Circular ensembles

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Dyson introduced the circular ensemble of eigenvalues of a Haar distributed unitary matrix that has pdf $\prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^2$. A generalization of this is the pdf

$$\prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^2 \cdot \prod_{k=1}^{n} e^{-V(\theta_k)}.$$

These are closely related to orthogonal polynomials w.r.t. the weight $e^{-V(\theta)}$ on the unit circle. Basic facts can be found in Mehta's book (1).

Riemann-Hilbert techniques apply naturally to the study of scaling limits of the spacings in the case when V is analytic. How to go beyond analytic V?

Miller and McLaughlin (2) have a generalization of R-H techniques for non-analytic V and apply it to asymptotics of orthogonal polynomials w.r.t e^{-V}. These techniques have not been applied to the study of correlation functions yet. Specific projects:

1. Concrete question: How many derivatives of V are needed to deduce sine kernel correlations?

2. If V is k times continuously differentiable but the $(k+1)^{\text{st}}$ derivative has a jump discontinuity at a point, the scaled correlations at that point may have a different kernel which should form a universality class of its own? This would be a new phenomenon. See also the website http://www.math.lsa.umich.edu/ millerpd/ResearchAndPublications/CP.html

References

[1] Mehta, Madan Lal. (2004). Random matrices. Third edition. Pure and Applied Mathematics (Amsterdam), 142. Elsevier/Academic Press, Amsterdam, 2004.

[2] McLaughlin, K. T.-R. and Miller, P. D. (2006). The $\overline{\partial}$ steepest descent method and the asymptotic behavior of polynomials orthogonal on the unit circle with fixed and exponentially varying nonanalytic weights. *Int. Math. Res. Pap.* Art. ID 48673, 1–77. arxiv:math.CA/0406484