

Time Series Y_t

Stationary $E(Y_t) = \mu$ (constant)

$\text{Cov}(Y_t, Y_{t-h}) = \gamma(h)$

“Toeplitz” covariance matrix

$$\begin{pmatrix} \gamma(0) & \gamma(1) & \gamma(2) \\ \gamma(1) & \gamma(0) & \gamma(1) \\ \gamma(2) & \gamma(1) & \gamma(0) \end{pmatrix}$$

e_t : white noise—uncorrelated $(0, \sigma^2)$

$$Y_t = \mu + \sum_{j=0}^{\infty} w_j e_{t-j} \quad [\text{Wold representation}]$$

$$\gamma(h) = \sigma^2 \sum_{j=0}^{\infty} w_j w_{j+h}$$

Example:

$$(Y_t - \mu) = e_t + \rho e_{t-1} + \rho^2 e_{t-2} \dots$$

$$\gamma(h) = \left(\frac{\sigma^2}{1-\rho^2} \right) \rho^{|h|}$$

$$\rho(Y_t - \mu) = e_t + \rho [e_{t-1} + \rho e_{t-2} + \rho^2 e_{t-3} + \dots]$$

$$Y_t - \mu = \rho(Y_{t-1} - \mu) + e_t$$

“Autoregressive order 1”

Stationary? Yes if $|\rho| < 1$

Backshift $B(Y_t) = Y_{t-1}$

$$(1 - \rho B)(Y_t - \mu) = e_t$$

$$Y_t - \mu = (1 - \rho B)^{-1} e_t$$

$$(1 - X)^{-1} = 1 + X + X^2 + \dots \text{ if } |X| < 1!$$

$$Y_t - \mu = (1 + \rho B + \rho^2 B^2 + \dots) e_t$$

Moving Average Order 1

$$Y_t - \mu = e_t - \theta e_{t-1}$$

Clearly stationary

Forecast $\mu - \theta e_{t-1}$

How to get e_t ?

$$Y_t - \mu = (1 - \theta B) e_t$$

$$e_t = \sum_{j=0}^{\infty} \theta^j (Y_{t-j} - \mu)$$

Want $|\theta| < 1$ "invertible"

$$(Y_t - \mu) = \rho(Y_{t-1} - \mu) + e_t, Y_0 = \mu$$

$$\rho = 1 : Y_t = Y_{t-1} + e_t$$

Random Walk

No Mean Revision

Regress Y_t on $Y_{t-1} \rightarrow \hat{\rho}$ [OLS]

$$|\rho| < 1 \Rightarrow \sqrt{n} (\hat{\rho} - \rho) \sim N(0, 1 - \rho^2)$$

$$\rho=1 \Rightarrow n(\hat{\rho}-1) \sim \text{DF}$$

Extends to higher order and mixed models

ARIMA(p, d, q)

ARIMA(0,1,1) \equiv IMA(1,1)

$$Y_t - Y_{t-1} = e_t - \theta e_{t-1}$$

$$e_t = \sum_{j=0}^{\infty} \theta^j (Y_{t-j} - Y_{t-j-1})$$

$$= Y_t + (1-\theta)Y_{t-1} + \theta(1-\theta)Y_{t-2} + \dots$$

$$Y_t = e_t + (1-\theta) \sum_{i=0}^{\infty} \theta^i Y_{t-i} = e_t + \hat{Y}_t \text{ where}$$

$$\hat{Y}_t = (1-\theta) \sum_{i=0}^{\infty} \theta^i Y_{t-i} \quad \text{EWMA} \quad [0 < \theta < 1]$$

$$\hat{Y}_t = (1-\theta)Y_{t-1} + \theta \hat{Y}_{t-1} \quad \text{Weighted Average!}$$

EWMA = winner in CHANCE paper.

Transfer function: Observe X_t

$$Y_t = \beta_0 + \beta_1 \sum w_j X_{t-j} + \text{noise}$$

Intervention:

(1) $X_t = 0, 0, 0, 1, 0, 0, 0 \dots$

(2) $X_t = 0, 0, 0, 1, 1, 1, 1, \dots$

$$Y_t = 10 + 8(1 - .5B)^{-1} X_t + e_t$$

(1) 10, 10, 10, 18, 14, 12, 11, 10, 5, ...

(2) 10, 10, 10, 18, 22, 24, 25, 25.5 \rightarrow 26

$$\left[26 = 10 + 8(1 - .5)^{-1} \right]$$

Examples:

Hawaiian Milk Scare

Cincinnati Telephone Info Change

9-11 AMR Stock Volume