SAMSI Credit Risk Working Group Presentation

Modeling Correlated Defaults: First Passage Model under Stochastic Volatility

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(Ongoing Project)
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Outline of This Talk

- Brief review of first passage model for single-name default

- Stochastic volatility effects on first passage model for single-name default

- Setup of first passage model under stochastic volatility for two-name correlated defaults

- Derivation for the approximate joint default/survival probability

- Model calibration (ongoing work)

First Passage Model for Single-Name Default

- Was first proposed by Black and Cox (1976), where the volatility for the underlying firm's value is taken to be constant.

- The firm defaults at the first time that the firm's value drops to or below some exogenously prespecified level — the default barrier/threshold.

- Closed-form formula exists for the default probability.

- The default probability and yield spread of the bond are close to zero for short maturities, which is contrary to market observations.

Stochastic Volatility Effects on Single-Name Default

- Proposed by Fouque, Sircar and Sϕlna (2004).

- No closed-form formula exists for the default probability.

- For fast and slow mean-reverting stochastic volatilities, closed-form formulas exist for the asymptotic default probability.

- Incorporating fast mean-reverting stochastic volatility can significantly increase the default probability and yield spread for short maturities.

- Incorporating slow mean-reverting stochastic volatility can increase the default probability and yield spread for long maturities.

Model Setup for Two-Name Correlated Defaults

Under physical measure $\mathbb{P}$,

$$\begin{aligned} \mathrm{d}X_t^{(1)} &= \mu_1 X_t^{(1)}\mathrm{d}t + f_1(Y_t, Z_t)X_t^{(1)}\mathrm{d}W_t^{(1)} \\ \mathrm{d}X_t^{(2)} &= \mu_2 X_t^{(2)}\mathrm{d}t + f_2(Y_t, Z_t)X_t^{(2)}\mathrm{d}W_t^{(2)} \\ \mathrm{d}Y_t &= \frac{1}{\epsilon}(m_Y - Y_t)\mathrm{d}t + \frac{\nu_Y\sqrt{2}}{\sqrt{\epsilon}}\mathrm{d}W_t^{(Y)} \\ \mathrm{d}Z_t &= \delta(m_Z - Z_t)\mathrm{d}t + \nu_Z\sqrt{2\delta}\mathrm{d}W_t^{(Z)} \end{aligned}$$

where $\mathrm{d}W_t^{(1)}\mathrm{d}W_t^{(2)} = 0$, f_1 and f_2 are bounded above and below away from 0. $\epsilon > 0$ and $\delta > 0$ are small parameters, corresponding to fast and slow mean-reversion, respectively.

Default Barrier:

$$B_i(t) = K_i \mathrm{e}^{\eta_i t},\ i = 1, 2.$$

Dynamics under EMM

Under the equivalent martingale measure (EMM) $\mathbb{P}^*$ chosen by the market,

$$
\begin{aligned}
\mathrm{d}X_t^{(1)} &= rX_t^{(1)}\mathrm{d}t + f_1(Y_t, Z_t)X_t^{(1)}\mathrm{d}\tilde{W}_t^{(1)} \\
\mathrm{d}X_t^{(2)} &= rX_t^{(2)}\mathrm{d}t + f_2(Y_t, Z_t)X_t^{(2)}\mathrm{d}\tilde{W}_t^{(2)} \\
\mathrm{d}Y_t &= \left[\frac{1}{\epsilon}(m_Y - Y_t) + \frac{\nu_Y\sqrt{2}}{\sqrt{\epsilon}}\Lambda_1(Y_t, Z_t)\right]\mathrm{d}t + \frac{\nu_Y\sqrt{2}}{\sqrt{\epsilon}}\mathrm{d}\tilde{W}_t^{(Y)} \\
\mathrm{d}Z_t &= \left[\delta(m_Z - Z_t) + \nu_Z\sqrt{2\delta}\Lambda_2(Y_t, Z_t)\right]\mathrm{d}t + \nu_Z\sqrt{2\delta}\mathrm{d}\tilde{W}_t^{(Z)}
\end{aligned}
$$

where $\mathrm{d}\tilde{W}_t^{(1)}\mathrm{d}\tilde{W}_t^{(2)} = 0$, r is the constant interest rate, and Λ_1 and Λ_2 are combined market prices of risk. They are assumed to be bounded and only dependent on volatility factors (Y_t, Z_t).

Goal: Joint Survival Probability

$$
\begin{aligned}
u^{\epsilon,\delta} &\equiv u^{\epsilon,\delta}(t, x_1, x_2, y, z) \\
&\equiv \mathbb{P}^* \left\{ \tau_t^{(1)} > T, \tau_t^{(2)} > T \,\middle|\, \mathbf{X}_t = (x_1, x_2), Y_t = y, Z_t = z \right\}
\end{aligned}
$$

where $\tau_t^{(i)} = \inf \left\{ s \geq t |\, X_s^{(i)} \leq B_i(s) \right\}$.

Obviously $u^{\epsilon,\delta}(t, x_1, x_2, y, z) = 0$ if $x_1 \leq B_1(t)$ or $x_2 \leq B_2(t)$.

Therefore we focus on the case where $x_1 > B_1(t)$ and $x_2 > B_2(t)$.

PDE Representation

In terms of partial differential equations (PDE),

$$\mathcal{L}^{\epsilon,\delta} u^{\epsilon,\delta} \equiv \left[\frac{\partial}{\partial t} + \mathcal{L}_{(\mathbf{X},Y,Z)}\right] u^{\epsilon,\delta} = 0, \quad x_1 > B_1(t), x_2 > B_2(t)$$

$$u^{\epsilon,\delta}(t, B_1(t), x_2, y, z) = 0, \quad x_2 > B_2(t)$$

$$u^{\epsilon,\delta}(t, x_1, B_2(t), y, z) = 0, \quad x_1 > B_1(t)$$

$$u^{\epsilon,\delta}(T, x_1, x_2, y, z) = 1, \quad x_1 > B_1(t), x_2 > B_2(t)$$

where $\mathcal{L}_{(\mathbf{X},Y,Z)}$ is the infinitesimal generator of $(\mathbf{X}_t, Y_t, Z_t)$ under measure $\mathbb{P}^*$.

Operator Decomposition

$$\mathcal{L}^{\epsilon,\delta} = \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\epsilon}}\mathcal{M}_3.$$

$$\begin{aligned}
\mathcal{L}_0 &= (m_Y - y)\frac{\partial}{\partial y} + \nu_Y^2\frac{\partial^2}{\partial y^2} \\
\mathcal{L}_1 &= \nu_Y\sqrt{2}\left[\rho_{1Y}f_1x_1\frac{\partial^2}{\partial x_1\partial y} + \rho_{2Y}f_2x_2\frac{\partial^2}{\partial x_2\partial y} - \Lambda_1\frac{\partial}{\partial y}\right] \\
\mathcal{L}_2 &= \frac{\partial}{\partial t} + \frac{1}{2}f_1^2x_1^2\frac{\partial^2}{\partial x_1^2} + \frac{1}{2}f_2^2x_2^2\frac{\partial^2}{\partial x_2^2} + rx_1\frac{\partial}{\partial x_1} + rx_2\frac{\partial}{\partial x_2} \\
\mathcal{M}_1 &= \nu_Z\sqrt{2}\left[\rho_{1Z}f_1x_1\frac{\partial^2}{\partial x_1\partial z} + \rho_{2Z}f_2x_2\frac{\partial^2}{\partial x_2\partial z} - \Lambda_2\frac{\partial}{\partial z}\right]
\end{aligned}$$

Expansion of $u^{\epsilon,\delta}$

$$u^{\epsilon,\delta} = u_0 + \sqrt{\epsilon}u_{1,0} + \sqrt{\delta}u_{0,1} + \cdots$$

$\tilde{u} \equiv u_0 + \sqrt{\epsilon}u_{1,0} + \sqrt{\delta}u_{0,1}$ will be our approximation for the true value $u^{\epsilon,\delta}$ — First order approximation.

Term u_0

$$\begin{aligned} \langle \mathcal{L}_2 \rangle u_0 &= 0, \quad x_1 > B_1(t), x_2 > B_2(t) \\ u_0(t, B_1(t), x_2; z) &= 0, \quad x_2 > B_2(t) \\ u_0(t, x_1, B_2(t); z) &= 0, \quad x_1 > B_1(t) \\ u_0(T, x_1, x_2; z) &= 1, \quad x_1 > B_1(t), x_2 > B_2(t) \end{aligned}$$

where $\langle \cdot \rangle$ denotes the average (in y) with respect to $\mathcal{N}(m_Y, \nu_Y^2)$.

Note that u_0 depends on z in a parametric way.

Define

$$\bar{\sigma}_i(z) = \sqrt{\langle f_i^2(\cdot, z) \rangle}, \; i = 1, 2.$$

They are effective volatilities.

Probabilistic Representation for u_0

$$u_0 = \bar{\mathbb{E}}\left\{ \mathbf{1}_{\left\{ \inf_{t \le s \le T} \bar{X}_s^{(1)}/B_1(s) > 1, \inf_{t \le s \le T} \bar{X}_s^{(2)}/B_2(s) > 1 \right\}} \middle| \bar{\mathbf{X}}_t = (x_1, x_2) \right\}$$

where under measure $\bar{\mathbb{P}}$,

$$\begin{aligned} d\bar{X}_t^{(1)} &= r\bar{X}_t^{(1)} dt + \bar{\sigma}_1(z)\bar{X}_t^{(1)} d\bar{W}_t^{(1)} \\ d\bar{X}_t^{(2)} &= r\bar{X}_t^{(2)} dt + \bar{\sigma}_2(z)\bar{X}_t^{(2)} d\bar{W}_t^{(2)} \end{aligned}$$

with $d\bar{W}_t^{(1)} d\bar{W}_t^{(2)} = 0$. In other words, $\bar{X}_t^{(1)}$ and $\bar{X}_t^{(2)}$ are independent Geometric Brownian Motions.

Final Solution of u_0

$$
\begin{aligned}
u_0 \;&=\; \left[\mathsf{N}\left(d_2^{+(1)}\right) - \left(\frac{x_1}{B_1(t)}\right)^{p_1} \mathsf{N}\left(d_2^{-(1)}\right)\right] \\
&\quad \times \left[\mathsf{N}\left(d_2^{+(2)}\right) - \left(\frac{x_2}{B_2(t)}\right)^{p_2} \mathsf{N}\left(d_2^{-(2)}\right)\right]
\end{aligned}
$$

where $\mathsf{N}(\cdot)$ is the standard normal cumulative distribution function, and

$$
\begin{aligned}
d_2^{\pm(i)} \;&=\; \frac{\pm \ln(x_i/B_i(t)) + (r - \eta_i - \bar{\sigma}_i^2/2)(T-t)}{\bar{\sigma}_i\sqrt{T-t}} \\
p_i \;&=\; 1 - \frac{2(r-\eta_i)}{\bar{\sigma}_i^2}.
\end{aligned}
$$

Term $u_{1,0}$

$$\begin{aligned} \langle \mathcal{L}_2 \rangle u_{1,0} &= \mathcal{A} u_0, \quad x_1 > B_1(t), x_2 > B_2(t) \\ u_{1,0}(t, B_1(t), x_2; z) &= 0, \quad x_2 > B_2(t) \\ u_{1,0}(t, x_1, B_2(t); z) &= 0, \quad x_1 > B_1(t) \\ u_{1,0}(T, x_1, x_2; z) &= 0, \quad x_1 > B_1(t), x_2 > B_2(t) \end{aligned}$$

where $\mathcal{A} = \langle \mathcal{L}_1 \mathcal{L}_0^{-1} (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle$. It can be shown that

$$\begin{aligned} \sqrt{\epsilon}\mathcal{A} &= V_{11}^{(2)} x_1^2 \frac{\partial^2}{\partial x_1^2} + V_{22}^{(2)} x_2^2 \frac{\partial^2}{\partial x_2^2} \\ &+ V_{111}^{(3)} x_1 \frac{\partial}{\partial x_1} \left(x_1^2 \frac{\partial^2}{\partial x_1^2} \right) + V_{222}^{(3)} x_2 \frac{\partial}{\partial x_2} \left(x_2^2 \frac{\partial^2}{\partial x_2^2} \right) \\ &+ V_{112}^{(3)} x_2 \frac{\partial}{\partial x_2} \left(x_1^2 \frac{\partial^2}{\partial x_1^2} \right) + V_{122}^{(3)} x_1 \frac{\partial}{\partial x_1} \left(x_2^2 \frac{\partial^2}{\partial x_2^2} \right) \end{aligned}$$

where all V's are small of order $\sqrt{\epsilon}$ and only dependent on z.

Transformation of $u_{1,0}$

$$v_{1,0} = \sqrt{\epsilon} u_{1,0} + (T-t)\sqrt{\epsilon}\mathcal{A}u_0.$$

Then

$$\begin{aligned} \langle \mathcal{L}_2 \rangle v_{1,0} = 0, &\quad x_1 > B_1(t), x_2 > B_2(t) \\ v_{1,0}(t, B_1(t), x_2; z) = g_1(t, x_2), &\quad x_2 > B_2(t) \\ v_{1,0}(t, x_1, B_2(t); z) = g_2(t, x_1), &\quad x_1 > B_1(t) \\ v_{1,0}(T, x_1, x_2; z) = 0, &\quad x_1 > B_1(t), x_2 > B_2(t) \end{aligned}$$

where

$$\begin{aligned} g_1(t, x_2) &= (T-t) \lim_{x_1 \downarrow B_1(t)} \sqrt{\epsilon}\mathcal{A}u_0 \\ g_2(t, x_1) &= (T-t) \lim_{x_2 \downarrow B_2(t)} \sqrt{\epsilon}\mathcal{A}u_0 \end{aligned}$$

Change of Variables

Let $\{\xi_s^{(i)}\}_{t \leq s \leq T}$ be two independent Brownian Motions with drifts $\bar{\mu}_i = (r - \eta_i - \bar{\sigma}_i^2/2)/\bar{\sigma}_i$ under measure $\bar{\mathbb{P}}$, and

$$\xi_t^{(i)} = \xi_i = \frac{1}{\bar{\sigma}_i} \ln(x_i/B_i(t)).$$

Denote

$$\begin{aligned} v_{1,0}(t, x_1, x_2; z) &= \bar{v}_{1,0}(t, \xi_1, \xi_2; z) \\ g_1(t, x_2) &= \bar{g}_1(t, \xi_2) \\ g_2(t, x_1) &= \bar{g}_2(t, \xi_1) \end{aligned}$$

Probabilistic Representation of $\bar{v}_{1,0}$

$$\begin{aligned}\bar{v}_{1,0} &= \bar{\mathbb{E}}\Big\{\bar{g}_1(\bar{\tau},\xi^{(2)}_{\bar{\tau}})\mathbf{1}_{(\bar{\tau}=\bar{\tau}_1)}\mathbf{1}_{(\bar{\tau}\le T)} \\ &\quad +\bar{g}_2(\bar{\tau},\xi^{(1)}_{\bar{\tau}})\mathbf{1}_{(\bar{\tau}=\bar{\tau}_2)}\mathbf{1}_{(\bar{\tau}\le T)}\Big|\,\xi^{(1)}_t=\xi_1,\xi^{(2)}_t=\xi_2\Big\}\end{aligned}$$

for $\xi_1 > 0$ and $\xi_2 > 0$, where

$$\begin{aligned}\bar{\tau}_i &= \inf\Big\{s\ge t|\,\xi^{(i)}_s\le 0\Big\},\ i=1,2. \\ \bar{\tau} &= \min\{\bar{\tau}_1,\bar{\tau}_2\}.\end{aligned}$$

Two Probability Densities

$$\bar{\mathbb{P}}\left\{\bar\tau \in \mathrm{d}s, \bar\tau = \bar\tau_1, \xi^{(2)}_{\bar\tau} \in \mathrm{d}\xi \,\middle|\, \xi^{(1)}_t = \xi_1, \xi^{(2)}_t = \xi_2\right\}$$
$$= \frac{\xi_1}{\pi(s-t)^2} \exp\left\{\bar\mu_2\xi - \frac{\xi^2}{2(s-t)}\right\} \sinh\left(\frac{\xi\xi_2}{2(s-t)}\right)$$
$$\exp\left\{-\frac{\xi_1^2+\xi_2^2}{2(s-t)} - \bar\mu_1\xi_1 - \bar\mu_2\xi_2 - (\bar\mu_1^2+\bar\mu_2^2)(s-t)/2\right\} \mathrm{d}\xi\mathrm{d}s$$

$\bar{\mathbb{P}}\left\{\bar\tau \in \mathrm{d}s, \bar\tau = \bar\tau_2, \xi^{(1)}_{\bar\tau} \in \mathrm{d}\xi \,\middle|\, \xi^{(1)}_t = \xi_1, \xi^{(2)}_t = \xi_2\right\}$ can be obtained by symmetry.

Term $u_{0,1}$

$$
\begin{aligned}
\langle\mathcal{L}_2\rangle u_{0,1} = -\langle\mathcal{M}_1\rangle u_0, &\quad x_1 > B_1(t), x_2 > B_2(t) \\
u_{0,1}(t, B_1(t), x_2; z) = 0, &\quad x_2 > B_2(t) \\
u_{0,1}(t, x_1, B_2(t); z) = 0, &\quad x_1 > B_1(t) \\
u_{0,1}(T, x_1, x_2; z) = 0, &\quad x_1 > B_1(t), x_2 > B_2(t)
\end{aligned}
$$

It can be shown that

$$
\begin{aligned}
\sqrt{\delta}\langle\mathcal{M}_1\rangle u_0 \;=\; & V_1^{(0)}\frac{\partial u_0}{\partial \sigma_1} + V_2^{(0)}\frac{\partial u_0}{\partial \sigma_2} \\
& + V_{11}^{(1)} x_1 \frac{\partial}{\partial x_1}\left(\frac{\partial u_0}{\partial \sigma_1}\right) + V_{12}^{(1)} x_1 \frac{\partial}{\partial x_1}\left(\frac{\partial u_0}{\partial \sigma_2}\right) \\
& + V_{21}^{(1)} x_2 \frac{\partial}{\partial x_2}\left(\frac{\partial u_0}{\partial \sigma_1}\right) + V_{22}^{(1)} x_2 \frac{\partial}{\partial x_2}\left(\frac{\partial u_0}{\partial \sigma_2}\right)
\end{aligned}
$$

where all V's are small of order $\sqrt{\delta}$ and only dependent on z.

Transformation of $u_{0,1}$

Define operators M_1 and M_2 such that

$$\sqrt{\delta}\langle\mathcal{M}_1\rangle u_0 = M_1\frac{\partial u_0}{\partial \sigma_1} + M_2\frac{\partial u_0}{\partial \sigma_2}.$$

Now define

$$\begin{aligned} v_{0,1} &= \sqrt{\delta}u_{0,1} - (T-t)\left(M_1\frac{\partial u_0}{\partial \sigma_1} + M_2\frac{\partial u_0}{\partial \sigma_2}\right) \\ &\quad +\frac{1}{2}(T-t)^2\left(M_1\bar{\sigma}_1 x_1^2\frac{\partial^2 u_0}{\partial x_1^2} + M_2\bar{\sigma}_2 x_2^2\frac{\partial^2 u_0}{\partial x_2^2}\right) \end{aligned}$$

PDE for $v_{0,1}$

$$\begin{aligned} \langle \mathcal{L}_2 \rangle v_{0,1} = 0, &\quad x_1 > B_1(t), x_2 > B_2(t) \\ v_{0,1}(t, B_1(t), x_2; z) = g_3(t, x_2), &\quad x_2 > B_2(t) \\ v_{0,1}(t, x_1, B_2(t); z) = g_4(t, x_1), &\quad x_1 > B_1(t) \\ v_{0,1}(T, x_1, x_2; z) = 0, &\quad x_1 > B_1(t), x_2 > B_2(t) \end{aligned}$$

where

$$\begin{aligned} g_1(t, x_2) &= -(T-t) \lim_{x_1 \downarrow B_1(t)} \left(M_1 \frac{\partial u_0}{\partial \sigma_1} + M_2 \frac{\partial u_0}{\partial \sigma_2} \right) \\ &\quad + \frac{1}{2}(T-t)^2 \lim_{x_1 \downarrow B_1(t)} \left(M_1 \bar{\sigma}_1 x_1^2 \frac{\partial^2 u_0}{\partial x_1^2} + M_2 \bar{\sigma}_2 x_2^2 \frac{\partial^2 u_0}{\partial x_2^2} \right) \\ g_2(t, x_1) &= -(T-t) \lim_{x_2 \downarrow B_2(t)} \left(M_1 \frac{\partial u_0}{\partial \sigma_1} + M_2 \frac{\partial u_0}{\partial \sigma_2} \right) \\ &\quad + \frac{1}{2}(T-t)^2 \lim_{x_2 \downarrow B_2(t)} \left(M_1 \bar{\sigma}_1 x_1^2 \frac{\partial^2 u_0}{\partial x_1^2} + M_2 \bar{\sigma}_2 x_2^2 \frac{\partial^2 u_0}{\partial x_2^2} \right) \end{aligned}$$

Solution for $v_{0,1}$

By doing the same type of change of variables as in the case of $v_{1,0}$, $v_{1,0}$ can be represented as an expectation, which is essentially a double integral.

Model Calibration (ongoing work)

- Calibrate to individual default probability (or yield spread, CDS data,etc.) to obtain estimates for those parameters entirely determined by individual firms.

- Calibrate to default correlation (or implied correlation from Gaussian copula) to obtain estimates for those parameters solely connected to the dependency structure.

The End

Thank You