The largest eigenvalue of Deformed random matrix ensembles

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Plan

- I. Introduction
  Motivation for the study of the largest eigenvalue of (deformed) random matrices.

- II. Deformed Ensembles.

- III. Universality results for deformed Wigner matrices.

- IV. A few words about sample covariance matrices.

Example: applications in finance and statistics

A portfolio $\mathcal{P}$ of N assets with weight w_i, $i = 1, \ldots, N$.
Σ the covariance matrix of the returns.

$$\text{Daily variance of return to be minimized : } R^2 = \sum_{i,j=1}^{N} w_i w_j \Sigma_{ij}.$$

Optimal $\mathcal{P}$: maximal weight on eigenvectors associated to the smallest eigenvalues of Σ.

Problems : Given Y, the $N \times p$ matrix of returns observed during a length p period and the sample covariance matrix YY^*,

1. estimate for Σ unknown in general
2. find a way to erase the effect of large eigenvalues.
3. Principal components analysis in mathematical statistics.

Deformed ensembles- Spiked models (Johnstone 2001)

Let $k \in \mathbb{N}$ and $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_k > 0$ be given (ind. of N). Consider a sequence (W_N), $N \geq k$, of $N \times N$ deterministic matrices s.t.

$$W_N = U_N \text{diag}(\pi_1, \ldots, \pi_k, 0, \ldots, O) U_N^*, \text{ with } U_N \in \mathbb{U}(N).$$

Let μ (resp. μ') be a centered probability distribution on $\mathbb{C}$ (resp. on $\mathbb{R}$) s.t. $\int |x|^2 d\mu(x) = \sigma^2$ and $\int |x|^2 d\mu'(x) < \infty$.

- **Sample covariance matrices.** Let X be a $N \times p$ $(p = p(N))$ random matrix with i.i.d. entries of distribution μ.

$$M_N = \frac{1}{p} \Sigma^{1/2} X X^* \Sigma^{1/2}, \text{ where } \Sigma = Id + W_N.$$

- **Deformed Hermitian random matrices.** Let $H_{ij}, 1 \leq i < j \leq N$ be independent random variables of distribution μ (μ' on the diagonal).

$$H_N = \frac{1}{\sqrt{N}} (H_{ij}) + W_N, \text{ of size } N \times N.$$

Questions

What is the influence of the deformation on the limiting spectral properties?
influence of k and the values $\pi_i, i = 1, \ldots, k$.

Limiting spectral law unchanged.
Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ be the ordered eigenvalues of H_N (resp. M_N) and $\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}$. Then if $\int |x|^4 d\mu^{(')} < \infty$, (Wigner case)

$$\lim_{N \to \infty} \mu_N = \mu_\sigma, \text{ with } \frac{d\mu_\sigma}{dx} = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}\, 1_{[-2\sigma, 2\sigma]}(x).$$

Similar results for sample covariance matrices (Marchenko-Pastur distribution if $\lim_{N\to\infty} p/N \,\exists$).

Question: what about the largest eigenvalues?

Phase transition for the largest eigenvalue: $\mu = \mathcal{N}(0, \sigma^2)$.

Theorem: Assume that π_1 is simple (for ease). Let λ_1^G be the largest eigenvalue of H_N^G.

- (1) If $\pi_1 < \sigma$, then, $\lim_{N\to\infty} P\left(N^{2/3}\left(\frac{\lambda_1^G}{\sigma} - 2\right) \leq x\right) = F_2(x)$, where F_2 is the well-known Tracy Widom distribution.

- (2) If $\pi_1 = \sigma$, then, $\lim_{N\to\infty} P\left(N^{2/3}\left(\frac{\lambda_1^G}{\sigma} - 2\right) \leq x\right) = F_{1+2}^{TW}(x)$. (generalized Tracy-Widom distribution which depends on the multiplicity of π_1).

- (3) If $\pi_1 > \sigma$ then, $\lim_{N\to\infty} P\left(\sigma^2(\pi_1)N^{1/2}\left(\frac{\lambda_1^G}{\sigma} - C(\pi_1)\right) \geq x\right) = P\left(\mathcal{N}(0, \sigma^2(\pi_1)) \geq x\right)$,

$$\text{where } C(\pi_1) = \frac{\pi_1}{\sigma} + \frac{\sigma}{\pi_1} \quad \text{and} \quad \sigma^2(\pi_1) = \frac{\pi_1^2}{\pi_1^2 - \sigma^2}.$$

1st such results for complex Wishart Ensembles: Baik, Ben Arous, P. (2004).

Universality results à la Soshnikov

Joint work with D. Féral, University of Toulouse (2006)
If μ is a symmetric distribution s.t.

- (H1) $\int |x|^2 d\mu(x) = \sigma^2$ and $\exists\, C > 0$ $\int |x|^{2k} d\mu(x) \leq (Ck)^k, \forall\, k$.

- $W_N = \dfrac{\pi_1}{N} J$ with $J_{ij} = 1, \forall i, j$: a specific rank one perturbation.

Let λ_1 be the largest eigenvalue of $H_N = \frac{1}{\sqrt{N}}(H_{ij}) + W_N$. Then, (1) to (3) hold.

⋆ Method: computations of moments $E\left(Tr(H_N^{s_N})\right)$, s_N large: same as for $\mu = \mathcal{N}(0, \sigma^2)$.
⋆ Strongly suspect that universality does not hold if $W_N = \text{diag}(\pi_1, 0, \cdots, 0)$.

Extensions

Complex ensembles

Aim: extend the results of Baik Ben Arous P. to ensembles of the same class.

1) Universality results for a "white" random sample covariance matrix

$$M_N = \frac{1}{p} X X^*$$

where X_{ij} are i.i.d. random variables and $\lim_{N\to\infty} \frac{p}{N} > 1$.

Extension of a preliminary result of A. Soshnikov (2001).

2) Study of other Deformed Ensembles

Extensions

Real ensembles

1) Universality results for real random matrices (Wigner case): lack of results for symmetric random matrices with i.i.d. entries (deformed GOE).
Yet if $W_N = \frac{\pi_1}{N} J$ and $F_1^{TW}(x)$ is the limiting distribution of a (non Deformed) GOE

$$\lim_{N \to \infty} P \left(N^{2/3} \left(\frac{\lambda_1}{\sigma} - 2 \right) \leq x \right) = F_1^{TW}(x), \text{ if } \pi_1 < \sigma.$$

2) Real Deformed Wishart matrices
D. Paul (2004) for a deformation of rank one. Gaussian fluctuations of the largest eigenvalue if $\pi_1 > \pi_1^c$.