Edgeworth Type Expansion of the distribution of the Largest Eigenvalue in Classical Random Matrix Ensembles

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Introduction

The limiting distribution function for the largest eigenvalues (the Tracy-Widom distribution) in Classical Random Matrix Ensembles

$$F_\beta(t) = \lim_{N\to\infty} F_{N,\beta}(t) = \lim_{N\to\infty} P_\beta(\lambda_{\max}^{\beta} \le t) \tag{1}$$

have found many applications outside their initial discovery in random matrix theory.
In these applications it is important to have correction terms to the limiting distribution.
(For example, in statistics the sample size is always finite; and to assess quantitatively the range of validity of limit laws, one needs finite N correction terms.)
Thus the need of an expansion of $F_{N,\beta}$ in terms of N.

Introduction contd

From the representation

$$F_{N\beta}(t) = \begin{cases} \det(I-\sigma_{N,2}) = E_2(0,J), & \text{for } \beta = 2; \\ \sqrt{\det(I-\sigma_{N,\beta})} = E_\beta(0,J), & \text{for } \beta = 1,4 \ . \end{cases} \quad (2)$$

as a Fredholm determinant of the integral operator with kernel $\sigma_{N,\beta}(x,y)$ on the set $J = [t, \infty)$.
Where for the unitary ensemble $\beta = 2$,

$$\sigma_{N2}(x,y) = \begin{cases} K_N(x,y) = \sum_{k=0}^{N-1} \varphi_k(x)\varphi_k(y), & \text{Guassian} \\ \\ K_N^\alpha(x,y) = \sum_{k=0}^{n-1} \frac{\phi_k^\alpha(x)\phi_k^\alpha(y)}{\Gamma(k+1)\Gamma(\alpha+k+1)}, & \text{Laguerre} \end{cases} ,$$

the orthogonal ensemble

$$\sigma_{N,1}(x,y) := \begin{pmatrix} \mathrm{K}_N + \psi \otimes \varepsilon\varphi & \mathrm{K}_N\mathrm{D} - \psi \otimes \varphi \\ \varepsilon\mathrm{K}_N - \varepsilon + \varepsilon\psi \otimes \varepsilon\varphi & \mathrm{K}_N + \varepsilon\varphi \otimes \psi \end{pmatrix} \quad (3)$$

and for the Symplectic ensemble

$$\sigma_{N,4}(x,y) := \frac{1}{2}\begin{pmatrix} \mathrm{K}_N + \psi \otimes \varepsilon\varphi & \mathrm{K}_N\mathrm{D} - \psi \otimes \varphi \\ \varepsilon\mathrm{K}_N + \varepsilon\psi \otimes \varepsilon\varphi & \mathrm{K}_N + \varepsilon\varphi \otimes \psi \end{pmatrix} \quad (4)$$

Introduction contd

With

$$DS(x) := S^{'}(x) \text{ is the differentiation operator}, \quad \varepsilon(x) = \frac{1}{2}\mathrm{sgn}(x)$$

$$\varphi_n(x) = \frac{1}{(2^n n! \sqrt{\pi})^{1/2}} H_n(x)\, e^{-x^2/2}, \quad \phi_n^{\alpha}(x) = x^{\alpha/2}\, e^{-x/2}\, L_n^{\alpha}(x),$$

are obtained from the orthogonalizing the sequence $x^n e^{-V(x)}$

$$\varphi(x) = (\frac{N}{2})^{\frac{1}{4}}\, \varphi_n(x), \text{ and } \quad \psi(x) = (\frac{N}{2})^{\frac{1}{4}}\, \varphi_{n-1}(x)$$

The representations

$$\begin{aligned} E_1(0,J)^2 &= E_2(0,J)\det(I-(K_N+RK_N)(1-\chi)\varepsilon\chi D) \\ &= E_2(0,J)\left\{(1-\tilde{v}_\varepsilon)(1-\frac{1}{2}\mathcal{R}_1) - \frac{1}{2}(q_\varepsilon - c_\varphi)\mathcal{P}_1\right\} \end{aligned}$$

$$\begin{aligned} E_4(0,J/\sqrt{2})^2 &= E_2(0,J)\det(I-\frac{1}{2}(K_N+R)K_N\varepsilon[\chi D]) \\ &= E_2(0,J)\left\{(1-\tilde{v}_\varepsilon)(1+\frac{1}{2}\mathcal{R}_4) + \frac{1}{2}q_\varepsilon\mathcal{P}_4\right\} \end{aligned}$$

Unitary Case

The goal for $\beta = 2$ is to use the results of Tracy and Widom from " Airy kernel and Painlevé II" as a starting point of our analysis.
We need an expansion of the kernel $\sigma_{N,2}$ with respect to N in terms of known quantities.
Making use of the following expansion,
if we set $\xi = (4n + 2\alpha + 2c)^{\frac{1}{2}} + \frac{X}{2^{\frac{2}{3}} n^{\frac{1}{6}}}, \quad \alpha > -1$, with X and c bounded. As $n \to \infty$,

$$e^{-\xi^2/2} L_n^\alpha(\xi^2) = (-1)^n 2^{-\alpha-\frac{1}{3}} n^{-\frac{1}{3}} \quad \Big\{ \mathrm{Ai}(X) + \frac{(c-1)}{2^{\frac{1}{3}}} \mathrm{Ai}'(X) n^{-\frac{1}{3}} +$$

$$\left[\frac{2 - 10c + 5c^2 - 5\alpha}{10 \cdot 2^{\frac{2}{3}}} X \mathrm{Ai}(X) + \frac{X^2}{20 \cdot 2^{\frac{2}{3}}} \mathrm{Ai}'(X) \right] n^{-\frac{2}{3}} +$$

$$\Big[\Big(\frac{5\alpha - 15c\alpha + 2c^3 - 15c^2 - 56c - 6}{60} + \frac{c-1}{40} X^3 \Big) \mathrm{Ai}(X) +$$

$$\frac{(c-1)(5(c-2)c - 3(2+5\alpha))}{60} X \mathrm{Ai}'(X) \Big] n^{-1} + O(n^{-\frac{4}{3}}) \mathrm{Ai}(X) \Big\}$$

Unitary Case contd

We find that for
$x = (2(n + c_G))^{\frac{1}{2}} + 2^{-\frac{1}{2}} n^{-\frac{1}{6}} X$ and $y = (2(n + c_G))^{\frac{1}{2}} + 2^{-\frac{1}{2}} n^{-\frac{1}{6}} Y$
with X, Y and c_G bounded,

$$K_n(x,y)\,dx = \Big\{ K_{\mathrm{Ai}}(X,Y) - c_G \mathrm{Ai}(X)\mathrm{Ai}(Y) n^{-\frac{1}{3}} +$$

$$\frac{1}{20}\Big[(X+Y)\mathrm{Ai}'(X)\mathrm{Ai}'(Y) - (X^2 + XY + Y^2)\mathrm{Ai}(X)\mathrm{Ai}(Y) +$$

$$\frac{-20c_G^2 + 3}{2}(\mathrm{Ai}'(X)\mathrm{Ai}(Y) + \mathrm{Ai}(X)\mathrm{Ai}'(Y))\Big] n^{-\frac{2}{3}} + O(n^{-1}) E(X,Y) \Big\}\, d\,X. \quad (5)$$

The error term, $E(X,Y)$, is the kernel of an integral operator on $L^2(J)$ which is trace class for any Borel subset J of the reals that is bounded away from minus infinity.

Unitary Case contd

For $x = 4(n + c_L) + 2\alpha + 2(2n)^{\frac{1}{3}} X$ and $y = 4(n + c_L) + 2\alpha + 2(2n)^{\frac{1}{3}} Y$ with X, Y and c_L bounded,

$$K_n^{\alpha}(x,y)\,dx = \Big\{ K_{\mathrm{Ai}}(X,Y) - 2^{\frac{2}{3}} c_L \mathrm{Ai}(X)\mathrm{Ai}(Y) n^{-\frac{1}{3}} +$$

$$\frac{2^{\frac{1}{3}}}{10}\Big[\ (X^2 + XY + Y^2)\mathrm{Ai}(X)\mathrm{Ai}(Y) - (X+Y)\mathrm{Ai}'(X)\mathrm{Ai}'(Y) -$$

$$(10c_L^2 - 1)(\mathrm{Ai}(X)\mathrm{Ai}'(Y) + \mathrm{Ai}'(X)\mathrm{Ai}(Y))\Big] n^{-\frac{2}{3}} + O(n^{-1}) F(X,Y) \Big\}\, dX. \tag{6}$$

The error term $F(X, Y)$ is the kernel of an integral operator on $L^2(J)$ which is trace class for any Borel subset J of the reals which is bounded away from minus infinity.

Justification

The relation between the Hermite and Laguerre polynomials

$$H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{-\frac{1}{2}}(x^2), \;\; H_{2n+1}(x) = (-1)^n 2^{2n+1} n! x L_n^{\frac{1}{2}}(x^2)$$

suggest the following change of variables
$x \;:=\; \xi$ for the determination of φ_n.
and $x \;:=\; \xi^2$ for the determination of the ϕ_n

Unitary Case contd

We get:

$$\text{for } t = \begin{cases} (2(n+c_G))^{\frac{1}{2}} + 2^{-\frac{1}{2}} n^{-\frac{1}{6}} s\ , & \text{for GUE}_n\ ; \\ 4(n+c_L) + 2\alpha + 2(2n)^{\frac{1}{3}} s & \text{for LUE}_n \end{cases} \qquad (7)$$

as $n \longrightarrow \infty$

$$F_{n,2}^{G,L}(t) = F_2(s)\{1 + a_{c_{G,L},2}^{G,L}\, u_0(s)\, n^{-\frac{1}{3}} + b_2^{G,L} E_{c_{G,L},2}^{G,L}(s)\, n^{-\frac{2}{3}}\} + O(n^{-1}) \qquad (8)$$

uniformly in s.
If in addition,

$$c_G^2 + c_L^2 = \frac{1}{4},\ \text{then } E_{c_G,2}^{G}(s) = E_{c_L,2}^{L}(s) = E_{c,2}(s),\ \text{and}$$

$$F_{n,2}^{G,L}(t) = F_2(s)\{1 + a_{c_{G,L},2}^{G,L}\, u_0(s)\, n^{-\frac{1}{3}} + b_2^{G,L} E_{c,2}(s)\, n^{-\frac{2}{3}}\} + O(n^{-1}). \qquad (9)$$

Orthogonal and Symplectic cases

For the orthogonal and Symplectic Matrix Ensembles, we need an expansion of

$$(1-\tilde{v}_\varepsilon)(1-\frac{1}{2}\mathcal{R}_1)-\frac{1}{2}(q_\varepsilon-c_\varphi)\mathcal{P}_1 \qquad \text{for the orthogonal case, and} \quad (10)$$

$$(1-\tilde{v}_\varepsilon)(1+\frac{1}{2}\mathcal{R}_4)+\frac{1}{2}q_\varepsilon\mathcal{P}_4 \qquad \text{for the symplectic case.} \quad (11)$$

Where

$$\mathcal{R}_1 := \int_{-\infty}^{t} R(x,t)\mathrm{d}x, \quad \mathcal{P}_1 := \int_{-\infty}^{t} P(x)\mathrm{d}x, \quad \mathcal{Q}_1 := \int_{-\infty}^{t} Q(x)\mathrm{d}x \quad (12)$$

and

$$\mathcal{R}_4 := \int_{-\infty}^{\infty} \varepsilon_t(x)R(x,t)\mathrm{d}x,\ \mathcal{P}_4 := \int_{-\infty}^{\infty} \varepsilon_t(x)P(x)\mathrm{d}x,\ \mathcal{Q}_4 := \int_{-\infty}^{\infty} \varepsilon_t(x)Q(x)\mathrm{d}x \quad (13)$$

All but $\varepsilon_t(x)$ are functions of n.

Thanks

THANKS