Observations on Objective Bayesian Inference

observed continuous random vector: $Y = (Y_1, \ldots, Y_n)$

distribution of Y depends on $\theta = (\theta^1, \ldots, \theta^{p+q})$

log likelihood function: $L(\theta)$

partition θ as $\theta = (\psi, \phi)$

ψ is a p-dimensional interest parameter

ϕ is a q-dimensional nuisance parameter

global mle: $\hat{\theta} = (\hat{\psi}, \hat{\phi})$

constrained mle: $\tilde{\theta}(\psi) = (\psi, \tilde{\phi}_\psi)$

log profile likelihood function: $M(\psi) = L\{\tilde{\theta}(\psi)\}$

likelihood ratio statistic: $W(\psi) = 2\{M(\hat{\psi}) - M(\psi)\}$

$$W(\psi) \sim \chi^2_p + O_p(n^{-1})$$

$$\mathrm{E}\{W(\psi)\} = p\left\{1 + \frac{b(\theta)}{n}\right\} + O_p(n^{-2})$$

Bartlett correction: $\dfrac{W(\psi)}{\mathrm{E}\{W(\psi)\}/p} \sim \chi^2_p + O_p(n^{-2})$

$$\frac{W(\psi)}{\sqrt{1 + \dfrac{b(\hat{\theta})}{n}}} \sim \chi^2_p + O_p(n^{-2})$$

$$\frac{W(\psi)}{\sqrt{1 + \dfrac{b(\psi, \tilde{\phi}_\psi)}{n}}} \sim \chi^2_p + O_p(n^{-2})$$

now suppose ψ is scalar

signed root statistic: $R(\psi) = \text{sgn}(\hat{\psi} - \psi)\sqrt{W(\psi)}$

$$R(\psi) \sim N(0,1) + O_p(n^{-1/2})$$

$$\mathrm{E}\{R(\psi)\} = \frac{m(\theta)}{\sqrt{n}} + O(n^{-3/2}),$$
$$\mathrm{var}\{R(\psi)\} = 1 + \frac{v(\theta)}{n} + O(n^{-2}),$$
third and higher cumulants are $O(n^{-3/2})$ or smaller

$$R(\psi) - \mathrm{E}\{R(\psi)\} \sim N(0,1) + O_p(n^{-1})$$

$$R(\psi) - \frac{m(\hat{\theta})}{\sqrt{n}} \sim N(0,1) + O_p(n^{-1})$$

$$R(\psi) - \frac{m(\psi, \tilde{\phi}_\psi)}{\sqrt{n}} \sim N(0,1) + O_p(n^{-1})$$

identical results hold from a Bayesian perspective

$$\mathrm{E}\{R(\psi)\} = \frac{m(\theta)}{\sqrt{n}} + O(n^{-3/2}),$$

$$\mathrm{var}\{R(\psi)\} = 1 + \frac{v(\theta)}{n} + O(n^{-2}),$$

third and higher cumulants are $O(n^{-3/2})$ or smaller

$$\frac{R(\psi) - \mathrm{E}\{R(\psi)\}}{\sqrt{\mathrm{var}\{R(\psi)\}}} \sim N(0,1) + O_p(n^{-3/2})$$

$$\frac{R(\psi) - \dfrac{m(\psi, \tilde{\phi}_\psi)}{\sqrt{n}}}{\sqrt{1 + \dfrac{v(\psi, \tilde{\phi}_\psi)}{n}}} \sim N(0,1) + O_p(n^{-3/2})$$

$$\frac{R(\psi) - \dfrac{m(\hat{\theta})}{\sqrt{n}}}{\sqrt{1 + \dfrac{v(\hat{\theta})}{n}}} \sim N(0,1) + O_p(n^{-1})$$

Exact and approximate conditional p-values as percentages

for testing $H_0 : \psi = \psi_0$ in the common log-odds ratio problem

ψ_0	Mid $p-$value	Full $p-$value	Relaxed $p-$value	$R(\psi)$ sim	$R(\psi)$ stand	$R^*(\psi)$	$R^*(\psi)$ discrete
-9.2101	0.100	0.200	0.198	0.195	0.000	0.000	0.000
-7.5999	0.500	0.996	0.978	0.944	0.000	0.000	0.000
-6.9057	1.000	1.982	1.937	1.840	0.000	0.000	0.001
-5.9862	2.500	4.890	4.744	4.366	0.001	0.002	0.042
-5.2871	5.000	9.572	9.235	8.214	0.280	0.037	0.411
-4.5803	10.000	18.368	17.648	15.034	4.469	0.428	2.939
-1.5929	10.000	16.703	6.584	7.886	5.971	12.426	23.140
-1.3143	5.000	8.644	3.074	3.918	2.800	6.016	12.494
-1.0841	2.500	4.429	1.454	2.083	1.323	2.954	6.708
-0.8265	1.000	1.814	0.549	0.555	0.495	1.175	2.942
-0.6555	0.500	0.919	0.264	0.310	0.237	0.591	1.577
-0.3103	0.100	0.188	0.049	0.061	0.044	0.123	0.370

$$\frac{R(\psi) - \dfrac{m(\hat{\theta})}{\sqrt{n}}}{\sqrt{1 + \dfrac{v(\hat{\theta})}{n}}} \sim N(0,1) + O_p(n^{-1})$$

$$\frac{R(\psi) - \dfrac{m(\hat{\theta})}{\sqrt{n}}}{\sqrt{1 + \dfrac{v'(\hat{\theta})}{n}}} \sim N(0,1) + O_p(n^{-3/2}),$$

$$\mathrm{var}\left\{ R(\psi) - \frac{m(\hat{\theta})}{\sqrt{n}} \right\} = 1 + \frac{v'(\theta)}{n} + O(n^{-2})$$

$$\frac{R(\psi) - \dfrac{m(\psi, \tilde{\phi}_\psi)}{\sqrt{n}}}{\sqrt{1 + \dfrac{v(\psi, \tilde{\phi}_\psi)}{n}}} \sim N(0,1) + O_p(n^{-3/2})$$

$$R^*(\psi) = R(\psi) + \frac{1}{R(\psi)} \log\left\{ \frac{U_F(\psi)}{R(\psi)} \right\}$$

$$L(\theta; Y) = L(\theta; \hat{\theta}, A) = L(\psi, \phi; \hat{\psi}, \hat{\phi}, A),\ A \text{ ancillary}$$

$$U_F(\psi) = \frac{\left| L_{;\hat{\theta}}(\hat{\psi}, \hat{\phi}) - L_{;\hat{\theta}}(\psi, \tilde{\phi}_\psi) \quad L_{\phi;\hat{\theta}}(\psi, \tilde{\phi}_\psi) \right|}{\left\{ \left| -L_{\phi\phi}(\psi, \tilde{\phi}_\psi) \right| \left| -L_{\theta\theta}(\hat{\psi}, \hat{\phi}) \right| \right\}^{1/2}}$$

$$U_B(\psi) = L_\psi(\psi, \tilde{\phi}_\psi) \frac{\left| -L_{\phi\phi}(\psi, \tilde{\phi}_\psi) \right|^{1/2}}{\left| -L_{\theta\theta}(\hat{\psi}, \hat{\phi}) \right|^{1/2}} \frac{\pi(\hat{\psi}, \hat{\phi})}{\pi(\psi, \tilde{\phi}_\psi)}$$

$1-\alpha$ Bayesian percentage point for ψ: $\hat{\psi}^B_{1-\alpha}$

$R(\psi) - \mu_B \sim N(0,1) + O_p(n^{-1})$ in Bayesian sense,
μ_B posterior mean of $R(\psi)$

$\Phi\{R(\hat{\psi}^B_{1-\alpha}) - \mu_B\} = 1-\alpha + O(n^{-1})$,
i.e., $R(\hat{\psi}^B_{1-\alpha}) - \mu_B = z_{1-\alpha} + O_p(n^{-1})$

typically, $\mathrm{pr}(\psi \leq \hat{\psi}^B_{1-\alpha}) = 1-\alpha + O(n^{-1/2})$

prior $\pi(\theta)$ is noninformative (objective) provided
$\mathrm{pr}(\psi \leq \hat{\psi}^B_{1-\alpha}) = 1-\alpha + O(n^{-1})$

$R(\psi) - \mu_F \sim N(0,1) + O_p(n^{-1})$ in frequentist sense,
μ_F frequentist mean of $R(\psi)$

$\hat{\psi}^B_{1-\alpha}$ is approximate frequentist confidence limit having
coverage error $O(n^{-1})$ provided
$\Phi\{R(\hat{\psi}^B_{1-\alpha}) - \mu_F\} = 1-\alpha + O(n^{-1})$,
i.e., $R(\hat{\psi}^B_{1-\alpha}) - \mu_F = z_{1-\alpha} + O_p(n^{-1})$

if the prior $\pi(\theta)$ is such that $\mu_B = \mu_F + O_p(n^{-1})$,

then $R(\hat{\psi}^B_{1-\alpha}) - \mu_B = z_{1-\alpha} + O_p(n^{-1})$

and $R(\hat{\psi}^B_{1-\alpha}) - \mu_F = z_{1-\alpha} + O_p(n^{-1})$

a condition for $\pi(\theta)$ to be noninformative is

$\mu_B = \mu_F + O_p(n^{-1})$

differentiation is denoted by subscripts:

$$L_r(\theta) = \partial L(\theta)/\partial\theta^r,\ L_{rs}(\theta) = \partial^2 L(\theta)/\partial\theta^r\theta^s, \text{ etc.}$$

define $\hat{L}_r = L_r(\hat{\theta}) = 0$, $\hat{L}_{rs} = L_{rs}(\hat{\theta})$, etc.

define $\lambda_r = \mathrm{E}\{L_r(\theta)\} = 0$, $\lambda_{rs} = \mathrm{E}\{L_{rs}(\theta)\}$, etc.

let $l_r = L_r(\theta) - \lambda_r = L_r(\theta)$, $l_{rs} = L_{rs}(\theta) - \lambda_{rs}$, etc.

λ's are assumed to be of order $O(n)$

l's have expectation 0 and are assumed to be $O_p(n^{1/2})$

joint cumulants of l_r, l_{rs}, etc. are of order $O(n)$

extend the λ-notation:

$$\lambda_{r,s} = \mathrm{E}(l_r l_s),\ \lambda_{r,s,t} = \mathrm{E}(l_r l_s l_t),\ \lambda_{rs,t} = \mathrm{E}(l_{rs} l_t)$$

Bartlett identities: differentiate $\int \exp\{L(\theta)\}dy = 1$

$$\lambda_r = 0, \quad \lambda_{rs} + \lambda_{r,s} = 0,$$

$$\lambda_{rst} + \lambda_{rs,t}[3] + \lambda_{r,s,t} = 0$$

diifferentiate $\lambda_{rs} = \int l_{rs} \exp\{L(\theta)\}dy$ to obtain

$$\lambda_{rs/t} = \partial\lambda_{rs}/\partial\theta^t = \lambda_{rst} + \lambda_{rs,t}$$

define matrix inverses $(\hat{L}^{rs}) = (\hat{L}_{rs})^{-1}$, $(\lambda^{rs}) = (\lambda_{rs})^{-1}$

define $\hat{V}^{rs} = \hat{L}^{rs} - \hat{L}^{r1}\hat{L}^{s1}/\hat{L}^{11}$, $\nu^{rs} = \lambda^{rs} - \lambda^{r1}\lambda^{s1}/\lambda^{11}$

$$
\begin{aligned}
R(\psi) &= T(\psi) + \frac{1}{6}\{T(\psi)\}^2\left\{\frac{\hat{M}_{111}}{(-\hat{M}_{11})^{3/2}}\right\} + O_p(n^{-1}),\\
\hat{M}_{11} &= \frac{1}{\hat{L}^{11}},\\
\hat{M}_{111} &= \hat{L}_{rst}\frac{\hat{L}^{r1}\hat{L}^{s1}\hat{L}^{t1}}{(\hat{L}^{11})^3}\\
T(\psi) &= (\hat{\psi}-\psi)(-\hat{M}_{11})^{1/2} = \frac{(\hat{\psi}-\psi)}{(-\hat{L}^{11})^{1/2}},
\end{aligned}
$$

$$
\begin{aligned}
\pi_{\psi|Y}(\psi) &\propto \exp\{B(\psi) + M(\psi)\},\\
B(\psi) &= -\frac{1}{2}\log\left\{\frac{\left|-L_{\psi\psi}(\psi,\tilde{\phi}_\psi)\right|}{\left|-L_{\psi\psi}(\hat{\psi},\hat{\phi})\right|}\right\} + \log\left\{\frac{\pi(\psi,\tilde{\phi}_\psi)}{\pi(\hat{\psi},\hat{\phi})}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\pi_{T(\psi)|Y}(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}\Bigg\{1 &- \frac{\hat{B}_1}{(-\hat{M}_{11})^{1/2}}t\\
&-\frac{1}{6}\frac{\hat{M}_{111}}{(-\hat{M}_{11})^{3/2}}t^2 + O(n^{-1})\Bigg\},\\
\hat{B}_1 = -\frac{1}{2}\hat{L}_{rst}\frac{\hat{L}^{r1}\hat{V}^{st}}{\hat{L}^{11}} &+ \left.\frac{\partial\log\pi(\theta)}{\partial\theta^r}\right|_{\theta=\hat{\theta}}\frac{\hat{L}^{r1}}{\hat{L}^{11}},
\end{aligned}
$$

$$
\begin{aligned}
\mu_B &= -\frac{1}{2}\hat{L}_{rst}\frac{\hat{L}^{r1}\hat{V}^{st}}{(-\hat{L}^{11})^{1/2}} + \frac{\partial \log \pi(\theta)}{\partial \theta^r}\bigg|_{\theta=\hat{\theta}} \frac{\hat{L}^{r1}}{(-\hat{L}^{11})^{1/2}} \\
&\qquad + \frac{1}{3}\hat{L}_{rst}\frac{\hat{L}^{r1}\hat{L}^{s1}\hat{L}^{t1}}{(-\hat{L}^{11})^{3/2}} + O_p(n^{-1}) \\
&= -\frac{1}{2}\lambda_{rst}\frac{\lambda^{r1}\nu^{st}}{(-\lambda^{11})^{1/2}} + \frac{\partial \log \pi(\theta)}{\partial \theta^r}\frac{\lambda^{r1}}{(-\lambda^{11})^{1/2}} \\
&\qquad + \frac{1}{3}\lambda_{rst}\frac{\lambda^{r1}\lambda^{s1}\lambda^{t1}}{(-\lambda^{11})^{3/2}} + O_p(n^{-1})
\end{aligned}
$$

$$
\begin{aligned}
R(\psi) &= -(-\lambda^{11})^{-1/2}(\lambda^{r1}l_r + \tfrac{1}{2}\lambda^{r1}\lambda^{su}\nu^{tv}\lambda_{rst}l_u l_v \\
&\qquad + \tfrac{1}{6}\lambda^{r1}\tau^{su}\tau^{tv}\lambda_{rst}l_u l_v - \lambda^{r1}\lambda^{st}l_{rs}l_t \\
&\qquad\qquad + \tfrac{1}{2}\lambda^{r1}\tau^{st}l_{rs}l_t) + O_p(n^{-1}), \\
\tau^{rs} &= \lambda^{r1}\lambda^{s1}(\lambda^{11})^{-1} = \lambda^{rs} - \nu^{rs}
\end{aligned}
$$

$$
\begin{aligned}
\mu_F &= \frac{1}{2}\lambda_{rst}\frac{\lambda^{r1}\nu^{st}}{(-\lambda^{11})^{1/2}} - \frac{1}{6}\lambda_{rst}\frac{\lambda^{r1}\lambda^{s1}\lambda^{t1}}{(-\lambda^{11})^{3/2}} \\
&\quad + \lambda_{rs,t}\frac{\lambda^{r1}\lambda^{st}}{(-\lambda^{11})^{1/2}} + \frac{1}{2}\lambda_{rs,t}\frac{\lambda^{r1}\lambda^{s1}\lambda^{t1}}{(-\lambda^{11})^{3/2}} + O(n^{-1})
\end{aligned}
$$

the condition $\mu_B = \mu_F + O(n^{-1})$ yields:

$$\frac{\partial \log \pi(\theta)}{\partial \theta^r} \frac{\lambda^{r1}}{(-\lambda^{11})^{1/2}} = \lambda_{rst} \frac{\lambda^{r1}\lambda^{st}}{(-\lambda^{11})^{1/2}} + \lambda_{rs,t} \frac{\lambda^{r1}\lambda^{st}}{(-\lambda^{11})^{1/2}} + \frac{1}{2}\lambda_{rst} \frac{\lambda^{r1}\lambda^{s1}\lambda^{t1}}{(-\lambda^{11})^{3/2}} + \frac{1}{2}\lambda_{rs,t} \frac{\lambda^{r1}\lambda^{s1}\lambda^{t1}}{(-\lambda^{11})^{3/2}}$$

$$\frac{\partial \log \pi(\theta)}{\partial \theta^r} \frac{\lambda^{r1}}{(-\lambda^{11})^{1/2}} = -\frac{\partial}{\partial \theta^r} \left\{ \frac{\lambda^{r1}}{(-\lambda^{11})^{1/2}} \right\}$$

under parameter orthogonality, $\lambda^{r1} = 0$ ($r \neq 1$):

$$\lambda^{11} = (\lambda_{11})^{-1},$$

$$\frac{\partial \log \pi(\theta)}{\partial \psi} (-\lambda_{11})^{-1/2} = -\frac{\partial}{\partial \psi} (-\lambda_{11})^{-1/2}$$

$$\pi(\psi, \phi) \propto \sqrt{-\lambda_{11}(\psi, \phi)}\, g(\phi)$$

posterior mode is $\hat{\psi} + \dfrac{\hat{B}_1}{(-\hat{M}_{11})} + O_p(n^{-3/2})$:

$$\hat{\psi} - \frac{1}{2}\lambda_{rst}\lambda^{r1}\lambda^{st} - \lambda_{rs,t}\lambda^{r1}\lambda^{st}$$
$$-\frac{1}{2}\lambda_{rs,t}\frac{\lambda^{r1}\lambda^{s1}\lambda^{t1}}{(-\lambda^{11})} + O_p(n^{-3/2})$$

posterior mean:

$$\hat{\psi} + \frac{\hat{B}_1}{(-\hat{M}_{11})} + \frac{1}{2}\frac{\hat{M}_{111}}{(-\hat{M}_{11})^2} + O_p(n^{-3/2})$$
$$= \hat{\psi} - \frac{1}{2}\lambda_{rst}\lambda^{r1}\lambda^{st} - \lambda_{rs,t}\lambda^{r1}\lambda^{st}$$
$$-\frac{1}{2}\lambda_{rst}\frac{\lambda^{r1}\lambda^{s1}\lambda^{t1}}{(-\lambda^{11})} - \frac{1}{2}\lambda_{rs,t}\frac{\lambda^{r1}\lambda^{s1}\lambda^{t1}}{(-\lambda^{11})} + O_p(n^{-3/2})$$

posterior median is $\bar{\psi}^B_{0.5} + O_p(n^{-3/2})$,

where either $R(\bar{\psi}^B_{0.5}) - \mu_B = 0$

or $R(\bar{\psi}^B_{0.5}) - \mu_F = 0$

μ_F can be obtained by simulation at $\theta = \hat{\theta}$;

alternatively, simulate the distribution of $R(\psi)$

under a noninformative prior, the posterior median is close to the parameter value whose observed level of significance is 50%

simulate the distribution of $R(\psi)$; what about other approximate pivots?

let $P(\psi) = (-\lambda^{11})^{-1/2}(-\lambda^{r1}l_r + \eta^{1rs}l_r l_s + \eta^{1rst}l_{rs}l_t)$, η^{1rs}, η^{1rst} are $O(n^{-2})$

$$\kappa_1 = (-\lambda^{11})^{-1/2}(-\eta^{1rs}\lambda_{rs} + \eta^{1rst}\lambda_{rs,t}) + O(n^{-1}),$$
$$\kappa_2 = 1 + O(n^{-1}),$$
$$\kappa_3 = (-\lambda^{11})^{-3/2}(\lambda_{rst}\lambda^{r1}\lambda^{s1}\lambda^{t1} + 3\lambda_{rs,t}\lambda^{r1}\lambda^{s1}\lambda^{t1} + 6\eta^{111} - 6\eta^{1rs1}\lambda_{rs,t}\lambda^{t1}) + O(n^{-1})$$

Cornish-Fisher p-value from $P(\psi)$:
$\Phi[P(\psi) - \frac{1}{6}\kappa_3\{P(\psi)\}^2 - \kappa_1 + \frac{1}{6}\kappa_3] + O(n^{-1})$

consider another pivot:

$$\breve{P}(\psi) = (-\lambda^{11})^{-1/2}(-\lambda^{r1}l_r + \breve{\eta}^{1rs}l_r l_s + \breve{\eta}^{1rst}l_{rs}l_t)$$

require p-values from $P(\psi)$ and $\breve{P}(\psi)$ agree to error of order $O_p(n^{-1})$

$$\eta^{1rst} = \breve{\eta}^{1rst}, \quad \eta^{1rs} - \eta^{1uv}\lambda_{uv}\tau^{rs} = \breve{\eta}^{1rs} - \breve{\eta}^{1uv}\lambda_{uv}\tau^{rs}$$

for $R(\psi)$:

$$\eta^{1rst} = \lambda^{r1}\lambda^{st} - \tfrac{1}{2}\lambda^{r1}\tau^{st},$$
$$\eta^{1rs} = -\tfrac{1}{2}\lambda^{t1}\lambda^{ru}\nu^{sv}\lambda_{tuv} - \tfrac{1}{6}\lambda^{t1}\tau^{ru}\tau^{sv}\lambda_{tuv},$$
$$\eta^{1rs} - \eta^{1uv}\lambda_{uv}\tau^{rs} = -\tfrac{1}{2}\lambda^{t1}\lambda^{ru}\nu^{sv}\lambda_{tuv} + \tfrac{1}{2}\lambda^{t1}\nu^{uv}\lambda_{tuv}\tau^{rs}$$

for $T(\psi) = \dfrac{(\hat{\psi} - \psi)}{(-\hat{L}^{11})^{1/2}}$:

$$\eta^{1rst} = \lambda^{r1}\lambda^{st} - \tfrac{1}{2}\lambda^{r1}\tau^{st},$$
$$\eta^{1rs} = -\tfrac{1}{2}\lambda^{t1}\lambda^{ru}\nu^{sv}\lambda_{tuv},$$
$$\eta^{1rs} - \eta^{1uv}\lambda_{uv}\tau^{rs} = -\tfrac{1}{2}\lambda^{t1}\lambda^{ru}\nu^{sv}\lambda_{tuv} + \tfrac{1}{2}\lambda^{t1}\nu^{uv}\lambda_{tuv}\tau^{rs}$$

prescription for conditional correctness: $\eta^{1rs1} = \frac{1}{2}\lambda^{r1}\lambda^{s1}$

for $R(\psi)$:

$$\eta^{1rst} = \lambda^{r1}\lambda^{st} - \tfrac{1}{2}\lambda^{r1}\tau^{st},$$

$$\eta^{1rs} = -\tfrac{1}{2}\lambda^{t1}\lambda^{ru}\nu^{sv}\lambda_{tuv} - \tfrac{1}{6}\lambda^{t1}\tau^{ru}\tau^{sv}\lambda_{tuv},$$

$$\eta^{1rs} - \eta^{1uv}\lambda_{uv}\tau^{rs} = -\tfrac{1}{2}\lambda^{t1}\lambda^{ru}\nu^{sv}\lambda_{tuv} + \tfrac{1}{2}\lambda^{t1}\nu^{uv}\lambda_{tuv}\tau^{rs}$$

for $T(\psi) = \dfrac{(\hat{\psi} - \psi)}{(-\hat{\lambda}^{11})^{1/2}}$:

$$\eta^{1rst} = \lambda^{1r}\lambda^{st},$$

$$\eta^{1rs} = -\tfrac{1}{2}\lambda^{1t}\lambda^{ru}\nu^{sv}\lambda_{tuv} + \tfrac{1}{2}\lambda^{1t}\tau^{ru}\lambda^{sv}\lambda_{tu,v},$$

$$\begin{aligned}\eta^{1rs} - \eta^{1uv}\lambda_{uv}\tau^{rs} &= -\tfrac{1}{2}\lambda^{1t}\lambda^{ru}\nu^{sv}\lambda_{tuv} + \tfrac{1}{2}\lambda^{1t}\nu^{uv}\lambda_{tuv}\tau^{rs} \\ &\quad + \tfrac{1}{2}\lambda^{1t}\tau^{ru}\lambda^{sv}\lambda_{tu,v} - \tfrac{1}{2}\lambda^{1t}\tau^{uv}\lambda_{tu,v}\tau^{rs}\end{aligned}$$

prescription for conditionality violated: $\eta^{1rs1} = \lambda^{r1}\lambda^{s1}$

Risk calculations for eigenvalues

We compare the performance of our estimator to that of other estimators proposed in the literature

Consider the following loss:

$$L(\lambda, \hat{\lambda}) = \sum_{i=1}^{m} \left(\frac{\lambda_i - \hat{\lambda}_i}{\lambda_i} \right)^2$$

We present risk calculations for three different scenarios:

$$\begin{aligned}
\lambda_I &= (1, 1, 1, 1, 1, 1, 1, 1, 1, 1) \\
\lambda_{II} &= (9.1,\ 0.1,\ 0.1,\ 0.1,\ 0.1,\ 0.1,\ 0.1,\ 0.1,\ 0.1,\ 0.1) \\
\lambda_{III} &= (10, 10, 10, 10, 10, 3, 3, 3, 3, 3)
\end{aligned}$$

Risk of estimators (p=10)

Percent reduction in average loss compared to MLE

	Ledoit	D&K	James-Stein	Bootstrap
Case I				
n=50	84%	4%	31%	53%
Case II				
n=50	-9%	0.5%	9%	49%
Case III				
n=50	---	2%	15%	33%

Risk of estimators (p=10)

Percent reduction in average loss compared to MLE

	Ledoit	D&K	James Stein	Bootstrap with iterate
Case I				
n=50	84%	4%	31%	64%
Case II				
n=50	-9%	0.5%	9%	58%
Case III				
n=50	---	2%	15%	29%