Incompressible Interface Formulation

Throughout this paper, the interface Γ is tracked by a parametric form $(X(s,t), Y(s,t)), 0 \le s \le L_b$, where s is the arc-length parameter of the initial configuration of the interface. The governing equations for an incompressible interface in Navier-Stokes flows can be written as follows.

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mu \nabla^2 \mathbf{u} + \mathbf{f}, && (1)\\
\nabla \cdot \mathbf{u} &= 0, && (2)\\
\mathbf{f}(\mathbf{x},t) &= \int_\Gamma \frac{\partial}{\partial s}(\sigma(s,t)\boldsymbol{\tau}(s,t))\,\delta(\mathbf{x}-\mathbf{X}(s,t))\,ds, && (3)\\
\frac{\partial \mathbf{X}(s,t)}{\partial t} = \mathbf{u}(\mathbf{X}(s,t),t) &= \int_\Omega \mathbf{u}(\mathbf{x},t)\,\delta(\mathbf{x}-\mathbf{X}(s,t))d\mathbf{x}, && (4)\\
\boldsymbol{\tau}(s,t) &= \frac{\partial \mathbf{X}}{\partial s}. && (5)
\end{aligned}$$

The surface tension $\sigma(s,t)$ on the interface is unknown such that the surface divergence

$$\nabla_s \cdot \boldsymbol{u}|_\Gamma = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\tau}} \cdot \boldsymbol{\tau}|_\Gamma = 0. \qquad (6)$$

One can further take derivatives explicitly so that

$$\frac{\partial}{\partial s}(\sigma\boldsymbol{\tau}) = \frac{\partial \sigma}{\partial s}\boldsymbol{\tau} + \sigma\frac{\partial \boldsymbol{\tau}}{\partial s} = \frac{\partial \sigma}{\partial s}\boldsymbol{\tau} + \sigma\,\kappa\,\boldsymbol{n}, \qquad (7)$$

where κ is the curvature of the interface and $\boldsymbol{n}$ is the unit outward normal. So the equations has an alternative form with jump conditions as

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mu \nabla^2 \mathbf{u}, && (8)\\
\nabla \cdot \mathbf{u} &= 0, && (9)\\
[\mathbf{u}] = 0, \qquad\qquad & \mu\left[\frac{\partial \mathbf{u}}{\partial \mathbf{n}}\right] = -\frac{\partial \sigma}{\partial s}\boldsymbol{\tau} && (10)\\
[p] = \sigma\,\kappa, \qquad\qquad & \left[\frac{\partial p}{\partial \mathbf{n}}\right] = \frac{\partial^2 \sigma}{\partial s^2}. && (11)
\end{aligned}$$

Once again, the surface tension σ needs to be determined so that the equation (6) must be satisfied on the interface. An augmented approach can be applied to solve the problem.