Financial Extremes

Jaya Bishwal
Department of Mathematics and Statistics
University of North Carolina at Charlotte
E-mail: J.Bishwal@uncc.edu
http://www.math.uncc.edu/~jpbishwa

Outline

- Basic Extremes
- Diffusion Extremes
- Interest Rate Extremes
- Stochastic Volatility Extremes
- Long Memory Extremes

Basic Extremes

Extreme Value Distributions

Frechet: $\Phi_\alpha(x) = \exp(-x^{-\alpha}),\ x > 0$

Gumbel: $\Lambda(x) = \exp(-e^{-x}),\ x \in \mathbb{R}$

Weibull: $\Psi_\alpha(x) = \exp(-(-x)^\alpha),\ x \le 0$

$\overline{F}(u) = 1 - F(u) = P(X > u)$.

Regular Variation

A positive measurable function f on $(0, \infty)$ is *regularly varying* at ∞ with index α if

$$\lim_{t \to \infty} \frac{f(tx)}{f(t)} = x^\alpha,\ x > 0.$$

Notation: $f \in \mathcal{R}(\alpha)$

f is said to be *slowly varying* if $\alpha = 0$, *rapidly varying* if the above limit is 0 for $x > 1$ and ∞ for $0 < x < 1$.

$F \in \mathcal{L}(\gamma), \gamma \ge 0$ if for every $y \in \mathbb{R}$,

$$\lim_{x \to \infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = e^{\gamma y}.$$

Convolution Equivalent Distributions

Let X have d.f. F. $F \in \mathcal{S}(\gamma), \gamma \geq 0$ if $F \in \mathcal{L}(\gamma)$ and

$$\lim_{x \to \infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = 2\widehat{f}(\gamma)$$

where $\widehat{f}(\gamma) = Ee^{\gamma X}$ is the moment generating function of X at γ. The class $\mathcal{S} := \mathcal{S}(0)$ is the class of subexponential distributions.

Subexponential distributions are heavy tailed in the sense that no exponential moments exist. $\mathcal{S}$ contains all df.f.s F with regularly varying tails and is a much larger class. Distribution functions in $\mathcal{S}(\gamma)$ for some $\gamma > 0$ have exponential tails, hence are lighter tailed than suexponential distributions.

Fisher-Tippett Theorem

For an iid sequence, the limit distribution of the maxima is one of the three extreme value distributions: Frechet, Gumbel and Weibull.

Diffusion Extremes

Consider the Itô stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \;\; t > 0, \;\; X_0 = x$$

W : standard Brownian motion

$\mu \in \mathbb{R}$: *drift coefficient*

$\sigma > 0$: *diffusion coefficient* or *volatility*

Running maxima :

$$M_t = \max_{0 \le s \le t} X_s, \;\; t > 0$$

Scale function :

$$s(x) = \int_z^x \exp\left(-2\int_z^y \frac{\mu(t)}{\sigma^2(t)}dt\right) dy, \;\; x \in (l, r)$$

where z is any interior point in (l, r).

Speed measure : m has Lebesgue density

$$m'(x) = \frac{2}{\sigma^2(x)s'(x)}, \;\; x \in (l, r).$$

and total mass $|m| = m((l, r))$. s' is the Lebesgue density of s. X_t is ergodic and its stationary distribution is absolutely continuous with Lebesgue density

$$h(x) = \frac{m'(x)}{|m|}, x \in (l, r).$$

X_t satisfies the *usual conditions* which guarantees that X is ergodic with stationary density:

$$s(r) = -s(l) = \infty$$

$$|m| < \infty$$

Davis Theorem

Let $(X_t)_{t\geq 0}$ satisfy the usual conditions. Then for any initial value $X_0 = y \in (l, r)$ and any $u_t \uparrow r$,

$$\lim_{t\to\infty} |P^y(M_t \leq u_t) - F^t(u_t)| = 0.$$

where F is a df defined for any $z \in (l, r)$ by

$$F(x) = \exp\left(-\frac{1}{|m|s(x)}\right), \quad x \in (z, r).$$

Proof: Diffusion can be represented as an Ornstein-Uhlenbeck process after a random time-change. Then standard theory of extremes of Gaussian processes apply.

Davis Corollary

$$\overline{F}(x) \sim \left(|m| \int_z^x s'(y)dy\right)^{-1} \sim (|m|s(x))^{-1}, \quad x \uparrow r.$$

F is in the maximum domain of attraction of G : $F \in MDA(G)$

$$\frac{M_t - b_t}{a_t} \to^{\mathcal{D}} G, \quad t \to \infty.$$

$G \in \{\Phi_\alpha, \Lambda\}, \alpha > 0$

Φ_α is Frechet distribution, Λ is Gumbel distribution.

Main Theorem

If μ and σ are differentiable in some left neighborhood of r such that

$$\lim_{x \to r} \frac{d}{dx} \; \frac{\sigma^2(x)}{\mu(x)} = 0$$

$$\lim_{x \to r} \frac{\sigma^2(x)}{\mu(x)} \exp\left(-2 \int_z^x \frac{\mu(t)}{\sigma^2(t)} dt \right) = -\infty,$$

then

$$\overline{F}(x) \sim |\mu(x)| h(x), \; x \uparrow r.$$

Interest Rate Extremes

Vasicek Model

$$dX_t = (a - bX_t)dt + \sigma dW_t,\ \ X_0 = x,\ \ a \in \mathbb{R}, b > 0$$

$$X_t = \frac{a}{b} + (x - \frac{a}{b})e^{-bt} + \sigma \int_0^t e^{-b(t-s)} dW_s.$$

$$E(X_t) = \frac{a}{b} + (x - \frac{a}{b})e^{-bt} \to \frac{a}{b},\ \ t \to \infty.$$

$$Var(X_t) = \frac{\sigma^2}{2b}(1 - e^{-2bt}) \to \frac{\sigma^2}{2b},\ \ t \to \infty.$$

X_t has a normal stationary distribution $N(\frac{a}{b}, \frac{\sigma^2}{2b})$

Condition of the Main Theorem holds which gives

$$\overline{F}(x) \sim \frac{2b^2}{\sigma^2}(x - \frac{a}{b})^2 \overline{H}(x),\ \ x \to \infty.$$

where $\overline{H}(x)$ is the tail of a stationary normal distribution, hence F has heavier tail than H.

$F \in MDA(\Lambda)$ with norming constants

$$a_t = \frac{\sigma}{a\sqrt{b \log t}}$$

$$b_t = \frac{\sigma}{\sqrt{b}}\sqrt{\log t} + \frac{a}{b} + \frac{\sigma}{4\sqrt{b}} \frac{\log\log t + \log(\sigma^2 d/2\pi)}{\sqrt{\log t}}$$

Cox-Ingersoll-Ross (CIR) Model

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t}dW_t$$

$$E(X_t) = \frac{a}{b} + (x - \frac{a}{b})e^{-bt} \to \frac{a}{b},\ t \to \infty.$$

$$V(X_t) = \frac{a\sigma^2}{2b^2}(1-(1+(x-\frac{a}{b})\frac{2b}{a})e^{-2bt}+(x-\frac{a}{b})\frac{2b}{a}e^{-3bt}) \to \frac{a\sigma^2}{2b^2},\ t \to \infty$$

X_t has a Gamma stationary distribution $\Gamma(\frac{2a}{\sigma^2}, \frac{2b}{\sigma^2})$

Condition of the Main Theorem holds which gives

$$\overline{F}(x) \sim \frac{2ab}{\sigma^2}\overline{G}(x) \sim Ax\overline{H}(x),\ x \to \infty$$

where $\overline{G}(x)$ is the tail of a stationary gamma distribution $\Gamma(\frac{2a}{\sigma^2} + 1, \frac{2b}{\sigma^2})$ hence F has heavier tail than H.

$F \in MDA(\Lambda)$ with norming constants

$$a_t = \frac{\sigma^2}{2b}$$

$$b_t = \frac{\sigma}{2b}(\log t + \frac{2a}{\sigma^2}\log\log t + \log(\frac{b}{\Gamma(2a/\sigma^2)})).$$

Chan-Karloyi-Logstaff-Sanders (CKLS) Model

$$dX_t = (a - bX_t)dt + \sigma X_t^{\gamma} dW_t, \ \gamma \in [1/2, \infty)$$

$\frac{1}{2} < \gamma < 1$

$$E(X_t) = \frac{a}{b} + (x - \frac{a}{b})e^{-bt} \to \frac{a}{b}, \ t \to \infty, \quad b > 0$$

$$E(X_t) = \frac{a}{b} + (x - \frac{a}{b})e^{-bt} \to \infty, \ t \to \infty, \quad b < 0$$

$$E(X_t) = x + at \to \infty, \ t \to \infty, \quad b = 0$$

The lack of first moment indicates that for certain parameter values the model can capture very large fluctuations in the data, which will reflect also in the maxima.

Stationary density is

$$h(x) = \frac{2}{A\sigma^2} x^{-2\gamma} \exp\left(-\frac{2}{\sigma^2}\left(\frac{a}{2\gamma - 1}x^{-(2\gamma-1)} + \frac{b}{2 - 2\gamma}x^{2-2\gamma}\right)\right).$$

for some constant $A > 0$.

Condition of the Main Theorem holds.

$\overline{F}(x) \sim bxh(x) \sim Bx^{2(1-\gamma)}\overline{H}(x), \ x \to \infty$

$F \in MDA(\Lambda)$ with norming constants

$$a_t = \frac{\sigma^2}{2b}\left(\frac{\sigma^2(1-\gamma)}{b}\log t\right)^{\frac{2\gamma-1}{2-2\gamma}}$$

$$b_t = \left(\frac{\sigma^2(1-2\gamma)}{b}\log t\right)^{\frac{1}{2-2\gamma}} \left(1 - \frac{2\gamma-1}{(2-2\gamma)^2}\frac{\log\left(\frac{\sigma^2(1-\gamma)}{b}\log t\right)}{\log t}\right)$$

$$+a_t \log\left(\frac{2b}{A\sigma^2}\right)$$

$\gamma = 1$

The model has an explicit solution

$$X_t = e^{-(b+\frac{\sigma^2}{2})t+\sigma W_t}\left(x + a\int_0^t e^{(b+\frac{\sigma^2}{2})s-\sigma W_s} ds\right)$$

The stationary density is inverse gamma:

$$h(x) = \left(\frac{\sigma^2}{2a}\right)^{-\frac{2b}{\sigma^2}-1}\left(\Gamma\left(\frac{2b}{\sigma^2}+1\right)\right)^{-1} x^{-2b/\sigma^2-2}\exp\left(-\frac{2a}{\sigma^2}x^{-1}\right),$$

$x > 0$, $h \in \mathcal{R}(-2b/\sigma^2 - 2)$ and the tail $\overline{H}$ of the stationary distribution is regularly varying.

$$\overline{F}(x) \sim Bx^{-2b/\sigma^2-1}, \quad x \to \infty.$$

$F \in MDA(\Phi_{1+2b/\sigma^2})$ with norming constants

$$a_t \sim Ct^{1/(1+2b/\sigma^2)}$$

$$b_t = 0$$

$\gamma > 1$

h has the same form as in the case $\frac{1}{2} < \gamma < 1$ and $\overline{H} \in \mathcal{R}(-2\gamma + 1)$

$$\overline{F}(x) \sim (Ax)^{-1}, \ x \to \infty.$$

CKLS pointed out, most plausible value of $\gamma = 1.5$

Condition of the Main Theorem holds.

$F \in MDA(\Phi_1)$ with norming constants

$$a_t \sim t/A$$

$$b_t = 0$$

Stochastic Volatility Extremes

Levy-Ornstein-Uhlenbeck Volatility

Empirical volatility changes in time and exhibits tails which are heavier than normal. Empirical volatility has upward jumps and clusters on high levels. Levy driven Ornstein-Uhlenbeck models can capture heavy tails and volatility jumps and have volatility clusters if the driving Levy process has regularly varying tails.

Black-Scholes Model

$$dS_t = rS_t dt + \sigma S_t dW_t$$

Heston Model

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_t$$

$$dV_t = \lambda(a - V_t)dt + \sigma\sqrt{V_t} dB_t$$

$\lambda, a, \sigma > 0, \quad \lambda a \geq \frac{\sigma^2}{2}$

a is the long run mean, λ is the rate of mean reversion. Volatililty is a CIR Process, W and B are two independent Brownian motions for simplicity, they could be correlated to include *leverage*.

GARCH Model

$$dS_t = \sqrt{V_t} dW_t$$

$$dV_t = \lambda(a - V_t)dt + \sigma V_t dB_t$$

Volatility is a CKLS model with elasticity $\gamma = 1$.

Barndorff-Neilsen-Shephard Model

$$dS_t = (\mu + rS_t)dt + \sqrt{V_t} dW_t + \rho dL_{\lambda t}$$

$$dV_t = -\lambda V_t dt + \sigma dL_{\lambda t}$$

$$V_t = e^{-\lambda t} V_0 + \int_0^t e^{-\lambda(t-s)} dL_{\lambda s}$$

is a cadlag process.

If V_0 is independent of L and $V_0 =^d \int_0^\infty e^{-s} dL_s$ then the process is stationary. The stationary solution is

$$V_t = e^{-\lambda t} \int_{-\infty}^t e^{\lambda s} dL_{\lambda s}.$$

We are concerned with processes L which are heavy or semi-heavy tailed, i.e., whose tails decrease no faster than exponentially.

Define

$$M_h := \sup_{0 \le t \le h} V_t.$$

Theorem a) If $L_1 \in \mathcal{S} \cap MDA(\Phi_\alpha)$, then

$$P(M_h > x) \sim (\lambda h + \alpha^{-1}) P(L_1 > x), \quad x \to \infty.$$

b) If $L_1 \in \mathcal{S} \cap MDA(\Lambda)$, then

$$P(M_h > x) \sim \lambda h P(L_1 > x), \quad x \to \infty.$$

Running Maxima Theorem

a) If $L_1 \in \mathcal{S} \cap MDA(\Phi_\alpha)$, then

$$P(a_{\lambda T}^{-1} M_T \le x) = e^{-x^{-\alpha}}, \quad x > 0$$

where a_T is such that

$$\lim_{T \to \infty} T P(L_1 > a_T x) = x^{-\alpha}, \quad x > 0.$$

b) If $L_1 \in \mathcal{S} \cap MDA(\Lambda)$, then

$$P(a_{\lambda T}^{-1}(M_T - b_{\lambda T}) \le x) = e^{-e^{-x}}, \quad x \in \mathbb{R}.$$

Example: Positive Shot Noise Process

Let L be a positive compound Poisson process

$$L_t = \sum_{j=1}^{N_t} \xi_j$$

where $(N_t)_{t\geq 0}$ is a Poisson process on $\mathbb{R}_+$ with intensity $\mu > 0$ and jump times $(\Gamma_k)_{k\in\mathbb{N}}$. The process N is independent of the i.i.d. sequence of positive r.v.s $(\xi_k)_{k\in\mathbb{N}}$ with d.f. F. The resulting volatility process is then the positive shot noise process

$$\begin{aligned} V_t &= e^{-\lambda t} V_0 + \int_0^t e^{-\lambda(t-s)} dL_{\lambda s} \\ &= e^{-\lambda t} V_0 + \sum_{i=1}^{N_{\lambda t}} e^{-\lambda t + \Gamma_j} \xi_j . \end{aligned}$$

Running Maxima Theorem

a) Let V be a stationary version of the OU process where L is a positive, compound Poisson process. Assume $L_1 \in \mathcal{S}(\gamma), \gamma > 0$. Then

$$\lim_{T\to\infty} P(a_{\lambda T}^{-1}(M_T - b_{\lambda T}) \leq x) = e^{-e^{-x}}, \quad x \in \mathbb{R}.$$

b) Assume that V is a $\Gamma(\mu, \gamma)$-OU process. Then

$$\lim_{T\to\infty} P(a_{\lambda T}^{-1}(M_T - b_{\lambda T}) \leq x) = e^{-e^{-x}}, \quad x \in \mathbb{R}.$$

Long Memory Extremes

A stationary process with correlation function ρ exhibits long range dependence , if there exists a $H \in (0, 1/2)$ and l is a slowly varying function such that

$$\rho(h) \sim l(h)h^{-2H}, \quad h \to \infty.$$

Long range dependence implies that

$$\int_0^\infty \rho(h)dh = \infty.$$

Superposition of Ornstein-Uhlenbeck Processes

Barndorff-Neilsen and Shephard proposed supOU processes as volatility models. Emprirical volatility has long memory in the sense that the empirical auto correlation function decreases slower than exponential. The class of supOU processes can capture extremal clusters and long range dependence.

supOU Processes

$$V_t = \int_{\mathbb{R}_+ \times \mathbb{R}} e^{-r(t-s)} I_{[0,\infty)}(t-s)d\Lambda(r, \lambda s), \quad t \geq 0$$

where $\lambda > 0$ and Λ is an *infinitely divisible independently scattered random measure (i.d.i.s.r.m.)* which are extensions of OU type processes of the form

$$V_t = \int_{-\infty}^{t} e^{-\lambda(t-s)} dL_{\lambda s}$$

where $\lambda > 0$ and L is a Levy process. The time change by λ yields marginal distributions independent of λ. To guarantee that the volatility process V is positive, the Levy process L is chosen as a subordinator. The resulting price process has martingale term $dS_t = \sqrt{V_t} dB_t$, where B is a Brownian motion independent of L.

The *generating quadruple* $(m,\ \sigma^2,\ \nu, \pi)$ determines completely the distribution of Λ. The underlying driving Levy process

$$L_t = \Lambda(\mathbb{R}_+ \times [0, t])$$

has *generating triplet* $(m,\ \sigma^2,\ \nu)$. The underlying driving Levy process

$$L_t = \Lambda(\mathbb{R}_+ \times [0, t]),\ t \geq 0$$

Define the probability measure $\overline{\pi}(dr) := \lambda/r\pi(dr)$ and the idisrm $\overline{\Lambda}$ with generating quadruple $(m/\lambda, \sigma^2/\lambda, \nu\lambda, \overline{\pi})$.

Thus $\overline{\pi}$ is a probability measure on $\mathbb{R}_+$ with $\lambda := \int_{\mathbb{R}_+} r\overline{\pi}(dr)$. The distribution π governs the long range dependence of the model. Essentially the measure

π needs sufficient mass near 0. We write $\overline{\pi}(r) := \pi((0,r])$.

Then

$$X_t = \int_{-\infty}^{\infty} e^{-rt} \int_{-\infty}^{rt} e^s d\overline{\Lambda}(r,s).$$

Then $X = V$ $a.s.$

$$dX_t = \int_{\mathbb{R}_+} \{-rX(t,dr)dt + d\overline{\Lambda}(t,r)\}$$

where

$$X(t,B) = \int_B e^{-rt} \int_{-\infty}^{rt} e^s d\overline{\Lambda}(r,\lambda s).$$

Example: Let π be gamma distribution with density

$$\pi(dr) = \Gamma(2H+1)^{-1} r^{2H} e^{-r} dr$$

for $r > 0$ and $H > 0$. Then $\lambda = 2H$ and

$$\rho(h) = \Gamma(2H)^{-1} \int_0^{\infty} r^{2H-1} e^{-r(h+1)} dr = (h+1)^{-2H},\ h \geq 0.$$

The following theorem shows how long range dependence can be introduced in the supOU models.

Theorem

$$\overline{\pi}(r) \sim (2H)^{-1} l(r^{-1}) r^{2H}, \quad r \to 0$$

if and only if

$$\rho(h) \sim \Gamma(2H) l(h) h^{-2H}, \quad h \to \infty$$

Theorem

Define $M_T := \sup_{0\le t\le T} V_t$

a) Let $L_1 \in \mathcal{R}_{-\alpha}$ with norming constants $a_T > 0$ such that

$$\lim_{T\to\infty} TP(L_1 > a_T x) = x^{-\alpha}, \quad x > 0.$$

Then

$$\lim_{T\to\infty} P(a_{\lambda T}^{-1} M_T \le x) = e^{-x^{-\alpha}}, \quad x > 0.$$

b) Let $L_1 \in \mathcal{S}(\gamma) \cap MDA(\Lambda)$ with norming constants $a_T > 0$ and $b_T \in \mathbb{R}$ such that

$$\lim_{T\to\infty} TP(L_1 > a_T x + b_T) = e^{-x}, \quad x \in \mathbb{R}.$$

Then

$$\lim_{T\to\infty} P(a_{\lambda T}^{-1}(M_T - b_{\lambda T}) \le x) = e^{{Ee^{\gamma L_1}}^{-1} Ee^{\gamma V_0} e^{-x}}, \quad x \in \mathbb{R}.$$

Typical examples of d.f.s in $\mathcal{S} \cap MDA(\Lambda)$ are GIG, NIG, GH, CGMY. All these distributions are self-decomposable, which means that they are possible stationary distributions of OU-type processes and hence also supOU processes.

References

Borkovec, M. and Kluppelberg, C. (2007) : Extremal behavior of diffusion models in finance, *Stochastic Processes and their Applications*, **117**, 432-456.

Davis, R. (1982) : Maximum and minimum of one-dimensional diffusions,*Stochastic Processes and their Applications*, **13**, 1-9.

Fasen, V., Kluppelberg, C. and Lindner, A. (2007): Extremal behavior of stochastic volatility models, (Eds.) Grossinho, Shiryayev, Esquivel, Oliviera *Stochastic Finance*, 107-155, Springer.

Fasen, V. and Kluppelberg, C. (2007): Extremes of supOU processes, (Eds.) Benth, DiNunno, Lindstrom, Oksendal, Zhang *Stochastic Analysis and Applications: The Abel Symposium*, 340-359, Springer.