# The largest eigenvalue of Deformed random matrix ensembles

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Samsi Opening Workshop on High Dimensional Inference and Random Matrices

September, 17-20, 2006

## Plan

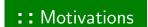
- I. Introduction

  Motivation for the study of the largest eigenvalue of (deformed) random matrices.
- II. Deformed Ensembles.

• III. Universality results for deformed Wigner matrices.

• IV. A few words about sample covariance matrices.

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# **Example: applications in finance and statistics**

A portfolio  $\mathcal{P}$  of N assets with weight  $w_i$ ,  $i = 1, \ldots, N$ .

 $\Sigma$  the covariance matrix of the returns.

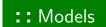
Daily variance of return to be minimized : 
$$R^2 = \sum_{i,j=1}^{N} w_i w_j \Sigma_{ij}$$
.

Optimal  $\mathcal{P}$ : maximal weight on eigenvectors associated to the smallest eigenvalues of  $\Sigma$ .

**Problems**: Given Y, the  $N \times p$  matrix of returns observed during a length p period and the sample covariance matrix  $YY^*$ ,

- 1. estimate for  $\Sigma$  unknown in general
- 2. find a way to erase the effect of large eigenvalues.
- 3. Principal components analysis in mathematical statistics.

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# Deformed ensembles- Spiked models (Johnstone 2001)

Let  $k \in \mathbb{N}$  and  $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_k > 0$  be given (ind. of N). Consider a sequence  $(W_N)$ ,  $N \geq k$ , of  $N \times N$  deterministic matrices s.t.

$$W_N = U_N \operatorname{diag}(\pi_1, \dots, \pi_k, 0, \dots, O)U_N^*, \text{ with } U_N \in \mathbb{U}(N).$$

Let  $\mu$  (resp.  $\mu'$ ) be a centered probability distribution on  $\mathbb C$  (resp. on  $\mathbb R$ ) s.t.  $\int\limits_{\bullet} |x|^2 d\mu(x) = \sigma^2 \text{ and } \int\limits_{} |x|^2 d\mu'(x) < \infty.$  • Sample covariance matrices. Let X be a  $N \times p$  (p = p(N)) random matrix with

i.i.d. entries of distribution  $\mu$ .

$$M_N = rac{1}{p} \Sigma^{1/2} X X^* \Sigma^{1/2}, ext{ where } \Sigma = Id + W_N.$$

**Deformed Hermitian random matrices.** Let  $H_{ij}$ ,  $1 \le i < j \le N$  be independent random variables of distribution  $\mu$  ( $\mu'$  on the diagonal).

$$H_N = \frac{1}{\sqrt{N}}(H_{ij}) + W_N$$
, of size  $N \times N$ .

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## Questions

What is the influence of the deformation on the limiting spectral properties? influence of k and the values  $\pi_i, i = 1, \ldots, k$ .

## Limiting spectral law unchanged.

Let  $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$  be the ordered eigenvalues of  $H_N$  (resp.  $M_N$ ) and  $\mu_N = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i}$ . Then if  $\int |x|^4 d\mu^{(')} < \infty$ , (Wigner case)

$$\lim_{N\to\infty}\mu_N=\mu_\sigma, \text{ with } \frac{d\mu_\sigma}{dx}=\frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2-x^2}\,\mathbf{1}_{[-2\sigma,2\sigma]}(x).$$

Similar results for sample covariance matrices (Marchenko-Pastur distribution if  $\lim_{N\to\infty} p/N \exists$ ).

Question: what about the largest eigenvalues?

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# Phase transition for the largest eigenvalue: $\mu = \mathcal{N}(0, \sigma^2)$ .

**Theorem:** Assume that  $\pi_1$  is simple (for ease). Let  $\lambda_1^G$  be the largest eigenvalue of  $H_N^G$ .

- (1) If  $\pi_1 < \sigma$ , then,  $\lim_{N \to \infty} P\left(N^{2/3} \left(\frac{\lambda_1^G}{\sigma} 2\right) \le x\right) = F_2(x)$ , where  $F_2$  is the well-known Tracy Widom distribution.
- (2) If  $\pi_1 = \sigma$ , then,  $\lim_{N \to \infty} P\left(N^{2/3}\left(\frac{\lambda_1^G}{\sigma} 2\right) \le x\right) = F_{1+2}^{TW}(x)$ . (generalized Tracy-Widom distribution which depends on the multiplicity of  $\pi_1$ ).

• (3) If 
$$\pi_1 > \sigma$$
 then,  $\lim_{N \to \infty} P\left(\sigma^2(\pi_1)N^{1/2}\left(\frac{\lambda_1^G}{\sigma} - C(\pi_1)\right) \ge x\right) = P\left(\mathcal{N}(0, \sigma^2(\pi_1)) \ge x\right)$ ,

where 
$$C(\pi_1) = \frac{\pi_1}{\sigma} + \frac{\sigma}{\pi_1}$$
 and  $\sigma^2(\pi_1) = \frac{\pi_1^2}{\pi_1^2 - \sigma^2}$ .

1st such results for complex Wishart Ensembles: Baik, Ben Arous, P. (2004).

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# Universality results à la Soshnikov

Joint work with D. Féral, University of Toulouse (2006) If  $\mu$  is a symmetric distribution s.t.

- (H1)  $\int |x|^2 d\mu(x) = \sigma^2$  and  $\exists C > 0$   $\int |x|^{2k} d\mu(x) \le (Ck)^k, \forall k$ .
- ullet  $W_N=rac{\pi_1}{N}J$  with  $J_{ij}=1, orall i,j$  : a specific rank one perturbation.

Let  $\lambda_1$  be the largest eigenvalue of  $H_N = \frac{1}{\sqrt{N}}(H_{ij}) + W_N$ . Then, (1) to (3) hold.

- $\star$  Method: computations of moments  $E\left(Tr(H_N^{s_N})\right)$ ,  $s_N$  large: same as for  $\mu=\mathcal{N}(0,\sigma^2)$ .
- \* Strongly suspect that universality does not hold if  $W_N = \text{diag}(\pi_1, 0, \cdots, 0)$ .

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## **Extensions**

## Complex ensembles

Aim: extend the results of Baik Ben Arous P. to ensembles of the same class.

1) Universality results for a "white" random sample covariance matrix

$$M_N = \frac{1}{p} X X^*$$

where  $X_{ij}$  are i.i.d. random variables and  $\lim_{N\to\infty} \frac{p}{N} > 1$ . Extension of a preliminary result of A. Soshnikov (2001).

2) Study of other Deformed Ensembles

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## **Extensions**

#### Real ensembles

1) Universality results for real random matrices (Wigner case): lack of results for symmetric random matrices with i.i.d. entries (deformed GOE).

Yet if  $W_N = \frac{\pi_1}{N}J$  and  $F_1^{TW}(x)$  is the limiting distribution of a (non Deformed) GOE

$$\lim_{N \to \infty} P\left(N^{2/3} \left(\frac{\lambda_1}{\sigma} - 2\right) \le x\right) = F_1^{TW}(x), \text{ if } \pi_1 < \sigma.$$

- 2) Real Deformed Wishart matrices
- D. Paul (2004) for a deformation of rank one. Gaussian fluctuations of the largest eigenvalue if  $\pi_1 > \pi_1^c$ .

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