Homework Problems 1

Turn in by October 25.

Linear stability. Consider a nonlinear system of differential equations $df/dt = G(f)$. Assume $f = z$ is a solution (z a constant). Linearization around z means substituting $f = z + \epsilon f_1 + O(\epsilon^2)$ on both sides of the equation, and take the derivitive $\frac{d}{d\epsilon}|_{\epsilon=0}$. So in this example, we would find $df_1/dt = G'(z)f_1$.

1. Hopf Bifurcation. Consider the system

$$\begin{aligned} \frac{dx}{dt} &= \beta x - y - x(x^2 + y^2) \\ \frac{dy}{dt} &= x + \beta y - y(x^2 + y^2) \end{aligned} \tag{1}$$

The origin is an equilibrium point. Linearize around the origin, and tell what kind of equilibrium point one has, depending on β. The notice the system for $\beta > 0$ that there exists a limit cycle of radius $\sqrt{\beta}$. [Hint: Change to polar coordinates.]

2. Consider the motion of a mass-spring system atop a frictional belt. The spring, with constant k, is attached to a wall at the left, the belt moves right. When the block is stationary, the restoring force of the spring equals frictional force of the belt and it moves with the speed of the belt v. Once the block moves (so $\frac{dx}{dt} \neq 0$), the frictional force is given by $g(\frac{dx}{dt} - v) = -k_1 \text{sgn}(\frac{dx}{dt} - v)$. If we define $k/m = \omega^2$ and $k_1/m = \alpha$, then for $\frac{dx}{dt} \neq 0$, we have

$$\frac{d^2x}{dt^2} + \omega^2 x + \alpha \text{sgn}(\frac{dx}{dt} - v) \ = \ 0$$

Describe the phase plane for this system. (Hint: Define the energy $\frac{1}{2}(\frac{dx}{dt})^2 + \frac{1}{2}\omega^2 x^2 \pm \alpha x = C$. Complete the square on the left.]

3. Savage developed a model for the flow of granular material in a smooth, steep walled hopper (here in 2 dimensions). Approximating $\cos(\theta)$ as 1 and ignoring wall friction in the defining equations, the defining equations are

$$\begin{aligned} \frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} - \rho g &= \rho u \frac{du}{dr} \\ \frac{d(ru)}{dr} &= 0 \end{aligned} \tag{2}$$

Here $\sigma_\theta = K\sigma_r$ with $K = \frac{1+\sin(\phi)}{1-\sin(\phi)}$. This system is supposed to hold in a domain $R_0 < r < R_1$. Solve the system. Show that the velocity is essentially independent of the 'head' if $R_1 >> R_0$.