An Introduction to Extreme Value Theory of Gaussian Random Fields

Yimin Xiao

Michigan State University & SAMSI

http://www.stt.msu.edu/~xiaoyimi

March 31, 2008

Outline

1. Introduction

2. Motivating examples

    2.1 Overflow probability in queueing systems

    2.2 Ruin probability

    2.3 Brain imaging

3. General techniques

    3.1 Metric entropy method (large deviation results)

    3.2 Double-sum method (power correction factors)

    3.3 Excursion set method (asymptotic expansions)

1. Introduction

Let $X = \{X(t), t \in T\}$ be a real valued stochastic process defined on an index set T, say $T \subseteq \mathbb{R}^N$. When $N > 1$, X is called a random field.

We are interested in studying the *excursion (or tail) probability*

$$\mathbb{P}\left(\sup_{t\in T} X(t) \geq u\right) \tag{1}$$

and the geometry of the *excursion set*

$$A_u := A_u(X,T) = \{t \in T : X(t) \geq u\}. \tag{2}$$

They are connected by

$$\mathbb{P}\left(\sup_{t\in T} X(t) \geq u\right) = \mathbb{P}\left(A_u \neq \emptyset\right).$$

When $N = 1$ and $X(t)$ is smooth, the number of connected components of A_u is related to the number of upcrossings, $N_u(T)$, by $X(t)$ of level u.

$$\# \text{ of connected components of } A_u \; = N_u(T) + \mathbf{1}_{\{X(0) \geq u\}}.$$

Moreover,

$$\begin{aligned} \mathbb{P}\left(\sup_{t \in [0,T]} X(t) \geq u \right) &\leq \mathbb{P}\left(N_u(T) \geq 1 \;\text{ or } X(0) \geq u \right) \\ &\leq \mathbb{E}\left(N_u(T) \right) + \mathbb{P}\left(X(0) \geq u \right). \end{aligned}$$

2. Motivating examples

2.1 Overflow probability in queueing systems

2.2 Ruin probability

2.3 Brain imaging

2.4 Other applications

2.1 Overflow probability in queueing systems

Consider a single server queueing system with the netput process $Y = \{Y(t), t \geq 0\}$ and $Y(0) = 0$.

Let $Z(t)$ be the queue length at time t.

It can be shown [see, e.g., Harrison (1985)] that, if Y has stationary increments, then

$$Z(t) \stackrel{d}{=} \sup_{0 \leq s \leq t} Y^*(s),$$

where $Y^*(s) = -Y(-s)$.

The steady-state probability (or overflow probability) is defined as

$$\lim_{t \to \infty} \mathbb{P}\{Z(t) \geq u\} = \mathbb{P}\left\{ \sup_{0 \leq s < \infty} Y^*(s) \geq u \right\}.$$

If the netput process Y is Brownian motion with a drift,

$$Y(t) = \sigma B(t) - ct \quad (c > 0,\ \sigma > 0),$$

then it is well known [cf. Harrison (1985)] that

$$\mathbb{P}\left\{\sup_{0 \le s < \infty} Y(s) \ge u\right\} = \exp\left(-\frac{2cu}{\sigma^2}\right).$$

Examples of netput processes Y:

• a fractional Brownian motion W^H with drift; Dębeki, Michna and Rolski (1998), Hüsler and Piterbarg (1999), $\cdots$.

• an integrated Gaussian process

$$Y(t) = \int_0^t \xi(s)ds - ct,$$

where $\{\xi(s), s \ge 0\}$ is a stationary Gaussian process; cf. Dębeki and Rolski (1995, 2002), Dębeki (2002), Dieker (2005), $\cdots$.

2.2 Ruin probability

Consider the risk process

$$U(t) = u + ct - S(t), \quad t \geq 0,$$

where $u \geq 0$ is the initial capital and $c > 0$ is a constant. In many cases, $S(t)$ can be assumed to be (or approximated by) a mean 0 Gaussian process.

The ruin probability with finite horizon is

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} (S(t) - ct) > u \right).$$

When $T = \infty$, this is called the ruin probability with infinite horizon.

Examples of $S(t)$ include Lévy processes, fractional Brownian motion and so on.

2.3 Brain imaging

Many studies of brain function with positron emission tomography (PET) involve the interpretation of subtracted PET images. The purpose of these studies is to see which areas of the brain show an increase in blood flow, or "activation," due to the stimulation condition.

Worsley, Evans, Marrett and Neelin (1992, 1993) showed that *the averaged image* can be modeled by a Gaussian random field $X(t)$ with a covariance function depending on the known resolution of PET camera.

The maximum of the random field $X(t)$ was used to test for activation at an unknown point in PET images.

The geometric characteristic of the excursion sets A_u was used to estimate the number of regions of activation. See Worsley (1995, 1996) for more information.

2.4 Other applications

Astrophysics:

◇ Gott, Mellot and Dickinson (1986), Rhoads, Gott and Postman (1994) studied the density of matter in the universe;

◇ Torres (1994) used the similar tools to study the fluctuations in the cosmic microwave which were discovered by Smoot, et al. (1994).

Microstructure modeling:

◇ Roberts and Garboczi (2001), Kozintsev and Kedem (2000) used excursion sets to generate realistic microstructure models.

3. General techniques

There are three main techniques for estimating the excursion probability (1):

• Metric entropy method: Dudley (1967), Borell (1975), Talagrand (1994), ...

• Double-sum method: Pickands (1969), Piterbarg (1996), ...

• Excursion set method: Adler (1976, 1981, 2000), Worsley (1995), Taylor (2001), Adler and Taylor (2007), ...

3.1 Large deviation results

As a first order approximation to (1), one studies the following

Problem: Find a positive function $\varphi(u)$ such that

$$\lim_{u\to\infty} \frac{1}{\varphi(u)} \log \mathbb{P}\left(\sup_{t\in T} X(t) \geq u\right) = -K,$$

where K is a positive constant.

• Advantages: It gives the correct logarithmic rate and, for Gaussian random fields, it can be done.

• Drawback: for some applications, the result is not precise enough.

The following theorem is originally due to Landau and Shepp (1970). See also Marcus and Sheep (1972).

Theorem 1 Let $X = \{X(t), t \in T\}$ be a centered Gaussian process with bounded sample paths a.s. Then

$$\lim_{u\to\infty} \frac{1}{u^2} \log \mathbb{P}\left(\sup_{t\in T} X(t) \geq u\right) = -\frac{1}{2\sigma_T^2}, \tag{3}$$

where $\sigma_T^2 = \sup_{t\in T} \mathbb{E}(X(t)^2)$.

Proof Half of the proof is easy: note that

$$\mathbb{P}\left(\sup_{t\in T} X(t) \geq u\right) \geq \sup_{t\in T} \mathbb{P}\left(X(t) \geq u\right),$$

which implies

$$\liminf_{u\to\infty} \frac{1}{u^2} \log \mathbb{P}\left(\sup_{t\in T} X(t) \geq u\right) \geq -\frac{1}{2\sigma_T^2}.$$

To prove the other half of Theorem 1, one can make use of the following important result in modern theory of Gaussian processes, which is due independently to Borell (1975) and Cirelson, Ibragimov and Sudakov (1976).

Theorem 2 [Borell's Inequality] Let $X = \{X(t), t \in T\}$ be a centered Gaussian process with bounded sample paths a.s. Then $m := \mathbb{E}(\sup_{t\in T} X(t)) < \infty$ and for all $u > m$,

$$\mathbb{P}\left\{\sup_{t\in T} X(t) > u\right\} \le 2\exp\left(-\frac{(u-m)^2}{2\sigma_T^2}\right).$$

The boundedness condition $m := \mathbb{E}(\sup_{t\in T} X(t)) < \infty$ can be verified by the metric entropy theorem due to Dudley (1967).

Theorem 3 [Dudley's Metric Entropy Condition] Let $X = \{X(t), t \in T\}$ be a centered Gaussian process with pseudo-metric

$$d(s,t) := \sqrt{\mathbb{E}(X(t) - X(s))^2}.$$

Then there exists a constant $K > 0$ such that

$$\mathbb{E}\left(\sup_{t\in T} X(t)\right) \le K \int_0^\infty \sqrt{\log N_d(T,\varepsilon)}\, d\varepsilon,$$

where $N_d(T,\varepsilon)$ is the ε-covering number of T under the metric d.

A necessary and sufficient condition for $\mathbb{E}(\sup_{t\in T} X(t)) < \infty$ was established in Fernique (1974) and Talagrand (1987).

Example 1: Fractional Brownian motion

An N-parameter fractional Brownian motion (FBM) with index $\alpha \in (0,1)$ is a centered Gaussian random field $X^\alpha = \{X^\alpha(t),\, t \in \mathbb{R}^N\}$ in $\mathbb{R}$ with covariance function

$$\mathbb{E}\left[X^\alpha(s)X^\alpha(t)\right] = \frac{1}{2}\left(|s|^{2\alpha} + |t|^{2\alpha} - |s-t|^{2\alpha}\right).$$

X^α has the following properties:

* Self-similarity: $\forall c > 0$,

$$\left\{X^\alpha(ct),\, t \in \mathbb{R}^N\right\} \overset{d}{=} \left\{c^\alpha X^\alpha(t),\, t \in \mathbb{R}^N\right\}.$$

* Isotropy & stationarity of increments:

$$\mathbb{E}\left[(X^\alpha(t) - X^\alpha(s))^2\right] = |t-s|^{2\alpha}.$$

The pseudo-metric d on $\mathbb{R}^N$ is

$$d(s,t) = |t-s|^\alpha.$$

Note that for any $\varepsilon > 0$,

$$d(s,t) \le \varepsilon \iff |s-t| \le \varepsilon^{1/\alpha}.$$

Hence, for every $M > 0$ and $T = [0, M]^N$, we have

$$N_d(T, \varepsilon) \asymp \left(\frac{M}{\varepsilon^{1/\alpha}}\right)^N .$$

The entropy integral

$$\int_0^\infty \sqrt{\log N_d(T,\varepsilon)}\, d\epsilon \le \int_0^{M^\alpha} \sqrt{\log\left(\frac{M}{\varepsilon^{1/\alpha}}\right)}\, d\varepsilon < \infty.$$

Consequently, we obtain

$$\lim_{u\to\infty} \frac{1}{u^2} \log \mathbb{P}\left(\sup_{t\in[0,M]^N} X^\alpha(t) \ge u\right) = -\frac{1}{2(\sqrt{N}\, M)^{2\alpha}}.$$

Example 2: Ruin probability with fractional BM

Let $N = 1$ and consider the ruin probability:

$$\mathbb{P}\left(\sup_{0\le t<\infty} (X^\alpha(t) - ct) > u\right).$$

Using the same method as above, one can show (try it!)

$$\lim_{u\to\infty} \frac{1}{u^{2-2\alpha}} \log \mathbb{P}\left(\sup_{0\le t<\infty} (X^\alpha(t) - ct) \ge u\right)$$

$$= -\frac{c}{2\alpha^{2\alpha}(1-\alpha)^{2-2\alpha}}.$$

3.2 Asymptotic results with power correction

By refining the entropy argument, one can improve the upper bound for the excursion probability given by Theorem 1; see Samorodnitsky (1987, 1988), Talagrand (1994).

Theorem 4 If for some constants $A \geq \sigma_T$, $\varepsilon_0 \in [0, \sigma_T]$ and $\beta > 0$ we have

$$N_d(T, \varepsilon) \leq \left(\frac{A}{\varepsilon}\right)^{\beta} \quad \forall\ \varepsilon < \varepsilon_0.$$

Then for all $u \geq \sigma_T^2(1 + \sqrt{\beta})/\varepsilon_0$ we have

$$\mathbb{P}\left(\sup_{t \in T} X(t) \geq u\right) \leq K \left(\frac{Au}{\sqrt{\beta}\sigma_T^2}\right)^{\beta} \overline{\Phi}\left(\frac{u}{\sigma_T}\right),$$

where $\overline{\Phi}(u) = \mathbb{P}\left(N(0,1) \geq u\right)$ it the tail probability of a standard normal r.v.

For some Gaussian random fields, the metric entropy method is still useful for studying the following

Problem: Find a positive function $\psi(u)$ such that

$$\lim_{u\to\infty} \frac{\mathbb{P}\left(\sup_{t\in T} X(t) \geq u\right)}{\psi(u)} = 1. \tag{4}$$

For example, Berman (1985) and Talagrand (1988) applied the metric entropy method to prove

Theorem 5 If the variance of $X = \{X(t), t \in [0,1]^N\}$ has a *unique maximum* at a point $\tau \in [0,1]^N$ [so $\sigma_T^2 = \mathrm{Var}\Big(X(\tau)^2\Big)$] and satisfies some technical conditions, then (4) holds with $\psi(u) = \mathbb{P}\{X(\tau) \geq u\}$.

Double-sum method

This method was initiated by Pickands (1969a, b) for stationary Gaussian processes. It was extended by Bickel and Rosenblatt (1973), Qualls and Watanabe (1973) and Piterbarg (1996) to Gaussian random fields.

The basic idea is to write $T = \bigcup_{k=1}^{n} T_k$, where T_k $(1 \leq k \leq n)$ generally depend on the exceedence level u.

Then

$$\begin{aligned}
\sum_{k=1}^{n} \mathbb{P}\left(\sup_{t \in T_k} X(t) \geq u\right) &\geq \mathbb{P}\left(\sup_{t \in T} X(t) \geq u\right) \\
&\geq \sum_{k=1}^{n} \mathbb{P}\left(\sup_{t \in T_k} X(t) \geq u\right) \\
&\quad - \sum_{j \neq k} \mathbb{P}\left(\sup_{t \in T_j} X(t) \geq u, \ \sup_{t \in T_k} X(t) \geq u\right).
\end{aligned}$$

Theorem 6 Let $\{X(t), t \in \mathbb{R}^N\}$ be a *stationary, isotropic* Gaussian random field with mean 0 and variance 1 such that

$$\mathbb{E}(X(s)X(t)) = 1 - |s-t|^{2\alpha} L(|s-t|) + o(|s-t|^{2\alpha} L(|s-t|))$$

as $|s-t| \to 0$, where $L(\cdot)$ is a slowly varying function at 0. Then for any bounded open set $T \subset \mathbb{R}^N$ with $\lambda_N(T) = \lambda_N(\overline{T})$,

$$\lim_{u\to\infty} \frac{\mathbb{P}\left(\sup_{t\in T} X(t) \ge u\right)}{\Psi(u)} = K_\alpha,$$

where K_α is the generalized Pickands' constant and $\Psi(u)$ is defined by

$$\Psi(u) = \lambda_N(T)(2\pi)^{-1} u^{-1} e^{-u^2/2} \left(\sigma^{-1}(u^{-1})\right)^{-N}.$$

In the above, $\sigma^2(r) = 2r^{2\alpha} L(r)$.

The conditions of Theorem 6 are quite restrictive. Some extensions have been established by Berman (1991), Piterbarg (1996), Dieker (2005).

3.3 Asymptotic expansion

For *smooth* Gaussian random fields, it is possible to establish an expansion of the form

$$\mathbb{P}\left(\sup_{t\in T} X(t) \geq u\right) = u^{\alpha} e^{-u^2/(2\sigma_T^2)} \left[\sum_{j=0}^{n} C_j u^{-j} + \text{ error}\right] \quad (5)$$

for large $u > 0$ and constant parameters α, σ_T, n and C_j depending on the *distribution of* X and the *geometry of* T.

This is the main theme of the recent book "Random Fields and Geometry" by R. Adler and J. Taylor (2007).

The key step is the following connection

$$\mathbb{P}\left(\sup_{t\in T} X(t) \geq u\right) = \mathbb{E}\left(\varphi(A_u)\right) + o\left(\mathbb{E}(\varphi(A_u))\right), \quad (6)$$

where $\varphi(A_u)$ denotes the *Euler characteristic* of A_u.

Euler characteristic

Let $A \subset \mathbb{R}^N$ be a nice set. For $N = 1, 2$, roughly,

$$\varphi(A) = \begin{cases} \#\text{of disjoint closed intervals in } A, & \text{if } N = 1, \\ \\ \#\text{of con. components} - \#\text{of holes}, & \text{if } N = 2. \end{cases}$$

For $N \geq 2$, $\varphi(A)$ can be defined by an iterative procedure.

If $X = \{X(t), t \in \mathbb{R}^N\}$ is a smooth, stationary and isotropic Gaussian random field, the expected Euler characteristic of the excursion set A_u, $\mathbb{E}\left(\varphi(A_u)\right)$, can be computed exactly.

Combining this with (6) yields expansion of the form (5).