$$L_{n,T,2}(\theta) = \frac{\theta}{2}(X_T^2 - T) - \frac{\theta^2}{2} \sum_{i=1}^{n} X_{t_{i-1}}^2 \Delta t_i$$

Higher Order AMLE

$$AMLE2 = arg \max_{\theta} L_{n,T,2}(\theta)$$

$$\theta_{n,T,2} = \frac{\frac{1}{2}(X_T^2 - T)}{\sum_{i=1}^{n} X_{t_{i-1}}^2 \Delta t_i}.$$

Results

$$|\theta_{n,T,1} - \theta_T| = O_P(\frac{1}{\sqrt{n}}).$$

$$|\theta_{n,T,2} - \theta_T| = O_P(\frac{1}{n}).$$

Model Discretization

Euler Scheme

$$\hat{X}_{t_i} = \hat{X}_{t_{i-1}} + a(t_{i-1}, \hat{X}_{t_{i-1}})(t_i - t_{i-1}) + b(t_{i-1}, \hat{X}_{t_{i-1}})\sqrt{t_i - t_{i-1}}Z_i$$

where $Z_i, i = 1, 2, \cdots, m$ are i.i.d. standard normal variables. The transition density of this scheme is normal.

Milstein Scheme

$$\begin{aligned}\tilde{X}_{t_i} &= \tilde{X}_{t_{i-1}} + a(t_{i-1}, \tilde{X}_{t_{i-1}})(t_i - t_{i-1}) \\ &\quad + b(t_{i-1}, \tilde{X}_{t_{i-1}})\sqrt{t_i - t_{i-1}}Z_i \\ &\quad + 2b(t_{i-1}, \tilde{X}_{t_{i-1}})b_x(t_{i-1}, \tilde{X}_{t_{i-1}})(Z_i^2 - 1)\end{aligned}$$

The transition density is noncentral chisquare.

$E|\hat{X}_{t_n} - X_{t_n}|^2 = O(\frac{1}{n})$ $\qquad$ $E|\hat{X}_{t_n} - X_{t_n}|^2 = O(\frac{1}{n^2})$

Conditional Least Squares

Uses Euler Scheme

$$Q_{n,T}(\theta) = \sum_{i=1}^{n} \left[X_{t_i} - X_{t_{i-1}} - \theta X_{t_{i-1}}\right]^2 .$$

$$\theta_{n,T} := arg\min_{\theta} Q_{n,T}(\theta)$$

$$\theta_{n,T} = \frac{\sum_{i=1}^{n} X_{t_{i-1}}(X_{t_i} - X_{t_{i-1}})}{\sum_{i=1}^{n} X_{t_{i-1}}^2 \Delta t_i}.$$

$$\theta_{n,T} = \theta_{n,T,1}.$$

Monte Carlo Pricing

In a risk neutral world, stock price S_t at time t follows the following linear Itô stochastic differential equation, known as the Black-Scholes model

$$dS_t = rS_t dt + \sigma S_t dW_t,\ t \geq 0$$

where $\{W_t\}$ is a standard Brownian motion, r is the risk-free interest rate and σ is the volatility. A simple application of Itô's formula to $\log S_t$ provides the exact solution of the equation given

$$S_t = S_0 \exp\{(r - \frac{1}{2}\sigma^2)t + \sigma W_t\}$$

where S_0 is the initial price of the stock. S_t is called Geometric Brownian motion.

Monte Carlo Simulation

The basic idea of simulating the paths of S goes back to the fact that the increments of Brownian motion are independent and normally distributed with zero mean and variance being the time difference.
Consider the time grid $0 = t_0 < t_1 < t_2 < \cdots < t_n$.
Then the exact discretization is

$$S_{t_{i+1}} = S_{t_i} \exp\left([r - \frac{1}{2}\sigma^2](t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i} Z_{i+1} \right)$$

$i = 1, 2, \cdots, n-1$, where Z_i are independent standard normal random variables.
The Euler approximation of the SDE is

$$S_{t_{i+1}} = rS_{t_i}(t_{i+1} - t_i) + \sigma S_{t_i}\sqrt{t_{i+1} - t_i} Z_{i+1}$$

Call Option

A call option is a financial contract between two parties, the buyer and the seller of the option. The buyer of the option has the right but not the obligation to buy an agreed quantity of the financial instrument (stock) at a certain time (expiration date) at a certain price (strike price).The seller is obligated to sell the financial instrument should the buyer so decide.

The buyer of a call option wants the price of the underlying instrument to go up. The seller either expects that it will not, or is willing to give up some of the upside (profit) from a price rise from the return from a premium and retaining the opportunity to make a a gain up to the strike price.

Black-Scholes Formula

Call option at time t is the expected discounted (at the risk free interest rate r pay-off

$$C_t = E[e^{-r(T-t)} \max(S_T - K, 0)]$$

where K is the strike price of the option and T is the time of maturity of the option
Black and Scholes calculated this and is known as the famous Black-Scholes option pricing formula

$$C_t = S_t \Phi(d_1) - e^{r(T-t)} K \Phi(d_2)$$

where

$$d_{1,2} = \frac{\log \frac{S_t}{K} + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

Φ is the normal distribution function.

Thanks for your attention!