Super fast tour on basics of compressed sensing

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What is compressed sensing about?

(i) Many signals of interest (sounds, images, etc...) are *compressible* in some well-chosen basis

(ii) Currently: make lots of measurements (e.g. 8Mpixels in a digital camera), expand on a well-chosen basis, retain and save only large coefficients.

(iii) Q: is it necessary to makes lot of measurements, if we already know they don't contain much information and we know how to efficiently compress them? CS: No!

Main players: E. Candes, D. Donoho, A. Gilbert, J. Romberg, M. Strauss, T. Tao, J. Tropp, many others, in the past and in the present...

The Fourier case

Theorem

For any $M > 0$, let

$$\#\{t : f(t) \neq 0\} \leq \frac{\alpha(M)}{\log N}\tau N$$

where Ω is a random set of frequencies (each frequency gets picked randomly with probability τ), with $\mathbb{E}[\#\Omega] = \tau N$. Then f can be reconstructed exactly with probability at least $1 - O(N^{-M})$. One can choose $\alpha(M) \sim \frac{1}{29.6(M+1)}$.

What is the reconstruction algorithm? Solve

$$\mathrm{argmin}_g ||g||_{\ell^1} \quad : \quad \hat{g}(k) = \hat{f}(k) \;\; k \in \Omega\,.$$

Compressible signals

Theorem (C-T,D)

Suppose $f \in R^N$, $|f|_{(n)}$, the n-th largest entry of the expansion of f on a basis, is smaller than $Rn^{-\frac{1}{p}}$, $p \in (0,1)$. Given K measurements $\{\langle f, \chi_k \rangle\}_{k=1,\dots,K}$, where χ_k are i.i.d. Gaussian random variables in R^N, the solution to

$$\mathrm{argmin}_{f^\#} ||\langle f^\#, \chi_k \rangle||_1 : \langle f^\#, \chi_k \rangle = \langle f, \chi_k \rangle ,$$

satisfies, with probability at least $1 - O(N^{-\frac{\rho}{\alpha}})$, where α is smaller than universal constant,

$$||f - f^\#||_2 \le C_{p,\alpha} R \left(\frac{K}{\log N} \right)^{\frac{1}{p} - \frac{1}{2}} .$$

Ex.: special classes of images with m pixels need, on a curvelet basis, $\sim m^{\frac{1}{4}} \log^{\frac{5}{2}}(m)$ coeffs.

A practical note

With matrices: assume the data is Ψx_0, where each column of Ψ is a basis vector, $||x_0||_p \leq R$, for some $p < 1$. Pick CS matrix Φ, $n \times m$, $n < m$, and measure $y = \Phi(\Psi x_0)$. Solve $\hat{x} = \mathrm{argmin}_x ||x||_1$ subject to $y = \Phi(\Psi x)$.
Can be made robust to noise, by solving the slightly relaxed problem $||\Phi(\Psi x_0) - \Phi(\Psi x)|| < \epsilon$, where ϵ is comparable to size (variance) of the noise.