Correlation Functions for Orthogonal Polynomial Random Matrix Ensembles

Ensembles of $N \times N$ Hermitian matrices

$$H = (H_{ij}) = (H_{ij}^R + i\, H_{ij}^I).$$

Measure

$$dH = \prod_i H_{ii} \prod_{i<j} dH_{ij}^R \prod_{i<j} dH_{ij}^I$$

Invariant under $H \to U^{-1}\, H\, U$ with U unitary. Consider probability measures

$$c\, e^{-\mathrm{tr}\, V(H)}\, dH;$$

called "unitary ensembles", or "orthogonal polynomial ensembles" for reasons to appear.

Divide out by unitary part. Find that if eigenvalues are $\lambda_1 < \cdots \lambda_N$ then density function is

$$P(x) = c_N \prod_{i<j} (x_i - x_j)^2 \prod_i w(x_i),$$

where $w(x) = e^{-V(x)}$.

The probability density that there are eigenvalues near $y_1, \cdots, y_n$ is the n-point correlation $R_n(y_1, \cdots, y_n)$ function given by

$$\frac{N!}{(N-n)!} \int \cdots \int P(y_1, \cdots, y_N)\, dy_{n+1} \cdots dy_N.$$

How to find a nice formula for these? Clue. If F is a symmetric function then

$$\mathbf{E}\,(F(\lambda_1, \cdots, \lambda_N))$$

$$= \int \cdots \int P(x_1, \cdots, x_N)\, F(x_1, \cdots, x_N)\, dx_1 \cdots dx_N.$$

If $F(x_1, \cdots, x_N) = \prod_{i \le n} \delta(x_i - y_i)$, symmetrized, then $R_n = \mathbf{E}\,(F(\lambda_1, \cdots, \lambda_N))$.

Consider F of the form

$$F(x) = \prod_{i=1}^{N} (1 + f(x_i)).$$

We'll find an integral operator K with kernel of the form

$$K(x, y) = \sum_{i=1}^{N} \varphi_i(x)\, \psi_i(y).$$

such that the expected value equals

$$\det\,(I + Kf),$$

where f here denotes multiplication by the function f. The determinant equals

$$\det\,(\delta_{ij} + (\varphi_i,\, \psi_j\, f))_{i,j=1}^{N}.$$

Start with

$$\mathbf{E}\left(\prod(1+f(\lambda_i))\right)$$

$$= c_N \int\cdots\int \prod_{i<j}(x_i - x_j)^2 \prod_i [w(x_i)\,(1+f(x_i))]\,dx.$$

The right side should equal 1 when $f = 0$.

General identity (Andréief 1883):

$$\int\cdots\int \det u_i(x_j)\,\det v_i(x_j)\,d\nu(x_1)\cdots d\nu(x_N)$$

$$= N!\,\det\left(\int u_i(x)\,v_j(x)\,d\nu(x)\right).$$

Taking $u_i(x) = v_i(x) = x^i$, $d\nu(x) = (1+f(x))\,w(x)\,dx$, we see that $\mathbf{E}\left(\prod(1+f(\lambda_i))\right)$ equals

$$c_N' \det\left(\int x^{i+j}\,(1+f(x))\,w(x)\,dx\right)_{i,j=0}^{N-1}.$$

Replacing x^i by $p_i(x)$, any polyomial of degree i, amounts to row and column operations. If we set $\varphi_i(x) = p_i(x)\sqrt{w(x)}$ then above becomes becomes

$$c_N'' \det\left(\int \varphi_i(x)\,\varphi_j(x)\,dx + \int \varphi_i(x)\,\varphi_j(x)\,f(x)\,dx\right).$$

Take the p_i to be the polynomials ON with respect to w, so the φ_i are ON with respect to Lebesgue measure. Taking $f=0$, get $c_N''=1$. So expected value equals

$$\det\Big(\delta_{ij}+(\varphi_i,\,\varphi_j\,f)\Big)=\det\left(I+Kf\right)$$

where K has kernel

$$K_N(x,y)=\sum_{i=0}^{N-1}\varphi_i(x)\,\varphi_i(y).$$

Hence "orthogonal polynomial ensembles".

If p_i arbitrary set

$$M=(m_{ij})=\left(\int\varphi_i(x)\,\varphi_j(x)\,dx\right),\quad M^{-1}=(\mu_{ij}),$$

$$\psi_i=\sum_j\mu_{ij}\,\varphi_j.$$

Factoring out M on the left, get

$$c_N'''\det\left(\delta_{ij}+\int\psi_i(x)\,\varphi_j(x)\,f(x)\,dx\right),$$

where $c_N'''=(\det M)\,c_N''$. Now taking $f=0$, get $c_N'''=1$ and

$$K_N(x,y)=\sum_i\varphi_i(x)\,\psi_i(y)=\sum_{i,j}\varphi_i(x)\,\mu_{ij}\,\varphi_j(y).$$

Special case $f = -\chi_J$: Probability that J contains no eigenvalues equals $\det(I - K\,\chi_J)$. If $J = (s,\, \infty)$ this is the distribution function for the largest eigenvalue.

Correlation functions. $R_n(y_1, \cdots, y_n)$ equals coefficient of $z_1 \cdots z_n$ in expansion of

$$\int \cdots \int P(x_1, \cdots, x_N) \prod_{i=1}^{N} \left[1 + \sum_{r=1}^{n} z_r\, \delta(x_i - y_r) \right] dx.$$

Integrals in matrix entries become sums, and above equals

$$\det\left(\delta_{rs} + K(y_r,\ y_s)\, z_s\right)_{r,s=1}^{n},$$

coefficient of $z_1 \cdots z_n$ equals

$$\det\left(K(y_r,\ y_s)\right)_{r,s=1}^{n}.$$

Orthogonal ensembles (real symmetric matrices). Formulas become $dH = \prod_{i \le j} dH_{ij}$,

$$\mathbf{E}\left(\prod (1 + f(\lambda_i)) \right)$$

$$= c_N \int \cdots \int \prod_{i<j} |x_i - x_j| \prod_i [w(x_i)\,(1 + f(x_i))]\, dx.$$

Assume N even and use (de Bruijn 1955)

$$\int_{x_1\le\cdots\le x_N}\cdots\int \det(u_i(x_j))\,dx_1\cdots dx_N$$

$$= \mathrm{Pf}\left(\int\int \mathrm{sgn}(y-x)\,u_i(x)\,u_j(y)\,dy\,dx\right)_{i,j=1}^{N}.$$

(Square of Pfaffian equals determinant.) Set $\varepsilon(x)=\frac{1}{2}\mathrm{sgn}(x)$, let p_i be arbitrary, $\varphi_i = p_i\sqrt{w}$, and find that the square of expected value equals determinant of the matrix with i,j entry

$$c_N \int\int \varepsilon(x-y)\,\varphi_i(x)\,\varphi_j(y)\,(1+f(x))\,(1+f(y))\,dy\,dx.$$

Let

$$M=\left(\int\int \varepsilon(x-y)\,\varphi_i(x)\,\varphi_j(y)\,dy\,dx\right),$$

$$M^{-1}=(\mu_{ij}), \qquad \psi_i=\sum_j \mu_{ij}\,\varphi_j,$$

and factor out M, so $\varphi_i(x)$ is replaced by $\psi_i(x)$. Define

$$(\varepsilon\varphi)(x)=\int \varepsilon(x-y)\,\varphi(y)\,dy.$$

What results is c'_N times the determinant of the matrix with i,j entry δ_{ij} plus

$$\int f\;[\psi_i\,\varepsilon\varphi_j - \varepsilon\psi_i\,\varphi_j - \varepsilon(f\psi_i)\,\varphi_j]\,dx.$$

Take $f=0$, get $c'_N = 1$. Integrand equals f times the matrix product

$$(-\varepsilon\psi_i - \varepsilon(f\psi_i) \qquad \psi_i) \quad \begin{pmatrix} \varphi_j \\ \\ \varepsilon\varphi_j \end{pmatrix}.$$

Determinant equals determinant of I plus operator with matrix kernel. (What is behind this is general identity $\det(I+AB) = \det(I+BA)$.) After some manipulation get that square of expected value equals $\det(I - K_N f)$ where (Dyson notation)

$$K_N(x,y) = \begin{pmatrix} S_N(x,y) & S_N D(x,y) \\ \\ IS_N(x,y) - \varepsilon(x-y) & S_N(y,x) \end{pmatrix},$$

where

$$S_N(x,y) = -\sum_{i,j} \varphi_i(x)\,\mu_{ij}\,\varepsilon\varphi_j(y),$$

$$IS_N(x,y) = -\sum_{i,j} \varepsilon\varphi_i(x)\,\mu_{ij}\,\varepsilon\varphi_k(y),$$

$$S_N D(x,y) = \sum_{i,j} \varphi_i(x)\,\mu_{ij}\;\varphi_j(y).$$

Therefore $R_n(y_1,\cdots,y_n)$ equals the coefficient of $z_1\cdots z_n$ in the expansion of

$$\sqrt{\det(\delta_{r,s} + K_N(y_r,y_s)\,z_s)}.$$

To evaluate, use keneral fact $\det(I+K) = \exp\{\text{tr}\log(I+K)\}$.

$$R_2(y_1, y_2) = \text{tr}K_N(y_1, y_1) \cdot \text{tr}\, K_N(y_2, y_2)$$
$$-\frac{1}{2}\,\text{tr}\,(K_N(y_1, y_2)\, K_N(y_2, y_1)).$$

Dyson showed that the correlation function can be interpreted as quaternion determinant.

Want M^{-1} to be as simple as possible. Choose p_i so that M is direct sum of $N/2$ copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Skew-orthogonal polynomials. Then $M^{-1} = -M$.

Scaling as $N \to \infty$. For the Gaussian unitary ensemble $w(x) = e^{-x^2}$. The eigenvalues fill out $(-\sqrt{2N}, \sqrt{2N})$, more or less. The p_i are normalized Hermite polynomials and their asymptotics shows that for fixed z

$$\frac{1}{\sqrt{2N}}\, K_N\left(z + \frac{x}{\sqrt{2N}},\, z + \frac{y}{\sqrt{2N}}\right)$$
$$\to \frac{1}{\pi}\,\frac{\sin(x-y)}{x-y} \quad \text{(sine kernel)}$$

(“bulk scaling”).

Largest eigenvalue $\approx \sqrt{2N}$.

$$\frac{1}{2^{1/2}N^{1/6}} K_N\left(\sqrt{2N}+\frac{x}{2^{1/2}N^{1/6}}, \sqrt{2N}+\frac{y}{2^{1/2}N^{1/6}}\right)$$

$$\to \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y)-\mathrm{Ai}'(x)\,A(y)}{x-y} \quad \text{(Airy kernel)}$$

(“edge scaling”). Thus scaling limit of correlation functions in bulk is $\det(K_{\mathsf{sine}}(x_i, x_j))$, and at edge is $\det(K_{\mathsf{Airy}}(x_i, x_j))$.

Universality theorems say that the same limiting kernels K_{sine} and K_{Airy} arise from bulk and edge scaling for “general” random matrix ensembles. Universality of bulk scaling was proved for large classes of orthogonal polynomial ensembles by Pastur-Scherbina (1996) and for both bulk and edge scaling by Deift-McLaughlin-Kriecherbauer-Venakides-Zhou (1997). For the symmetric matrix analogues with a special class of weights edge and bulk universality was proved by Deift-Gioev (2005). Edge scaling universality for Wigner ensembles (matrices with independent entries) was proved by Soshnikov (1999). Open problem: Universality in the bulk for other than orthogonal polynomial ensembles.