Satisficing measures for analysis of risky positions

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I. Motivation

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  - Why not use financial goals ("aspiration levels" or "targets") directly?

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- Normative perspective:
  - Castagnoli and LiCalzi (1996)
  - Bordley and LiCalzi (2000)
  - Tsetlin and Winkler (2007)

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  - Payne et al. (1980)
  - Diecidue and van de Ven (2005)

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- Axiomatically define a class of measures evaluating performance with respect to targets (*satisficing measures*)
- Impose "diversification-favoring" properties on the measures
- Show that these measures are *duals of risk measures*
- Describe some important examples
- Consider using these measures in the portfolio choice problem

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- **Overall goal**: $\Rightarrow$ create a set of target-based tools for risk management

II. Satisficing measures

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- A position $X \in \mathcal{X}$ achieves the target if and only if $X(\omega) \geq 0$
- *Notation*: For $X,\ Y \in \mathcal{X}$, $X \geq Y$ means $X(\omega) \geq Y(\omega)$ for all $\omega \in \Omega$ (e.g., $X \geq 0$ means always beat target).

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- What if every position always over/under performs the target?

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  - $\mathbb{P}\{Y \geq 0\} > 0$
  - $\mathbb{P}\{Z \geq 0\} = 0$

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- CSM $\subseteq$ QSM $\subseteq$ Satisficing measures

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  - Maximum aversion level such that we still "break even" under exponential utility
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- Family of coherent risk measures: $\rho(X) = \sup_{k \in (0,1]} \left\{ k \ : \ \mu_k(X) \leq 0 \right\}$,
  $\{\mu_k\}$ nondecreasing, is a CSM

III. Satisficing and risk duality

Value-at-risk and probability measures

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  $\mathbb{P}\{X \geq 0\} = \sup\{1 - \alpha \ : \ \mathrm{VaR}_\alpha(X) \leq 0\}$.
- Probability measure and VaR are *duals*:
  - VaR: fix tolerance level (α) and find smallest target subject to probability no smaller than $1 - \alpha$
  - Prob.: fix target and find smallest tolerance level (α) subject to $\mathrm{VaR}_\alpha(X)$ no larger than target

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Theorem. *A function $\rho : \mathcal{X} \to [0, \bar{\rho}]$, where $\bar{\rho} \in \{1, \infty\}$, is a satisficing measure if and only if there exists a family of risk measures $\{\mu_k \ : \ k \in (0, \bar{\rho}]\}$, nondecreasing in k, and $\mu_0 = -\infty$ such that*

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- Connects to Simon (1955): probabilities not known exactly!

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- **Theorem**: *For all positions $X \in \mathcal{X}$, $\rho_{CVaR}(X) \leq \mathbb{P}\{X \geq 0\}$ and $\rho_{CVaR}(X)$ is the best such lower bound among all distribution invariant CSMs.*

CVaR measure recognizes tail changes

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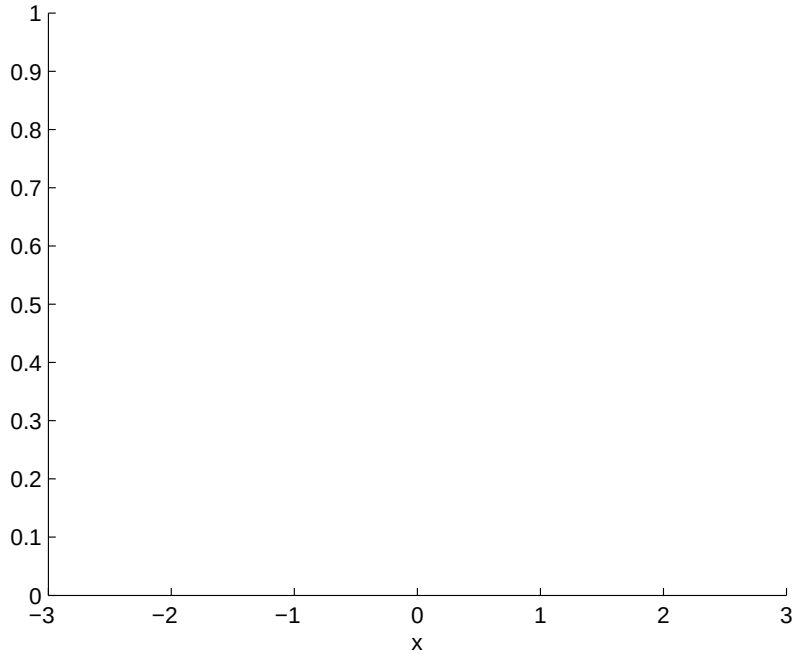

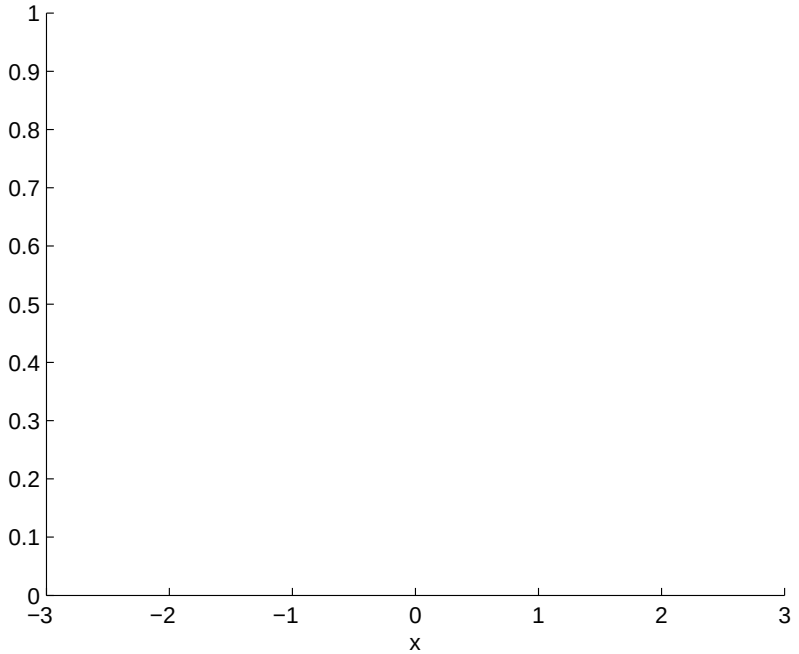

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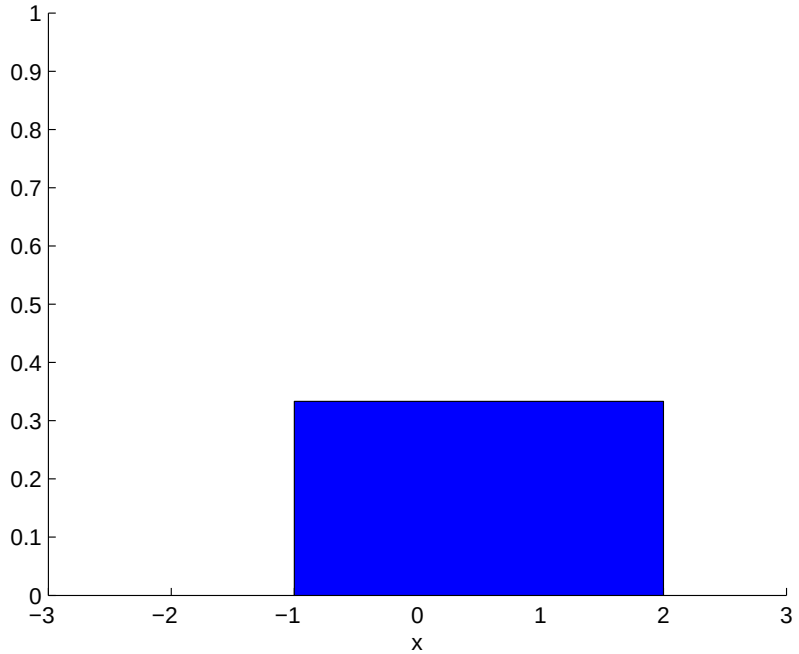

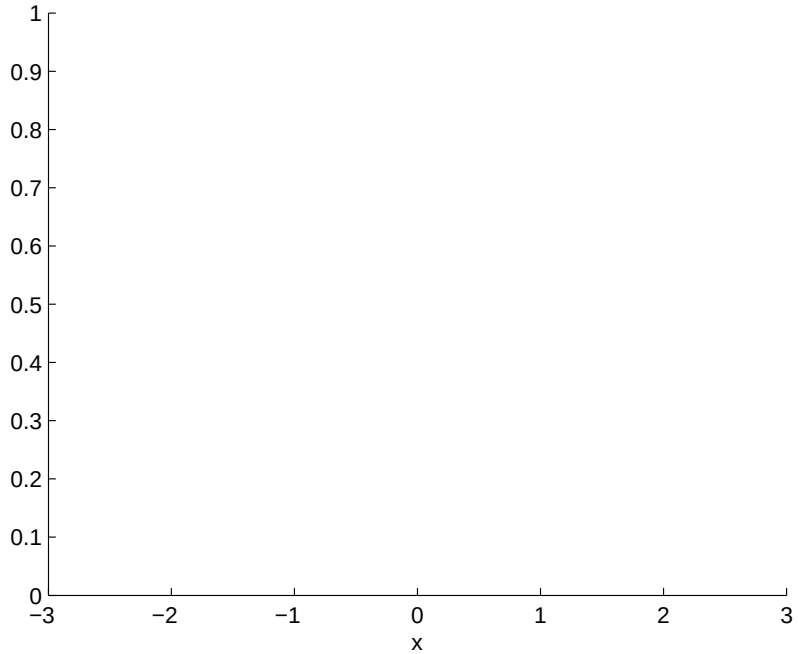

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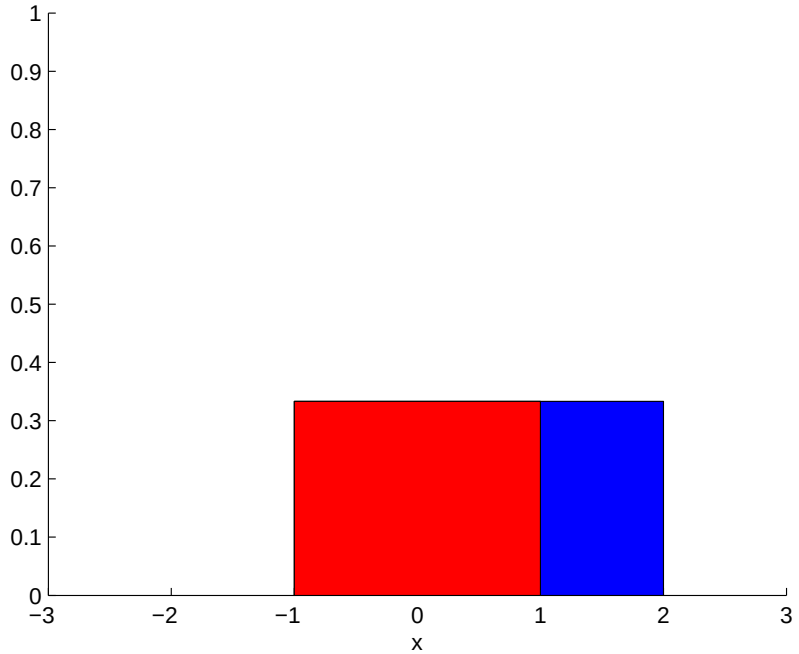

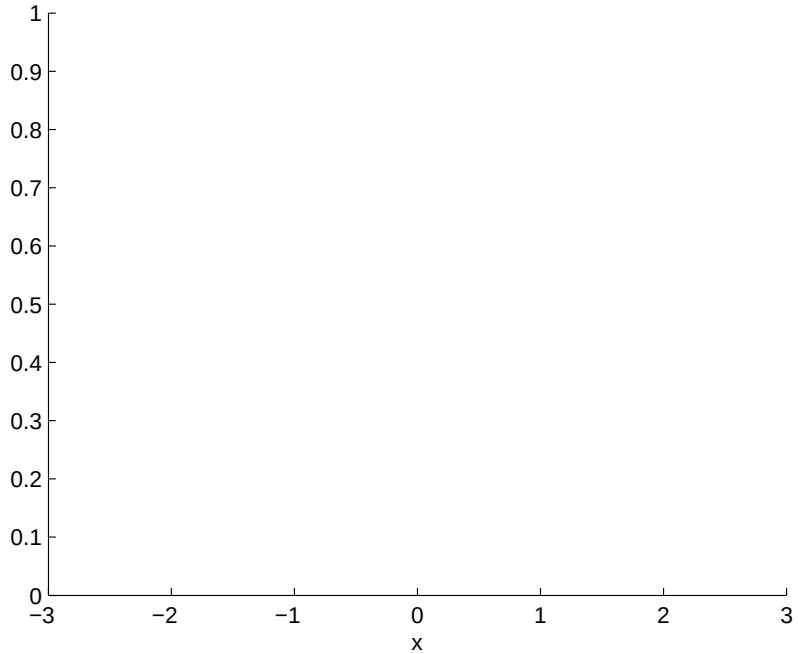

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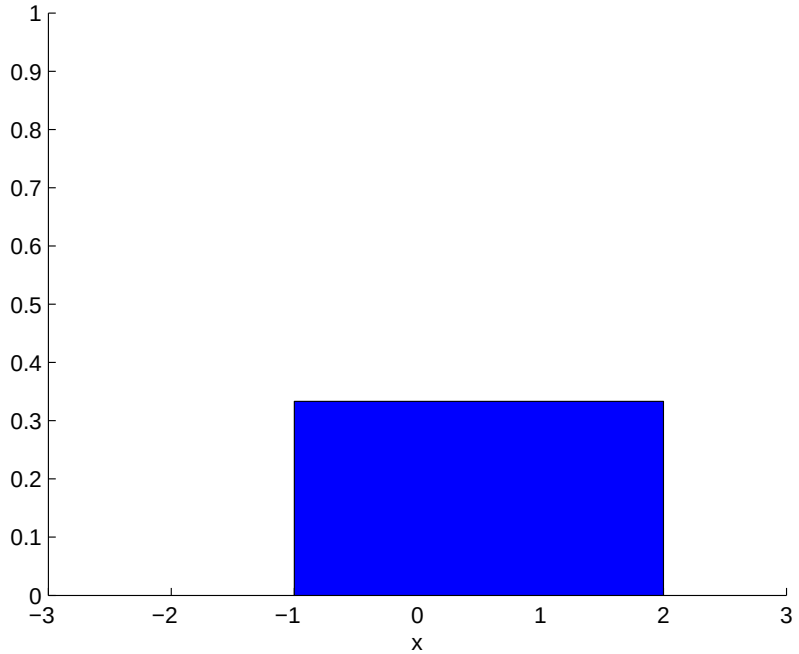

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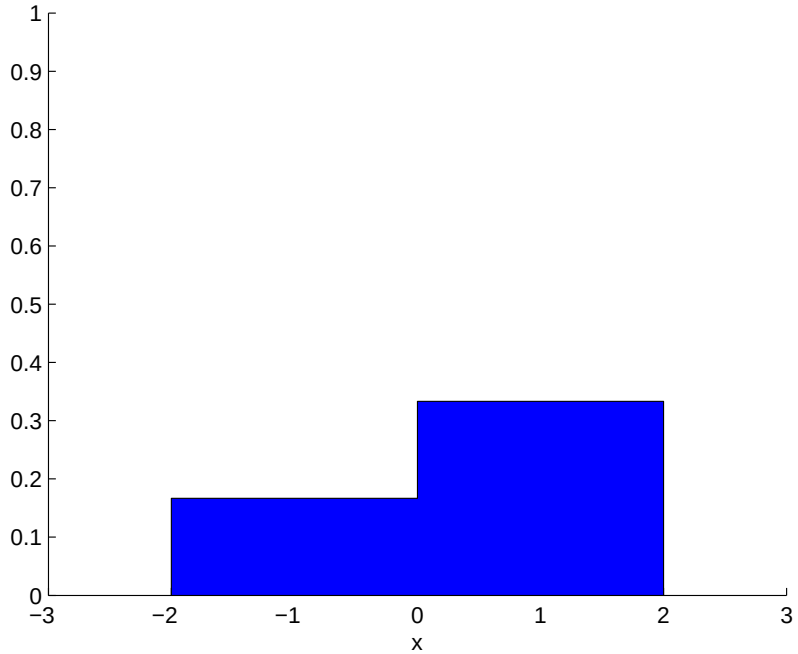

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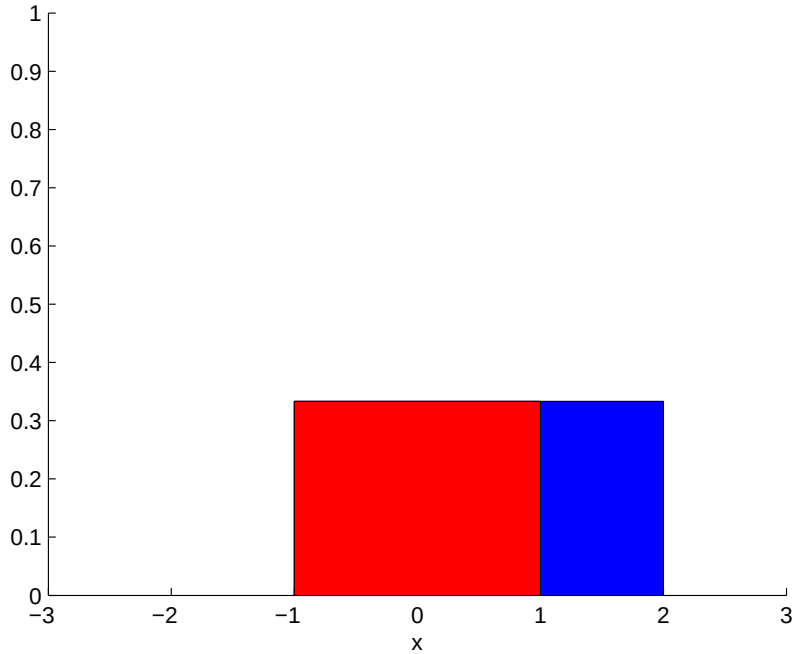

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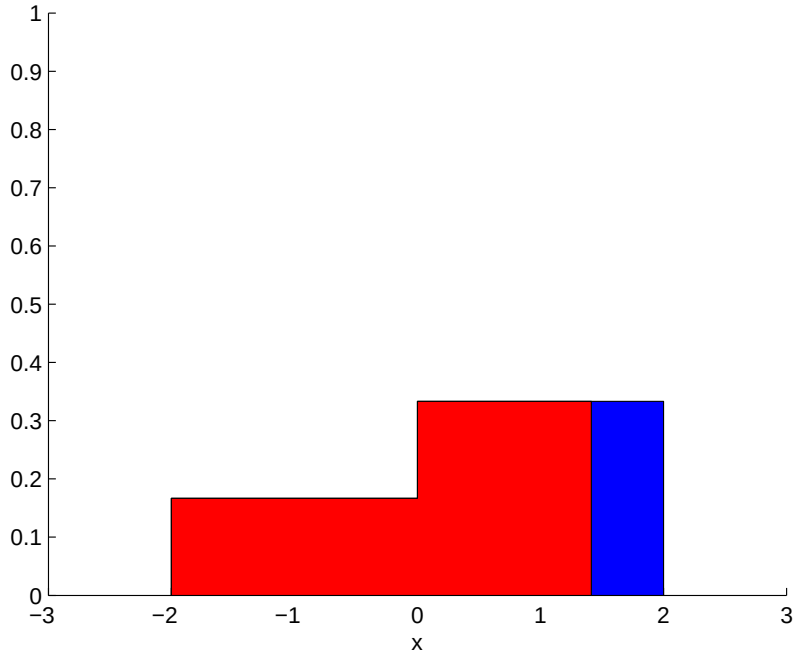

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III. Portfolio choice with satisficing measures

Tangent portfolios

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- Problem of interest:

$$\begin{aligned} \text{max.} \quad & \rho(\alpha X + (1-\alpha) r_f - r_f) \\ \text{s.t.} \quad & \mathbb{E}\left[\alpha X + (1-\alpha) r_f\right] = \gamma \quad (\star) \\ & X \in \mathcal{X},\ \alpha > 0. \end{aligned}$$

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- **Theorem**: *Suppose* $\mathbb{E}[X] > r_f$ *for all* $X \in \mathcal{X}$. *Let* $\rho(\cdot)$ *be a CSM and suppose the objective of Problem* $(\star)$ *is strictly positive for a given* $\gamma > r_f$. *Then the corresponding optimal, risky asset position* X^* *is also optimal to the problem.*

$$\begin{aligned} \textit{max.} \quad & \rho(X - r_f) \\ \textit{s.t.} \quad & X \in \mathcal{X}. \end{aligned}$$

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- Compare: $\rho =$ Sharpe ratio and $\rho = \rho_{\text{CVaR}}$

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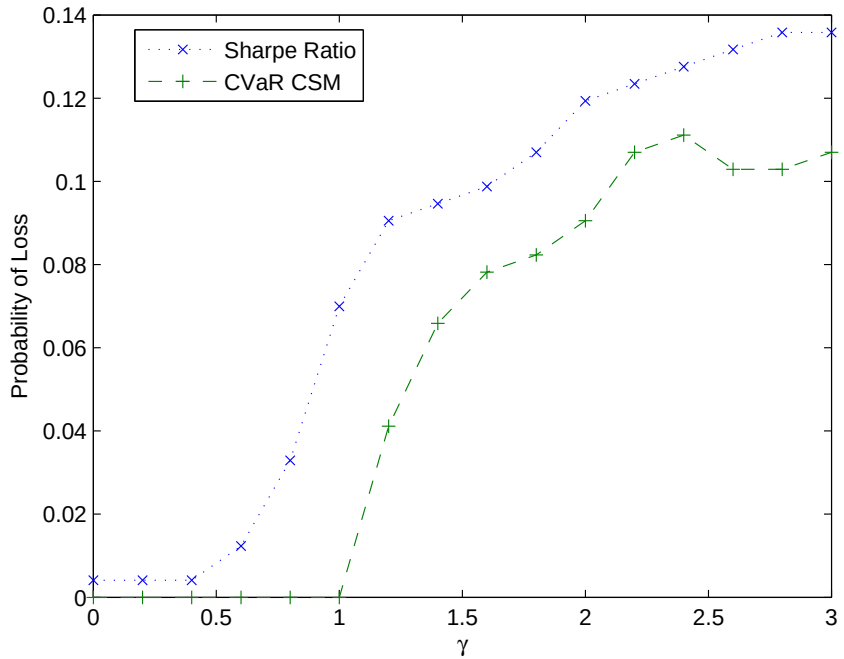

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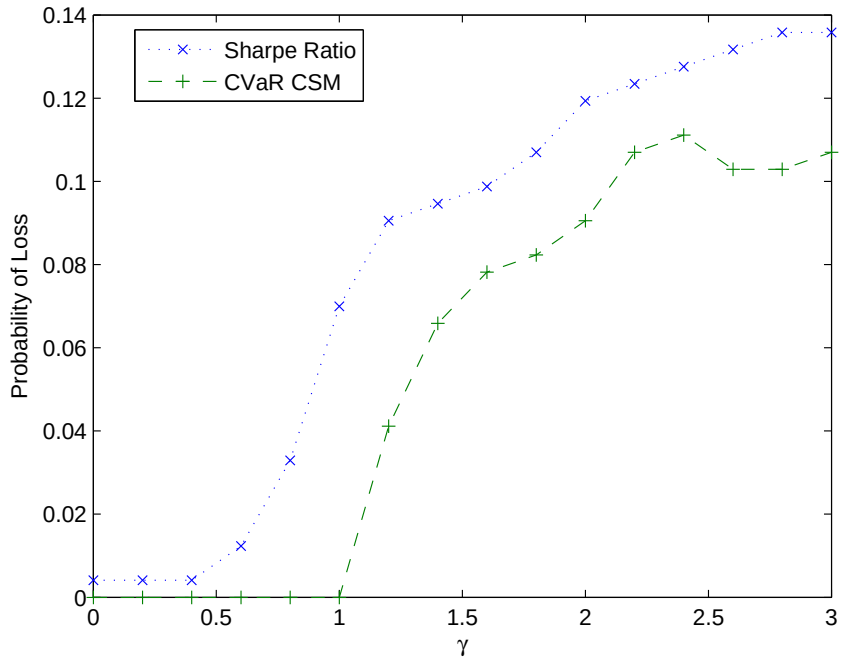

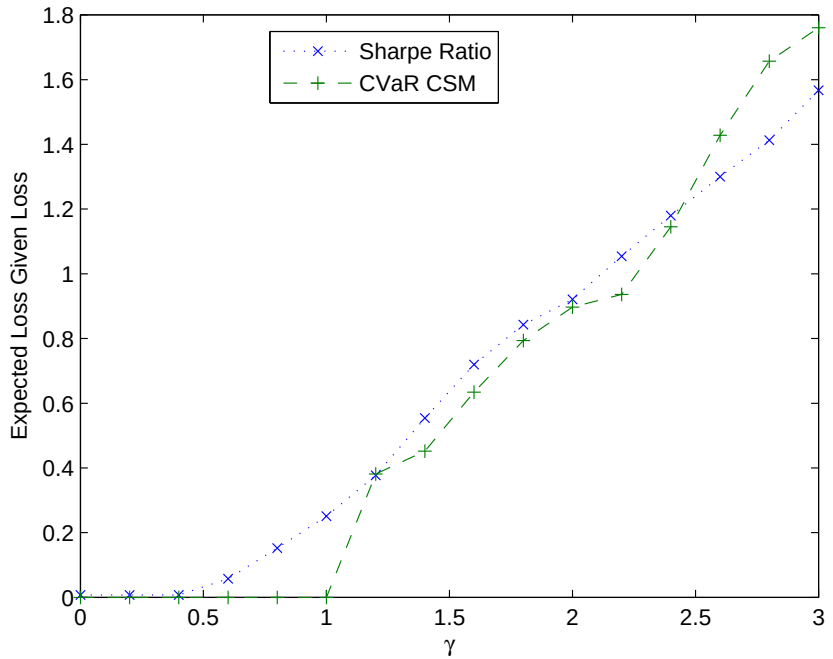

Numerical results #2

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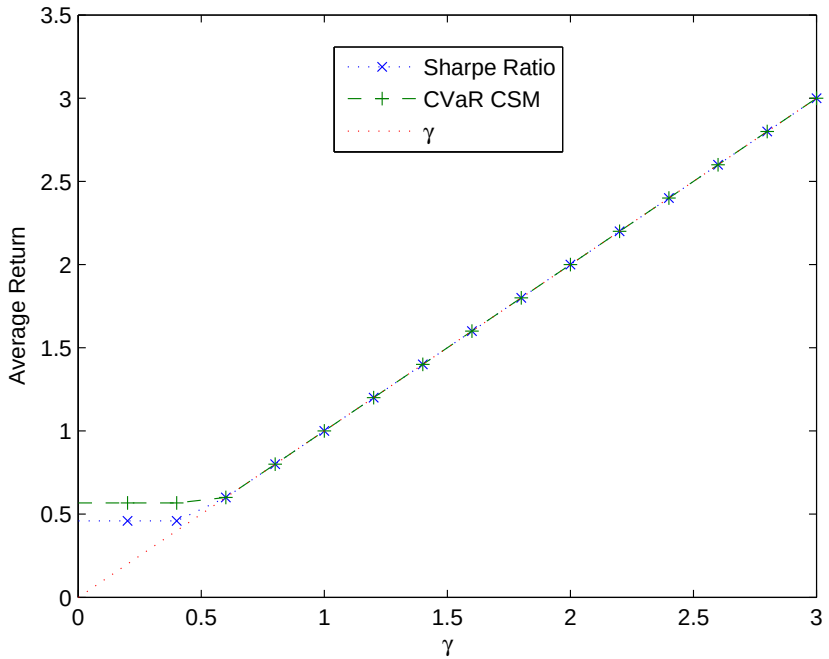

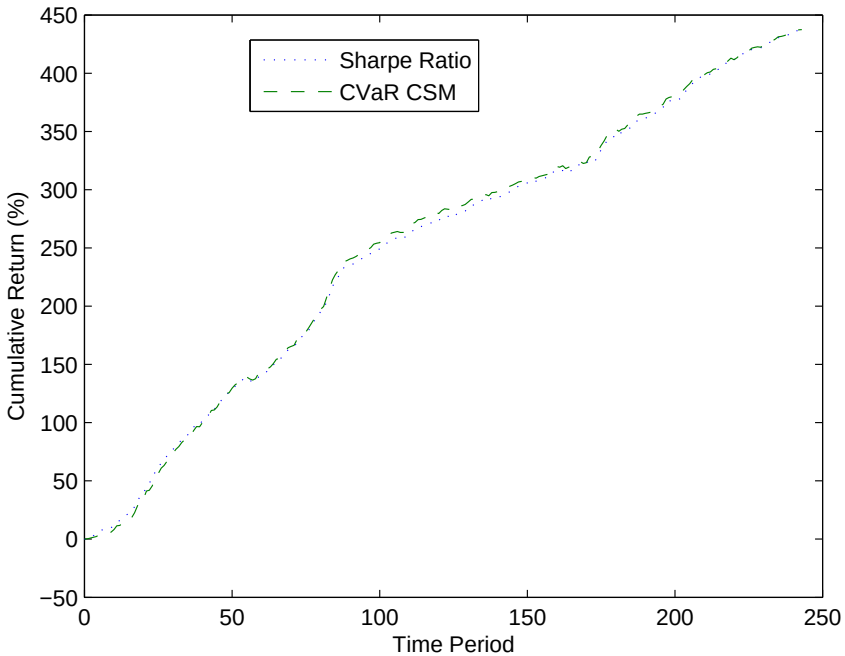

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  - Multiple targets