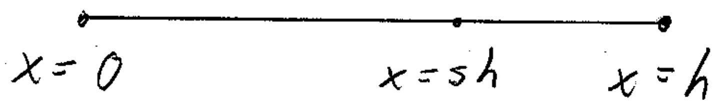
Bilinear Interpolation with Jump Conditions

First dimension one:

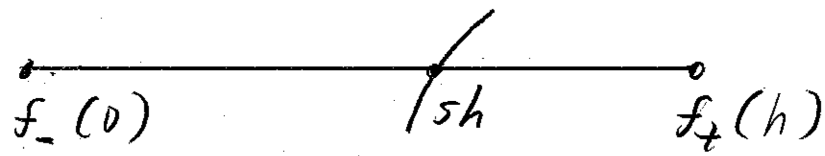


$$f(sh) = (1-s)f(0) + sf(h) + O(h^2)$$

if f is smooth.

Suppose f_- for $x \le sh$, f_+ for $x \ge sh$

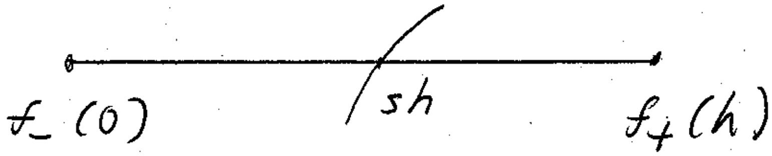
$[f] \equiv f_+(sh) - f_-(sh)$ etc.



We can assume f_- has a smooth extension to $0 \le x \le h$.

$$f_-(sh) = (1-s)f_-(0) + sf_-(h) + O(h^2)$$

$$= (1-s)f_-(0) + sf_+(h) - s[f](h) + O(h^2)$$



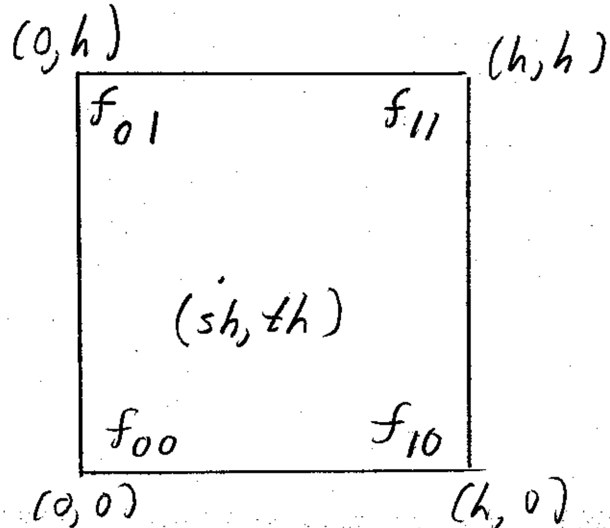
$$f_-(s) = (1-s)f_-(0) + s f_+(h) - s[f](h) + O(h^2)$$

$[f]$, $[f]'$ known at sh

$$[f](h) = [f](sh) + (h - sh)[f]'(sh) + O(h^2)$$

$$\boxed{\begin{aligned} f_-(s) &= (1-s)f_-(0) + s f_+(h) \\ &\quad - s[f] - hs(1-s)[f'] + O(h^2) \\ &\text{jumps at interface point } sh \end{aligned}}$$

A similar method works in 2D with bilinear interpolation:

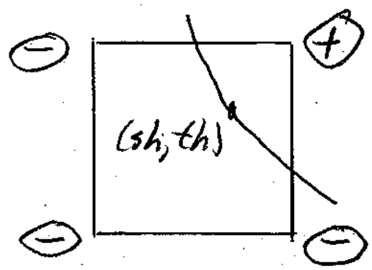


For f smooth

$$f(s,t) = (1-s)(1-t) f_{00} + s(1-t) f_{10} + (1-s)t f_{01} + st f_{11} + O(h^2)$$

Now suppose the interface goes through (sh, th)

Case 1



$$f_-(sh,th) = g(sh,th) - st[f] + O(h^2)$$

where $[f] = f_+(h,h) - f_-(h,h)$

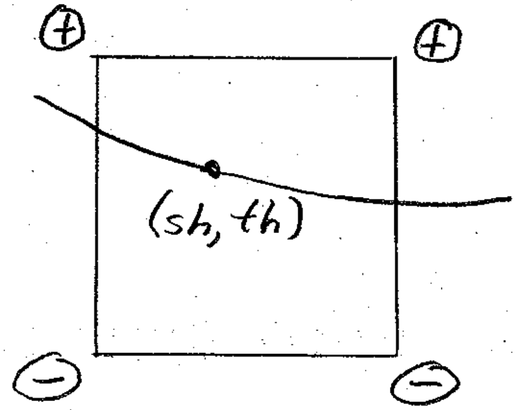
Assume jumps known at (sh, th).

$$[f](h,h) = [f](sh,th) + [f_x](h-sh) + [f_y](h-th) + O(h^2)$$

$g(sh,th)$ = 4-point interpolant from 4 corners, 3 $\ominus$, 1 $\oplus$

$$\boxed{f_-(sh,th) = g(sh,th) - st[f] - sth\left([f_x](1-s) + [f_y](1-t)\right) + O(h^2)}$$

Case 2



$q(sh,th)$ = bilinear interpolant from 4 corners, $2 \ominus$, $2 \oplus$

$$f_-(sh,th) = q(sh,th) + O(h^2)$$
$$\quad - (1-s)t[f](0,h) - st[f](h,h)$$

$$[f](0,h) = [f](sh,th) - sh[f_x] + (1-t)h[f_y] + O(h^2)$$
$$[f](h,h) = [f](sh,th) + (h-sh)[f_x] + (1-t)h[f_y] + O(h^2)$$

Combine. Terms with $[f_x]$ cancel.

$$\boxed{\begin{aligned} f_-(sh,th) = q(sh,th) &- t[f] \\ &- t(1-t)h[f_y] + O(h^2) \end{aligned}}$$

Compare with

D. V. Le, B.C. Khoo, J. Peraire, JCP 220 ('06), 109-38, Appendix A

M.-C. Lai, H.-C. Tseng, Computers + Fluids '07