Differential operator analogues of random matrix theory

October 3, 2006

Let $\lambda \in \mathbb{C}$ be a spectral parameter, and consider the eigenvalue problem

$$\epsilon \frac{d}{dx} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -i\lambda & \psi(x) \\ -\psi(x)^* & i\lambda \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

posed for $(u, v)^T \in L^2(\mathbb{R})$. Here ϵ is a positive parameter, and $\psi : \mathbb{R} \to \mathbb{C}$ is a function with rapid decay as $x \to \pm\infty$. By simple manipulations we may write this problem in the form $L(u, v)^T = \lambda(u, v)^T$, where L is a nonselfadjoint operator known as the Dirac operator or the Zakharov-Shabat operator. As was shown by Zakharov and Shabat (2), the spectral analysis of this operator is very important in the study of the Cauchy problem for the focusing nonlinear Schrödinger equation with initial data $\psi(x)$.

The continuous spectrum of the problem consists of the real line $\lambda \in \mathbb{R}$, and while the discrete spectrum is not generally constrained in any such way (due to nonselfadjointness) the eigenvalues must come in complex-conjugate pairs in the complex plane.

The paper of Zakharov and Shabat (2) contained the erroneous statement that if the potential $\psi(x)$ is a real-valued function then the discrete spectrum is confined to the imaginary axis of the complex λ-plane. It took 30 years to get the correct version of this statement: recently Klaus and Shaw (1) proved that if the potential $\psi(x)$ is real-valued *and has a single maximum*, then the discrete spectrum is purely imaginary. Moreover, there is an explicit constant $C > 0$ such that the number of eigenvalues on the imaginary axis in the upper half-plane is the integer part of $C\|\psi\|_1/\epsilon$, and these eigenvalues are confined to the imaginary interval $[0, i\psi_{\max}]$.

An interesting problem is then the following. Suppose $\psi(x)$ is a random potential of Klaus-Shaw type (part of the problem is to correctly formulate the meaning of this hypothesis). Then for $N = 1, 2, 3, \dots$, set $\epsilon = C\|\psi\|_1/N$, and we know from (1) that the eigenvalues in the upper half-plane have the form $\lambda = is_k$, $k = 1, \dots, N$, where r_k are random positive real numbers whose joint distribution is inherited from the distribution of the random "matrix elements" $\psi(x)$. This problem could be treated at first by simulation; of interest are the usual statistics of random matrix theory: the limiting empirical distribution of eigenvalues as $N \to \infty$, level spacing distribution, *etc.* Answers to questions of this type would be useful in the study of solutions of the focusing nonlinear Schrödinger equation with "noisy" initial data.

Also well-known (2) are the "inverse-scattering" formulae that allow the reconstruction of the potential $\psi(x)$ given (i) the discrete eigenvalues, (ii) certain "proportionality constants" associated with the discrete eigenvalues, and (iii) the "reflection coefficient" for the problem associated with the continuous spectrum. The reflection coefficient can be identically zero, and there is a sense in which the reflection coefficient is asymptotically negligible if ϵ is small (or equivalently, if N is large). All of this means that there is the possibility of writing down an explicit formula for the joint distribution of eigenvalues by computation of the Jacobian of the inverse-scattering formulae, which gives an avenue to treating this problem analytically as well as computationally.

References

[1] M. Klaus and J. K. Shaw, "Purely imaginary eigenvalues of Zakharov-Shabat systems," *Phys. Rev. E*, **65** (2002): 36607–36611.

[2] V. E. Zakharov and A. B. Shabat, "Exact theory of two-dimensional self-focusing and one-dimensional self-modulation

of waves in nonlinear media," *Sov. Phys. JETP*, **34** (1972): 62–69.