Some Experiments for β-ensembles

1. Random tridiagonal matrices
2. Metropolis-Hastings chain on eigenvalues
3. Auxilliary variables chain

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Dumitriu Edelman 2002

Random tridiagonal matrices

$$T = Q \Lambda Q^T, \quad q = \text{first row of } Q$$
$$\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$$

$$T = \begin{pmatrix} a_n & b_{n-1} & & \\ b_{n-1} & a_{n-1} & b_{n-2} & \\ & b_{n-2} & \ddots & \ddots \\ & & \ddots & a_1 \end{pmatrix}$$

$\exists$ bijection $(a,b) \longleftrightarrow (\lambda, q)$

restrict $b \geq 0$; restrict $\lambda_1 < \lambda_2 < \cdots < \lambda_n$

$$\Delta(A) = \prod_{i<j} |\lambda_j - \lambda_i| = \frac{\prod_{i=1}^{n-1} b_i^i}{\prod_{i=1}^{n} q_i}$$

$$\left[\prod_{i=1}^{n} q_i\right]^{-1} d\lambda\, dq = \left[\prod_{i=1}^{n-1} b_i^i\right]^{-1} db\, da$$

Ex: $Z_{\lambda q}^{-1} \prod_{i<j} (\lambda_j - \lambda_i)^\beta e^{-\frac{1}{2}\sum \lambda_i^2} f(\vec{q})\, dq\, d\lambda$

$$= Z_{a,b}^{-1} \frac{f(\vec{q})}{\left[\prod_{i=1}^{n} q_i\right]^{\beta-1}} \left[\prod_{i=1}^{n-1} b_i^{\beta i - 1} e^{-\frac{1}{2}[a_i^2 + 2b_i^2]}\right] da\, db$$

$b_i \overset{\text{den}}{\sim} c^{-1} b_i^{\beta i - 1} e^{-b_i^2}$

$a_i \overset{\text{den}}{\sim} c^{-1} e^{-\frac{1}{2} a_i^2}$

$O(n^2)$ time alg...

What happens at $a_i = 0 \ \forall i$?

Lem: λ an eigenvalue $\Rightarrow -\lambda$ an eigenvalue for $\vec{a} = \vec{0}$

Proof: ~~odd~~

~~$\begin{pmatrix} 0 & b_{n-1} \\ b_{n-1} & 0 \end{pmatrix}$ leaves odd subproblem~~

Say n = even

$$\det \begin{pmatrix} -\lambda & b_{n-1} & & \\ b_{n-1} & \lambda & \ddots & \\ & \ddots & & b_1 \\ & & b_1 & \lambda \end{pmatrix} = \lambda \det A_{n-1} - b_{n-1}^2 \det A_{n-2}$$

Note if n odd, $\det(A_n) = \lambda \cdot$ other stuff

$$\begin{pmatrix} \lambda & b_{n-1} & \\ b_{n-1} & \lambda & \\ & & \ddots \end{pmatrix}$$

Either 0 or $\square$

$\uparrow$ λ^2 $\quad$ $\uparrow$ A_{n-2} still odd

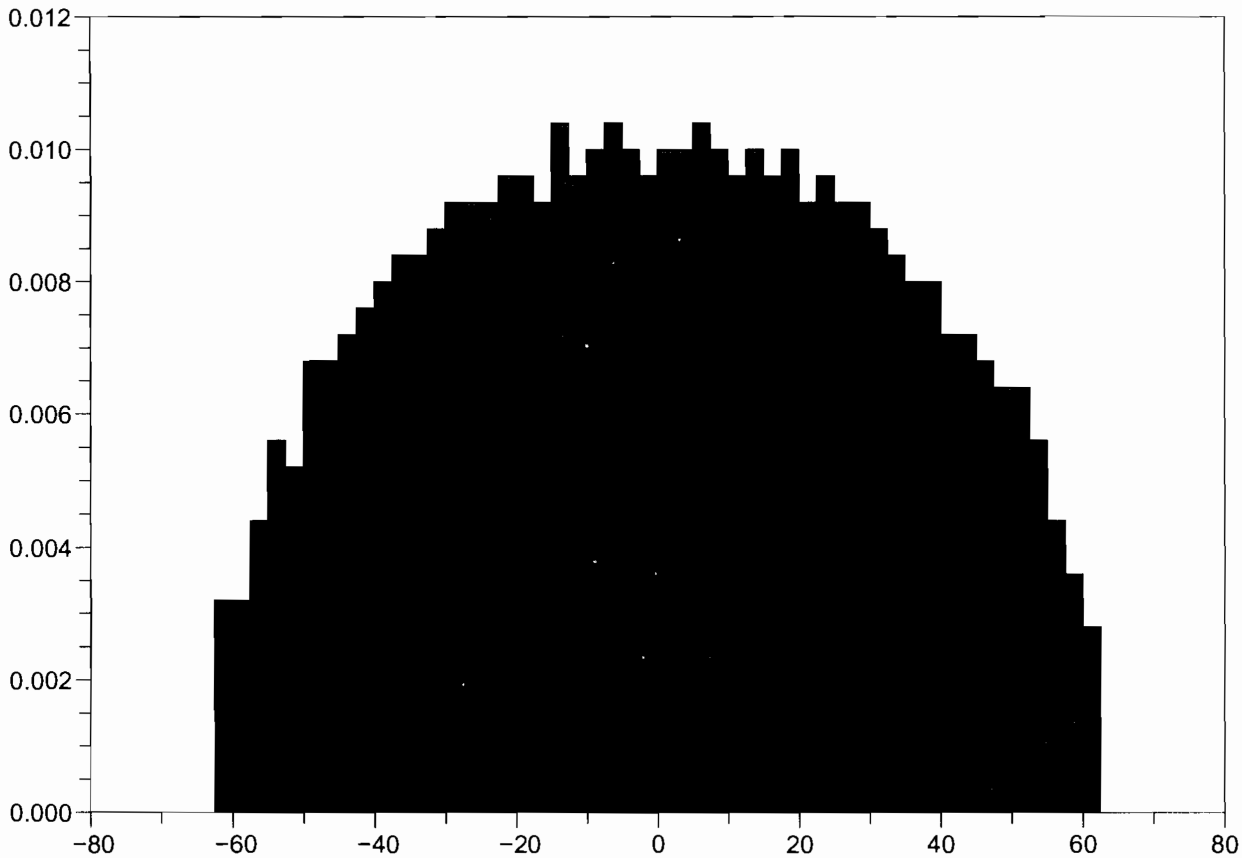
$\det(A_1) = \lambda$

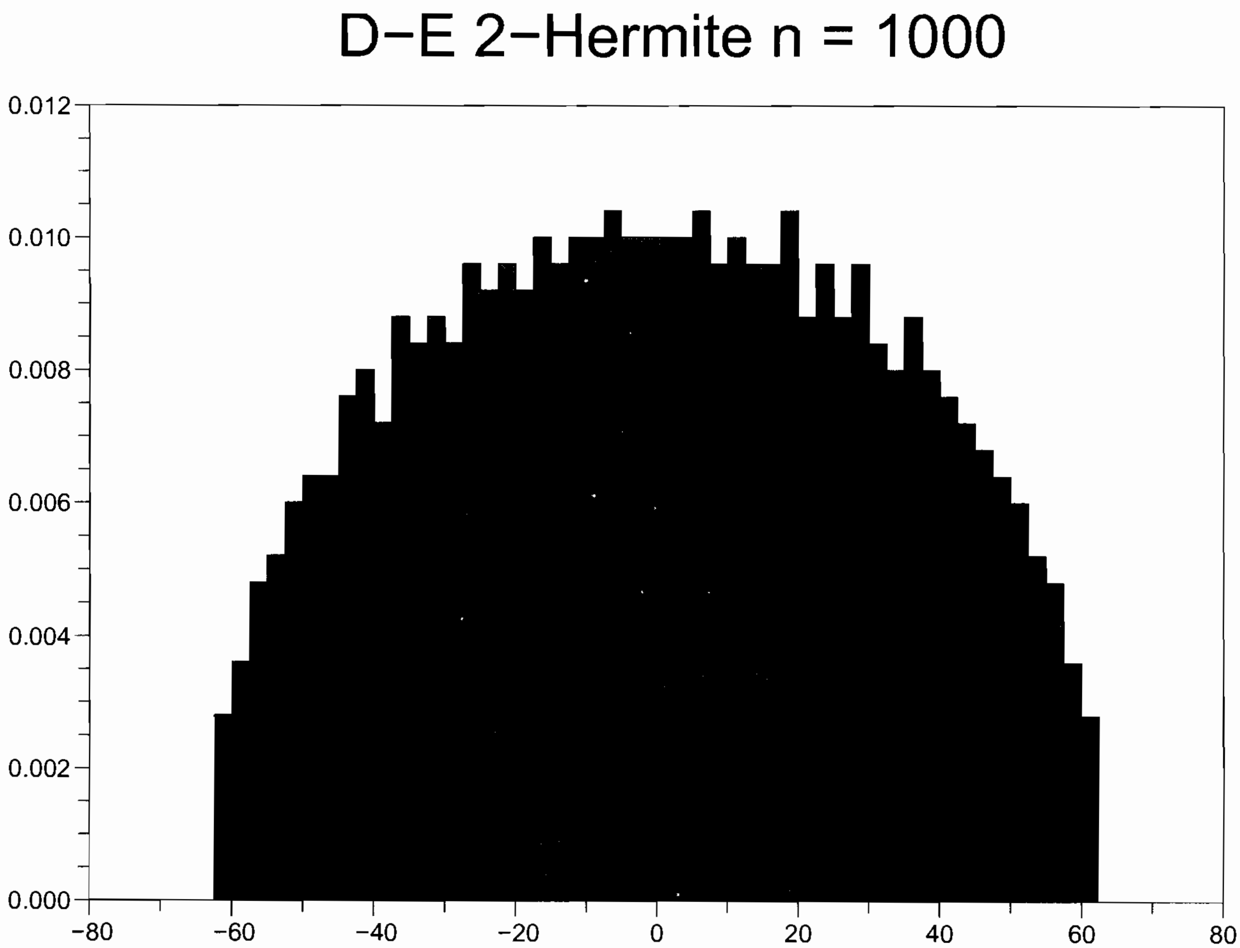
So ~~det~~ n even $\det(A_n) = \lambda^2$ stuff $+ b_{n-1}^2 \det A_{n-2}$

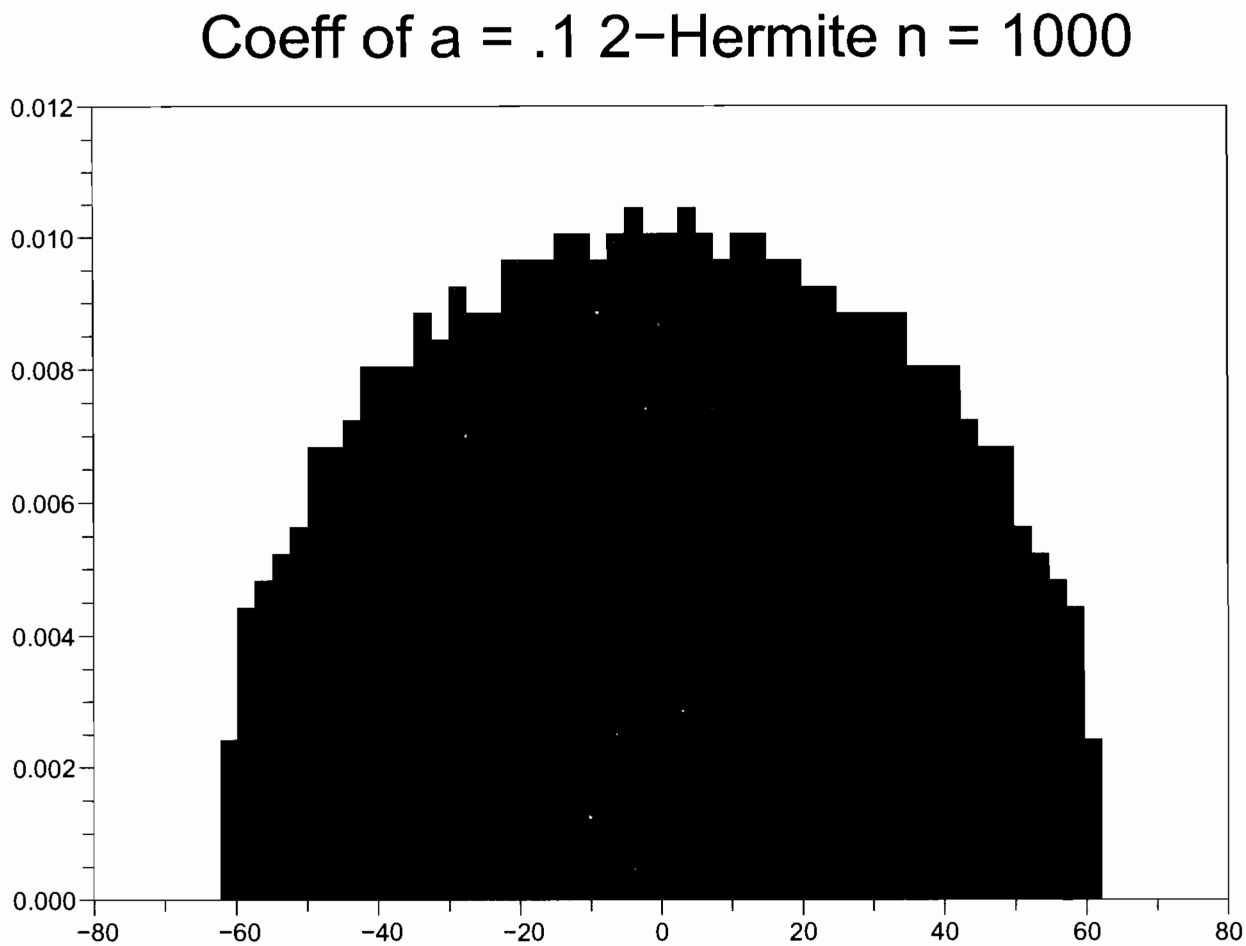
Hence $\lambda \Rightarrow -\lambda$ ev.

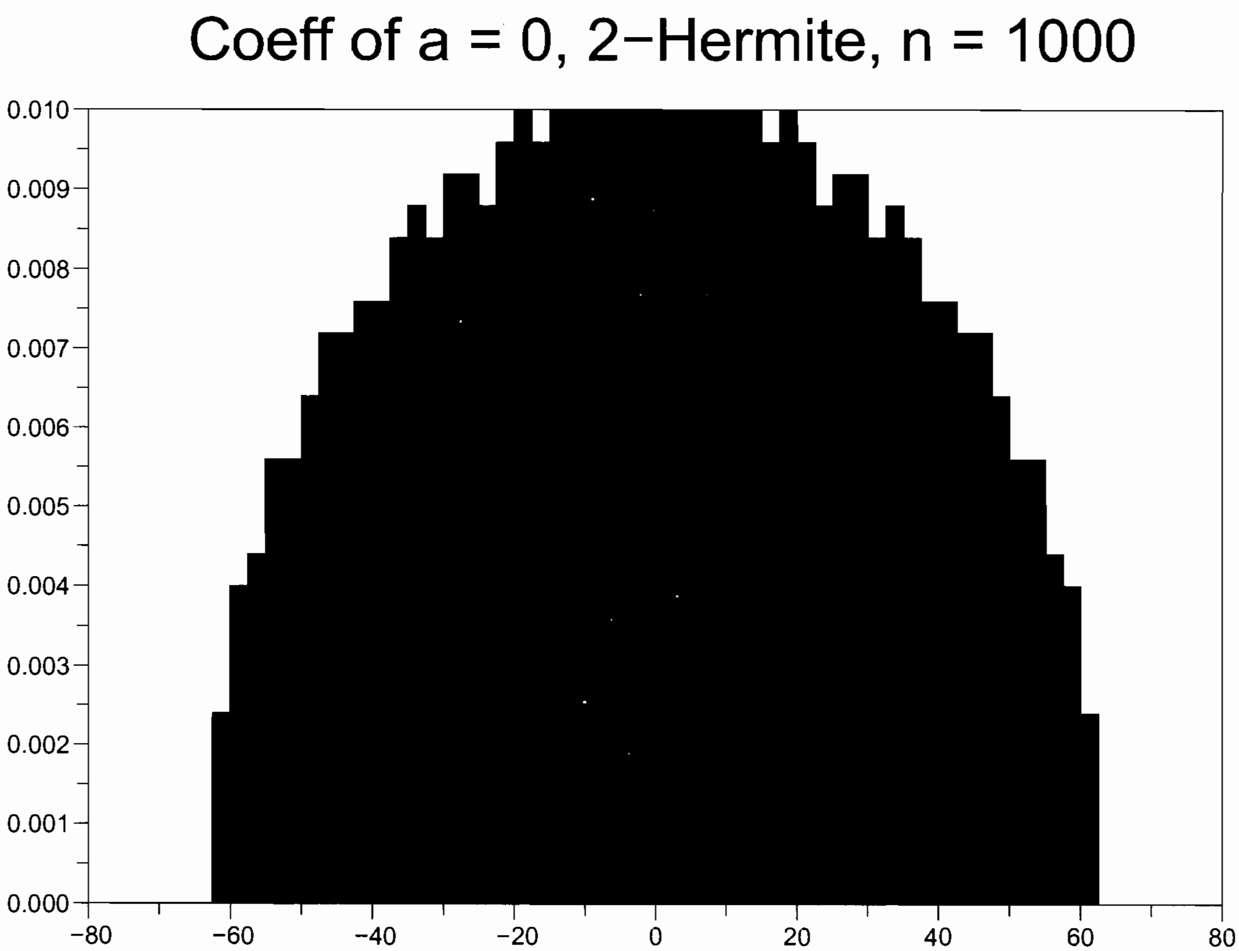
$\square$

2−Hermite n = 1000

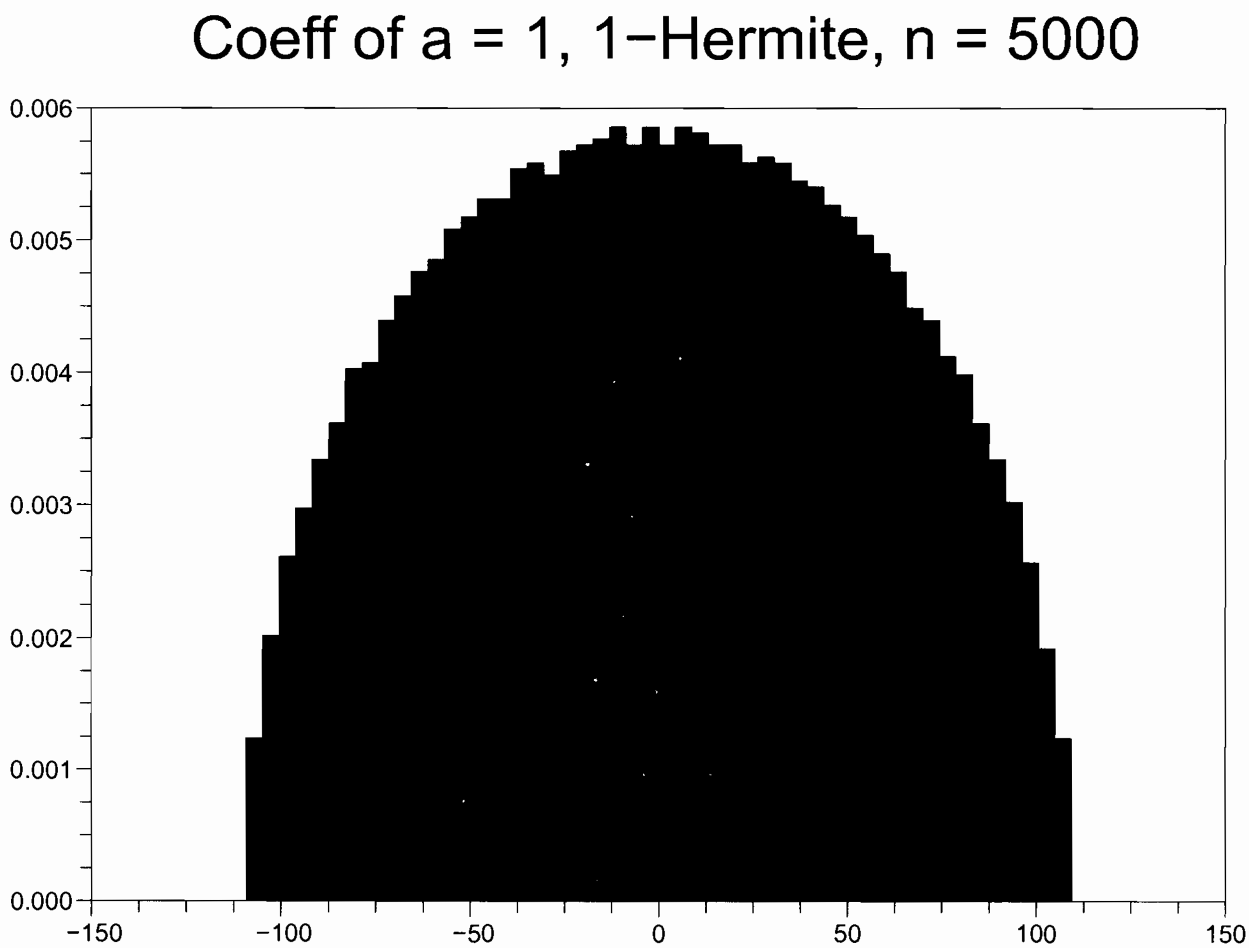


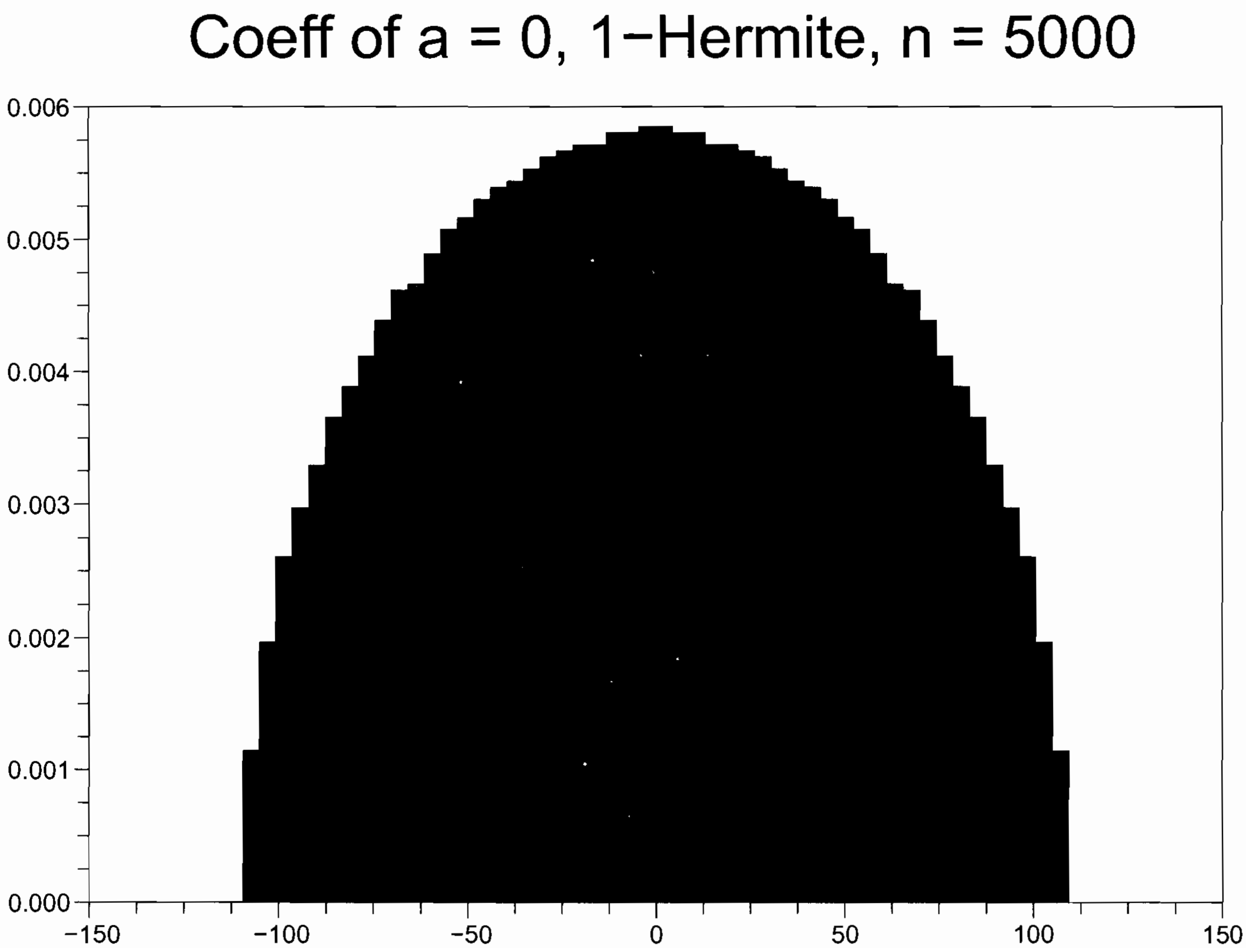

D−E 2−Hermite n = 1000
0.012
0.010
0.008
0.006
0.004
0.002
0.000
−80
−60
−40
−20
0
20
40
60
80

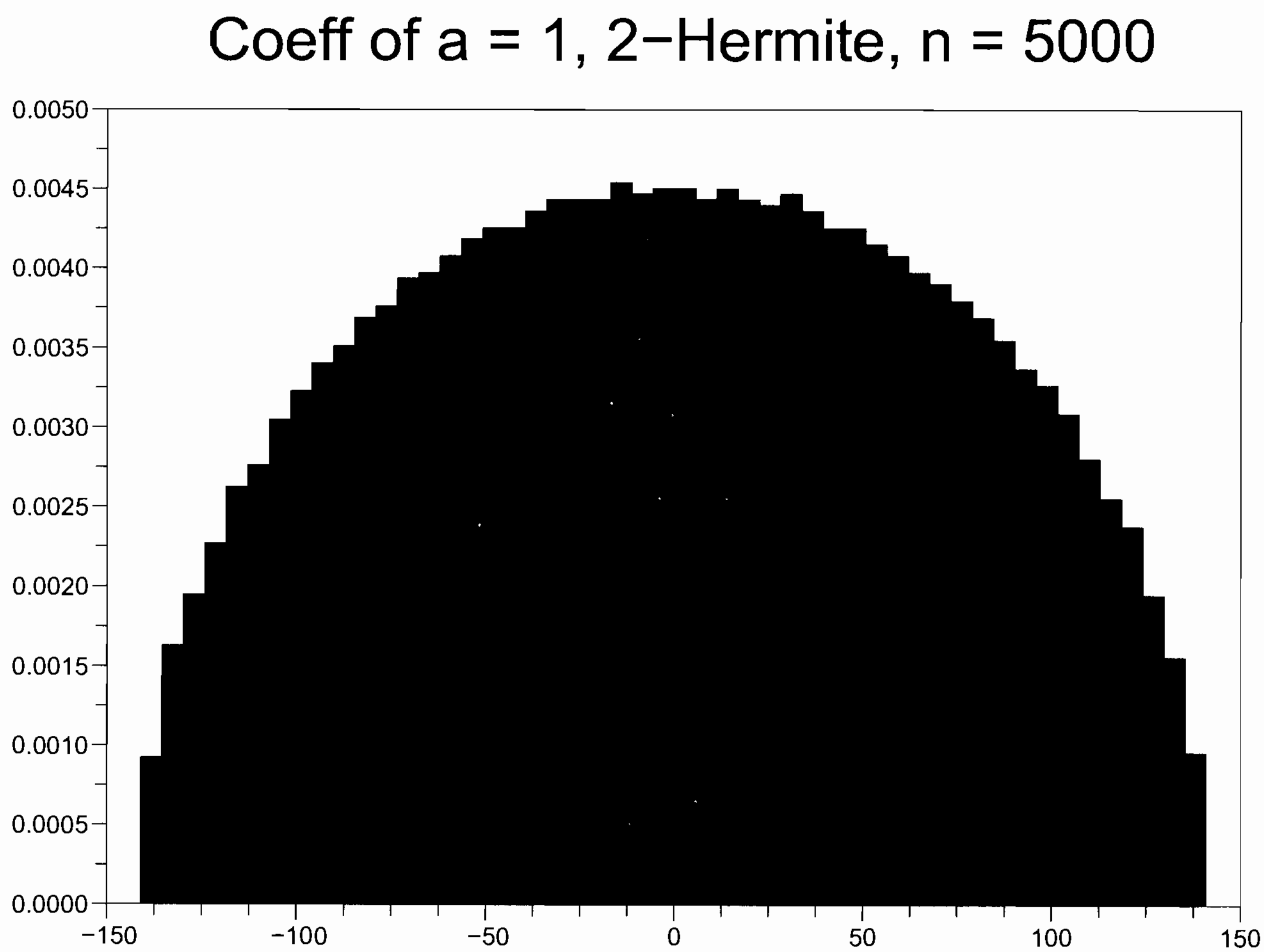

Coeff of a = .1 2−Hermite n = 1000
0.012
0.010
0.008
0.006
0.004
0.002
0.000
−80
−60
−40
−20
0
20
40
60
80



Coeff of a = 0, 2−Hermite, n = 1000
0.010
0.009
0.008
0.007
0.006
0.005
0.004
0.003
0.002
0.001
0.000
−80
−60
−40
−20
0
20
40
60
80


Coeff of a = 1, 1−Hermite, n = 5000
0.006
0.005
0.004
0.003
0.002
0.001
0.000
−150
−100
−50
0
50
100
150


Coeff of a = 0, 1−Hermite, n = 5000
0.006
0.005
0.004
0.003
0.002
0.001
0.000
−150
−100
−50
0
50
100
150


Coeff of a = 1, 2−Hermite, n = 5000
0.0050
0.0045
0.0040
0.0035
0.0030
0.0025
0.0020
0.0015
0.0010
0.0005
0.0000
−150
−100
−50
0
50
100
150

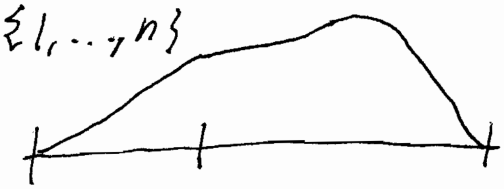
② Markov Chain

States $\lambda_1 < \lambda_2 < \cdots < \lambda_n$

Gibbs

1) Choose $i \sim Unif\{1, \ldots, n\}$

Ex: $i = 5$



2) Choose $\lambda_i \mid \lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n$.

MH: 1) (same)

2) ~~Pick~~ choose $\lambda_i' \leftarrow Unif[\lambda_i - 1, \lambda_i + 1]$

3) Choose $U \leftarrow Unif[0,1]$

4) If $U \le \pi(\lambda_i') / \pi(\lambda_i)$

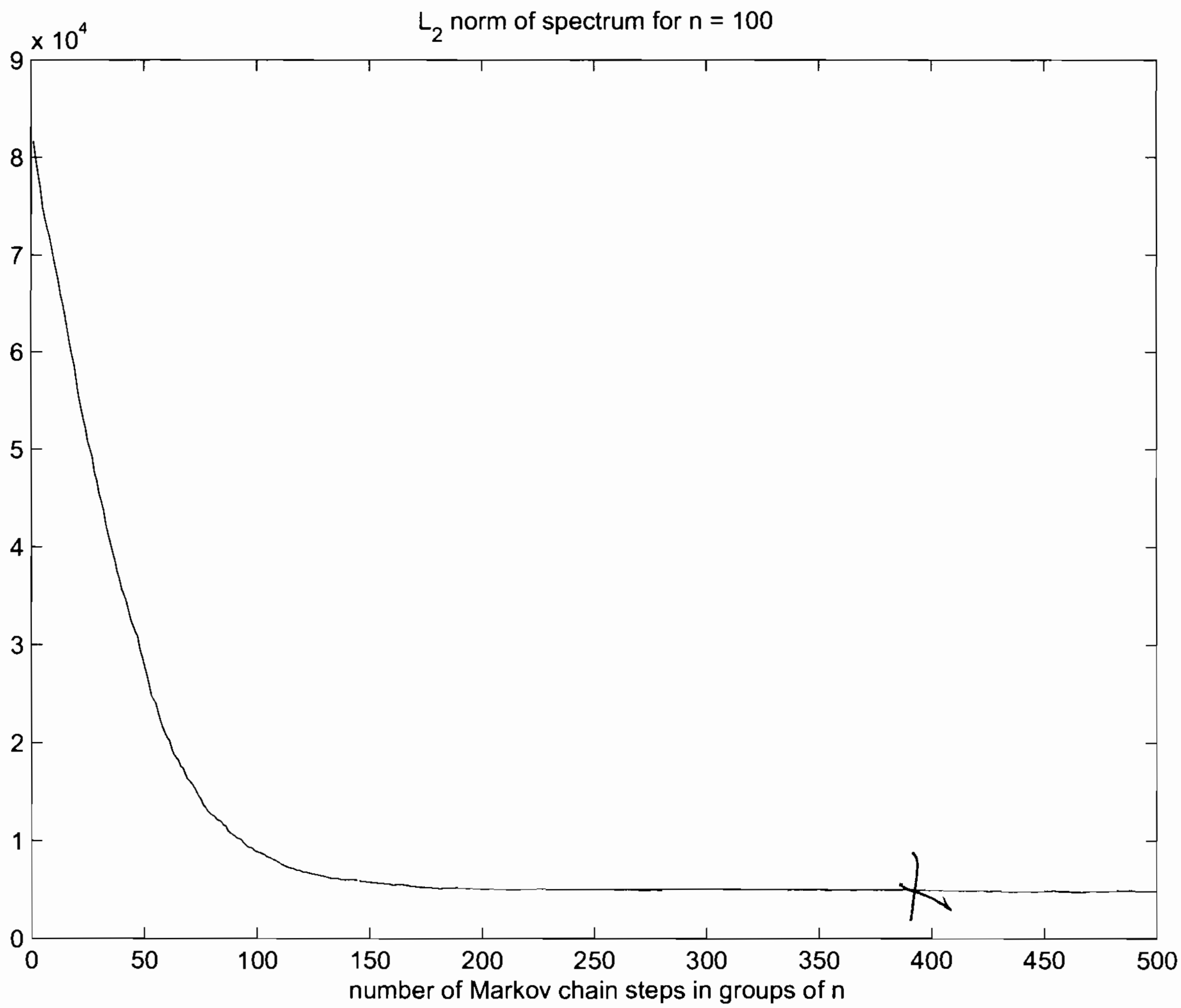
4a) accept move to λ_i'

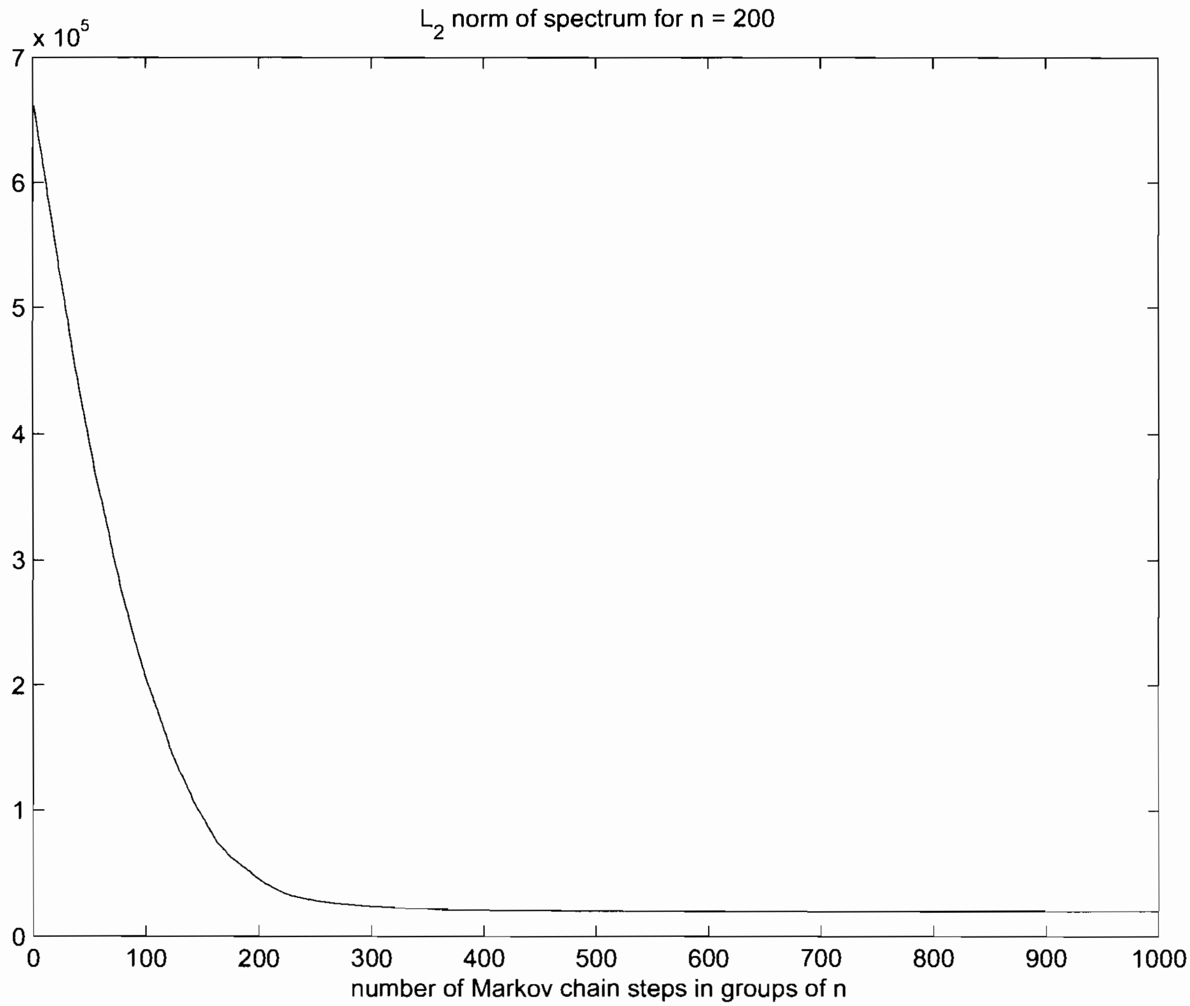
4b) Else

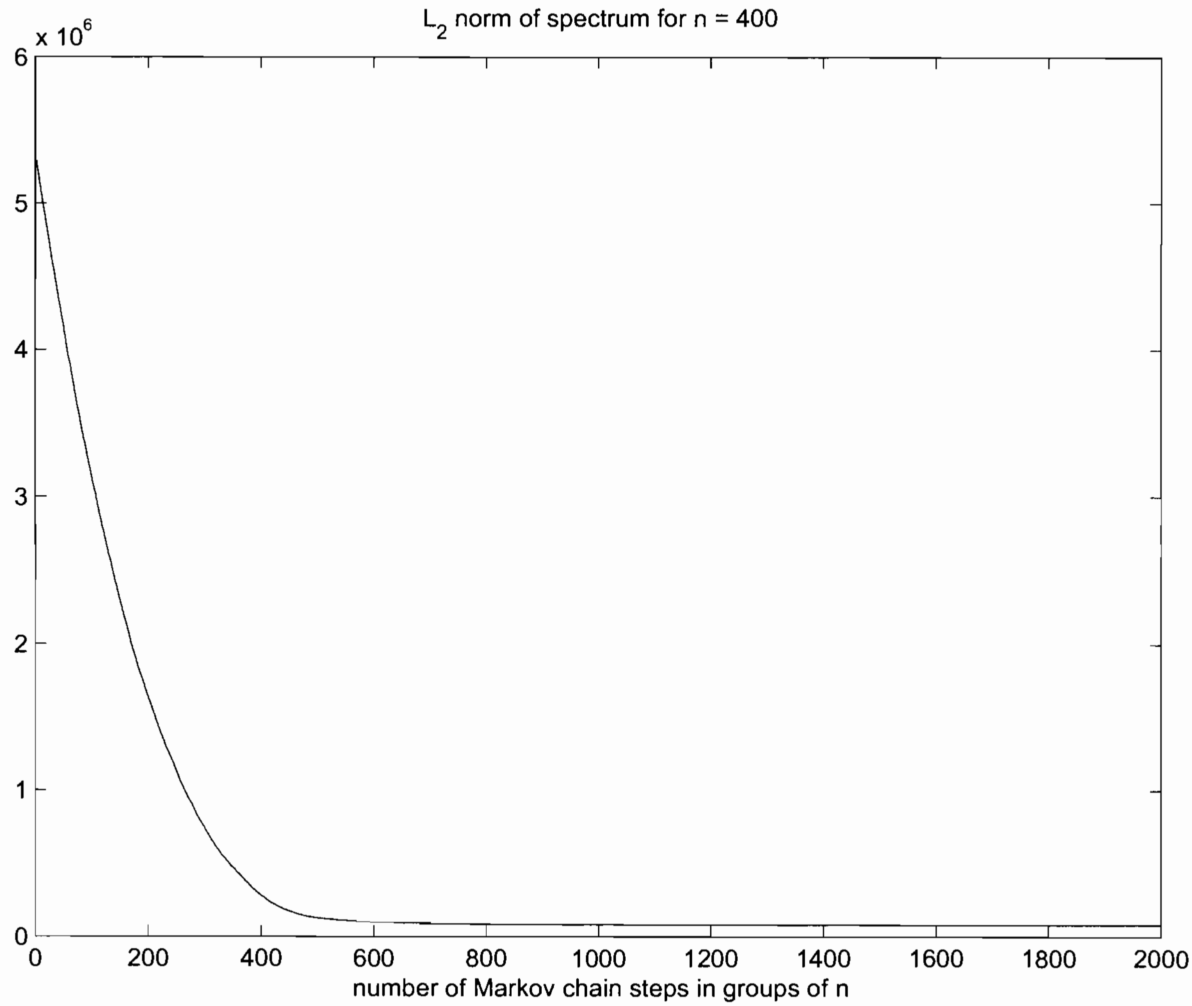
4c) · keep i^{th} e.v. at λ_i.

Takes at least n^2 steps to move.

Evidence $O(n^2)$ steps sufficient.


L$_2$ norm of spectrum for n = 100
x 10^4
9
8
7
6
5
4
3
2
1
0
0
50
100
150
200
250
300
350
400
450
500
number of Markov chain steps in groups of n


L_2 norm of spectrum for n = 200
x 10^5
7
6
5
4
3
2
1
0
0
100
200
300
400
500
600
700
800
900
1000
number of Markov chain steps in groups of n



L2 norm of spectrum for n = 400
x 10^6
0
1
2
3
4
5
6
0
200
400
600
800
1000
1200
1400
1600
1800
2000
number of Markov chain steps in groups of n

③ Create an auxilliary variable Y

$$Y|\lambda \sim \text{Unif}\left[0, e^{-V(\lambda)}\right].$$

Chain on (λ, Y)

1) Choose $\lambda | Y$ $\quad (\pi(\cdot \mid e^{-V(\lambda)} \geq Y))$
2) Choose $Y/\lambda \leftarrow \text{Unif}\left[0, e^{-V(\lambda)}\right]$.