Granular Avalanche Down an Incline

The risk of volcanic eruption is one that public safety officials throughout the world face several times each year. The US Geological Survey reports that there are about 50 volcanoes that erupt every year. In the 1980s, approximately 30,000 people were killed and almost a half million were forced from their homes because of volcanic activity.

Geophysical mass flows - debris flows, avalanches, landslides - are often initiated by volcanic activity. These flows can contain $O(10^6 - 10^7)\,\mathrm{m}^3$ or more of material. Typically this material is a mixture of soil and rocks in the centimeter size range, sometimes mixed with meter-sized boulders. These flows can be $O(10)\,\mathrm{m}$ deep and can run out at speeds of tens of meters per second, over distances of tens of kilometers. And often these flows contain a significant quantity of interstitial fluid. Here we describe a model of these flows.

Consequent to an eruption can be hazards including passive gas emission or the slow efflux of lava, or violent explosions accompanied by stratospheric plumes and pyroclastic flows of red-hot ash, rock and gas that race along the surface of the mountain. These flows can melt snow, creating a muddy mixture of ash, rock and water. Other potential hazardous activity includes giant debris flows, coarse block and ask flows, and avalanches.

Absent volcanic activity, intense rainfall on hillsides that are devoid of vegetation can cause massive mudslides. The 1998 mudflow at Casita Volcano in Nicaragua caused thousands of deaths. In the San Bernardino area of California during December 2003, hills that were scoarched by summertime wildfires gave way in massive mudslide destroying homes and businesses.

Exploiting the small depth of typical mass flows relative to their length, Savage and Hutter developed a "thin layer" model for granular flows down inclines (Savage and Hutter, *JFM* (1989)).

1 Modeling Flows

A thin layer of material is assumed to flow over a smooth basal surface; as we will see, motion normal to the bed can be neglected, and we ignore any erosion of the basal surface. By depth-averaging the mass and momentum balance laws, one derives a system of PDEs for the flow depth and the flow velocities. We consider the equations of motion in three space dimensions, called x, y and z, plus time; depth averaging will remove the z dimension, the direction normal to the basal surface, yielding a system in x, y and t.

Consider then a mass of solid material of constant specific density ρ^s, flowing on a surface described by $F_b(\mathbf{x}) = z - b(x,y) = 0$. Write v for the velocities of the material. When writing equations in component form, we use numerical superscripts for velocities in specific directions. Mass conservation may be written as

$$dt\rho^s + \nabla \cdot (\rho^s v) = 0 \tag{1}$$

The momentum equations takes the form

$$\rho^s(\partial_t v + (v \cdot \nabla)v) = -\nabla \cdot T^s + \rho^s g \tag{2}$$

To proceed, scale the independent and dependent variables, z by H, x, y by L, stresses by $\rho^s gH$ and time as $t^2 = L/g$. After clearing the coefficients, the continuity equations is unchanged: $\partial_x v^1 + \partial_y v^2 + \partial_z v^3 = 0$. However in the momentum equations, several terms are multiplied by $\epsilon = H/L$ which is assumed small:

$$\partial_t v^1 + v^1\partial_x v^1 + v^2\partial_y v^1 + v^3\partial_z v^1 = -(\epsilon\partial_x T^{s11} + \epsilon\partial_y T^{s21} + \partial_z T^{s31}) + g^x$$
$$\partial_t v^2 + v^1\partial_x v^2 + v^2\partial_y v^2 + v^3\partial_z v^2 = -(\epsilon\partial_x T^{s12} + \epsilon\partial_y T^{s22} + \partial_z T^{s32}) + g^y$$
$$\epsilon(\partial_t v^3 + v^1\partial_x v^3 + v^2\partial_y v^3 + v^3\partial_z v^3) = -(\epsilon\partial_x T^{s13} + \epsilon\partial_y T^{s23} + \partial_z T^{s33}) - g^z$$

We discard all terms multiplied by ϵ in the z-momentum equation, but retain the ϵ terms in the x- and y-equations. The justification for retaining this terms can be made a posteriori by examining the scale of the entire term ϵT^s. In particular, dropping these terms results in a trivial model; see the discussion in Savage-Hutter.

Only two terms remain in the z-momentum equation

$$\partial_z T^{s33} = -g^z$$

so the normal solids pressure is lithostatic -

$$T^{s33}(z) \;=\; g^z(h - z) \quad . \tag{3}$$

We now several assumptions to simplify the modeling. From the scaled equations, it is clear we can ignore all motion normal to the basal surface, so $v^3 = 0$ in the bulk of the flow. The assumption of Mohr-Coulomb plasticity is far too complex to use in a depth-averaged model. An assumption is made whereby all shear and longitudinal stresses are proportional to the normal stress; similar assumptions appear in soil mechanics and have their roots in the classic work of Terzaghi and Rankine. This issue will be detailed below.

As for boundary conditions, the upper surface of the flowing mass at $F_h(\mathbf{x}, t) = z - h(x, y, t) = 0$ is assumed to be a material surface and stress free. That is, the surface is defined by the motion of the material: $\partial_t h + v_x\partial_x h + v_y\partial_y h = v_z$. At the base of the mass, material is assumed to flow tangent to $F_b = 0$, and to satisfy a sliding friction law. This friction relation specifies that the shear traction S and the normal stress N are proportional: $S|_{F_b} = -\mathrm{sgn}(v)\, N|_{F_b} \tan(\phi_{bed})$, where ϕ_{bed} is the basal friction angle and the $-\mathrm{sgn}(v)$ specifies that the shear force opposes motion.

2 Depth Averaging

Integrate the incompressibility equation through the layer to find

$$\int_b^h (\partial_x v^1 + \partial_y v^2 + \partial_z v^3)\, dz \;=\; 0 \quad .$$

The Leibnitz formula allows the interchange of differentiation and integration. Then apply the kinematic conditions on the upper and lower surfaces to find

$$\partial_t \hat{h} + \partial_x \left(\hat{h}\overline{v}^1\right) + \partial_y \left(\hat{h}\overline{v}^2\right) = 0 \quad . \tag{4}$$

The depth-averaged velocities are defined by $\hat{h}\overline{v}^1 = \int_b^h v^1\,dz$, and $\hat{h}\overline{v}^2 = \int_b^h v^2\,dz$, and $\hat{h} = h - b$.

Now consider the equation of the motion. Notice that the depth averaged lithostatic stress is given by

$$\int_b^h \partial_z T^{s33}\,dz \;=\; -\int_b^h g^z \;=\; -\hat{h}g^z$$

The left-hand side of the x-momentum equation is

$$LHS = \partial_t v^1 + v^1\partial_x v^1 + v^2\partial_y v^1 + v^3\partial_z v^1\ .$$

Use the incompressibility equation, depth average, and use boundary conditions to find

$$\int_b^h LHS\,dz \;=\; \partial_t \int_b^h v^1\,dz + \partial_x \int_b^h v^1 v^1\,dz + \partial_y \int_b^h v^1 v^2\,dz + \partial_z \int_b^h v^1 v^3\,dz \tag{5}$$

We would like to write these terms as products of the average velocity, but can't. We may have, for example, $\int_b^h v^1 v^1\,dz = c\hat{h}(\overline{v}^1)(\overline{v}^1)$ for some c, where $c = 1$ for plug flow and $c = 6/5$ for parabolic flow.

As for the right hand side:

$$\int_b^h RHS\,dz = -\int_b^h (\epsilon\partial_x T^{s11} + \epsilon\partial_y T^{s12} + \partial_z T^{s13})\,dz + \int_b^h g^x\,dz \tag{6}$$

In order to proceed, several assumptions are made: The earth pressure relations for the solid phase is employed. To this end, basal shear stresses are assumed to be proportional to the normal stress:

$$T^{s*3} = -\frac{v^*}{||\mathbf{v}||}\tan(\phi_{bed})T^{s33} = \alpha_{*3}T^{s33}\quad ,$$

where $*$ can be either 1 or 2 (i.e., x or y), and the velocity ratio determines the force opposes motion in the $*$-direction, to the extend that force is mobilized in that direction. Likewise the diagonal stresses are taken to be proportional to the normal solid stress

$$T^{s**} = k_{ap}T^{s33} = \alpha_{**}T^{s33}\ ,$$

where the same index x or y is used in both $*$s. Finally, the T^{12} shear stress is determined by a Coulomb relation

$$T^{s12} = -\mathrm{sgn}(\partial_y v^1)\sin(\phi_{int})k_{ap}T^{s33} = \alpha_{12}T^{s33}$$

where the sgn function again ensures that friction opposes straining in the (x, y)-plane. The k_{ap} term is called the earth pressure coefficient, and takes on different values in the active and passive stress states. The active or passive state of stress is developed if an element of material is elongated or compressed in flow, and the formula for the corresponding states can be derived from the Mohr diagram

$$k_{ap} = 2\frac{1 \pm \left[1 - \cos^2\phi_{int}(1 + \tan^2\phi_{bed})\right]^{1/2}}{\cos^2\phi_{int}} - 1 \tag{7}$$

in which the negative square root corresponds to an active state ($\partial_x \overline{v}^1 + \partial_y \overline{v}^2 > 0$), and the positive root to the passive state ($\partial_x \overline{v}^1 + \partial_y \overline{v}^2 < 0$).

We find

$$
\begin{aligned}
RHS &= -\epsilon \int_b^h \partial_x \alpha_{11} T^{s33}\, dz - \epsilon \int_b^h \partial_y \alpha_{12} T^{s33}\, dz - \int_b^h \partial_z \alpha_{13} T^{s33}\, dz \\
+ \int_b^h g^x\, dz &= -\epsilon[\partial_x \int_b^h \alpha_{11} T^{s33}\, dz - \alpha_{11} T^{s33}|_{z=h} \partial_x h + \alpha_{11} T^{s33}|_{z=b} \partial_x b] \\
&\quad - \epsilon[\partial_y \int_b^h \alpha_{12} T^{s33}\, dz - \alpha_{12} T^{s33}|_{z=h} \partial_y h + \alpha_{12} T^{s33}|_{z=b} \partial_y b] \\
&\quad - \alpha_{13}[T^{s33}|_{z=h} - T^{s33}|_{z=b}] + \int_b^h g^x\, dz \quad .
\end{aligned}
\tag{8}
$$

One must use kinematic and stress condition at the upper surfaces which reads

$$
\mathbf{T} \cdot \mathbf{n} \Big|_{z=h(x,y,t)} = T_{xx} \Big|_{z=h(x,y,t)} \partial_x h + T_{yx} \Big|_{z=h(x,y,t)} \partial_y h - T_{zx} \Big|_{z=h(x,y,t)} = 0
$$

Now combining terms yields an x-momentum equation:

$$
\begin{aligned}
& \partial_t(\hat{h}\overline{v^1}) + \partial_x(\hat{h}\overline{v^1 v^1}) + \partial_y(\hat{h}\overline{v^1 v^2}) \\
= & \quad -\frac{1}{2}\epsilon \partial_x(\alpha_{11}\hat{h}^2(-g^z)) - \frac{1}{2}\epsilon \partial_y(\alpha_{12}\hat{h}^2(-g^z)) \\
& \quad + (-\epsilon\alpha_{11}\partial_x b - \epsilon\alpha_{12}\partial_y b + \alpha_{13})\hat{h}(-g^z) + \hat{h} g^x \quad .
\end{aligned}
\tag{9}
$$

The y-momentum has a similar form, and can be obtained by interchanging x and y and 1 and 2 in the x-equation.

Writing the relations for mass and momentum balance, the governing equations form a system of hyperbolic balance laws that can be written in matrix form

$$
\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = \mathbf{S} \tag{10}
$$

This system has a structure similar to the shallow water equations, except for the complexity of the dissipation terms on the right-hand side. The system is strictly hyperbolic and genuinely nonlinear away from the "vacuum state" $h = 0$. (Exercise: Show that, for the equations in 1 space dimension, the characteristic speeds are $\lambda_\pm = v^1 \pm \sqrt{\epsilon \alpha_{11} h}$.)