Computational Issues in Regime Switching Models

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Regime Switching Models

Suppose there are m discrete states or regimes

There is a controlled process S that, in regime r, is governed by

$$dS = \mu(S,r)\,dt + \sigma(S,r)\,dz$$

There is a flow of returns given by $f(S,r)$

There is a one-time reward of $R(S)$ from switching regimes

$R_{rq}(S)$ is the reward for switching from regime r to regime q when the stochastic state is S.

To avoid the possibility of infinite profits, it must be true that $R_{rq} + R_{qr} \leq 0$.

Also ensures that the switches must take place at isolated times or infinite switching costs are be incurred.

The agent uses a discount rate of $\rho(S)$

$$V(S,r) = \max E\left[\int_0^\infty \beta_t f(S_t, r_t) dt\right] + \sum_{i=1}^{\infty} \beta_{\tau_i} R(r_{\tau_i}, r^+_{\tau_i})$$

where τ_i are switch times and $\beta_t = e^{\int_0^t \rho(S_\tau) d\tau}$ is the discount factor

Solution Conditions

The solution can be characterized as a set, for each regime, of regions in the continuous state space on which no discrete switch is undertaken.

In the interior of the no-switch regions for regime r, the value function satisfies the Feynman-Kac equation

$$\rho V(S,r) = f(S,r) + \mu(S,r)V_S(S,r) + \tfrac{1}{2}\sigma^2(S,r)V_{SS}(S,r)$$

At the boundary points of the no-switch region, it is optimal to switch to one of the other regimes.

The value function must satisfy two conditions at such a point. First, the pre-switch value must equal the post-switch value plus the reward for switching (less the switching costs):

$$V(S^*,r) = V(S^*,q) + R_{rq}(S^*)$$

a condition known as value-matching.

At the optimal switching points the marginal values before switching must equal the marginal value after switching plus the marginal reward for switching (a condition known as smooth-pasting):

$$V_S(S^*,r) = V_S(S^*,q) + R'_{rq}(S^*)$$

Switching Models using the Complementarity Framework

Brekke and Oksendal (Theorem 3.4) have shown that the optimal value function $V(S,R)$ satisfies[2]

$$\rho(S)V(S,r) \geq f(S,R) + \mu(S,r)V_S(S,r) + \frac{\sigma^2(S,r)}{2}V_{SS}(S,r)$$

and the $m-1$ conditions

$$V(S,r) \geq V(S,x) + R_{rx}, \quad \forall x \neq r \tag{1}$$

Furthermore, one of these m conditions must be satisfied with equality at each (S,r).

Which one is satisfied with equality determines the optimal policy.

Thus, if $V(S,r) = V(S,x) + R_{rx}$, for some x, it is optimal at S to switch from r to x.

Otherwise it is optimal to remain in regime r and the first condition is satisfied with equality.

[2] If S is multidimensional, this condition is more accurately written as

$$\rho V \geq f + \sum_i \mu_i \frac{\partial V}{\partial S_i} + \frac{1}{2}\sum_i \sum_j \Sigma_{ij} \frac{\partial^2 V}{\partial S_i \partial S_j}$$

where $\Sigma_{ij} = \sum_k \sigma_{ik}\sigma_{jk}$. The simpler form is to avoid notational clutter.

Interpretation

The value function is the value of an asset that generates the payment flows.

By Ito's Lemma the expected rate of appreciation of the asset is

$$\frac{dE[V(S,r)]}{dt} = \mu(S,r)V_S(S,i) + \frac{\sigma^2(S,r)}{2}V_{SS}(S,r). \tag{2}$$

The total rate of return when regime r is active equals:
$f(S,r)$ the current return flow plus
$dE[V(S,r)]/dt$ the expected rate of capital appreciation

$$\rho(S)V(S,r) \geq f(S,r) + \mu(S,R)V_S(S,r) + \frac{\sigma^2(S,r)}{2}V_{SS}(S,r)$$

says that the rate of return obtainable by investing V dollars must be at least as great as the total rate of return generated by the assets if one remains in regime r.

$$V(S,r) \geq V(S,x) + R_{rx}, \quad \forall x \neq r \tag{3}$$

says that the value function must be at least as great as the value obtainable by switching regimes.

Mine Example

An example from Brekke and Oksendal

Consider a mine currently containing Q units of ore.

The mine is either idle ($r = 1$) or ore is extracted at rate hQ ($r = 2$) with a fixed cost of k incurred.

The transition equation for Q is thus

$$dQ = \begin{cases} 0 & \text{if } r = 1 \\ -hQdt & \text{if } r = 2 \end{cases} \tag{4}$$

The current price at which ore can be sold evolves according to a geometric Brownian motion

$$dP = \mu P dt + \sigma P dW \tag{5}$$

The flow of returns to the mine is

$$f(Q, P, R) = \begin{cases} 0 & \text{if } r = 1 \\ hQP - k & \text{if } r = 2 \end{cases} \tag{6}$$

The firm incurs fixed startup and shutdown costs of C_{12} and C_{21} and uses a fixed discount rate of ρ.

Mine Example - Solution

The solution conditions are

$$
\begin{aligned}
0 = \min \big(&\rho V(Q,P,1) - \mu P V_p(Q,P,1) - \tfrac{1}{2}\sigma^2 P^2 V_{PP}(Q,P,1), \\
&V(Q,P,1) - V(Q,P,2) + C_{12} \big)
\end{aligned} \tag{7}
$$

and

$$
\begin{aligned}
0 = \min \big(&\rho V(Q,P,2) - \mu P V_p(Q,P,2) - \tfrac{1}{2}\sigma^2 P^2 V_{PP}(Q,P,2) \\
&+ hQV_Q(Q,P,2) - (hQP - k),\; V(Q,P,2) - V(Q,P,1) + C_{21} \big)
\end{aligned} \tag{8}
$$

General PDE Methods for Switching Models

For now assume there are no exogenous jumps in S or R or any other complications

Recall that the solution condition for switching models is

$$0 = \min\left(\rho(S)V(S,i) - \mu(S,i)V'(S,i) + \frac{\sigma^2(S,i)}{2}V''(S,i) - f(S,i),\ \min_{j\neq i} V(S,i) - V(S,j) + R_{ij}(S)\right)$$

Suppose that we replace $V(S,i)$ with $\phi(S)\theta_i$

ϕ is a $1\times n$ vector of basis functions of your choosing

θ_i is an $n\times 1$ vector of coefficients to be determined

Furthermore, pick n nodal values of S: $s_k,\ k=1,\dots,n$

For each k we have the condition

$$0 = \min\left(A_{ki}\theta_i - f(s_k,i),\ \min_{j\neq i} \phi(s_k)\theta_i - \phi(s_k)\theta_j + R_{ij}(s_k)\right)$$

where

$$A_{ki} = \rho(s_k)\phi(s_k) - \mu(s_k,i)\phi'(s_k,i) + \frac{\sigma^2(s_k,i)}{2}\phi''(s_k,i)$$

General PDE Methods for Switching Models - continued

There are m conditions of this type for each s_k (one for each regime)

Notice that the linearity of the PDE and the linearity of the approximating function $\phi(S)\theta_i$ implies that the conditions are linear in θ_i

We can stack the conditions and vectorize the θ_i (so θ is an $nm \times 1$ vector

The result is a problem of the form

$$0 = \min(A_1\theta + b_1, \ldots, A_m\theta + b_m)$$

This is known as an Extended Vertical Linear Complementarity Problem (EVLCP)

It is a generalization of the more common LCP:

$$Mx + q \geq 0,\ x \geq 0,\ x^\top(Ax + b) = 0$$

Familiar to economists through Kuhn-Tucker conditions associated with inequality constrained optimization

Issues in Using the EVLCP Approach

Approximating functions (ϕ) must be chosen
- Value function is C^1 (or possibly C^2) so polynomial approximations are a poor choice
- Finite Element and Finite Difference methods work better
- Curse of dimensionality problems
- New approaches involve use of sparse grids and radial basis functions

Algorithm used to solve EVLCP (more on this to come)

Defining the decision rule
- Approximation at discrete nodes
- Poorest behavior at boundaries
- may need to smooth boundary

Solving EVLCPs

Projected Successive Over-relaxation (PSOR)
Modified Lemke Method (Complementary Pivoting)
Smoothing Newton

PSOR is essentially a function iteration approach:
  Pros: No linear solves needed
  Cons: Slow and exhibits convergence problems

Modified Lemke:
  Pros: Good convergence properties
  Cons: Very slow and requires linear solve

Smoothing Newton:
  Pros: Good convergence properties and relatively fast
  Cons: Requires linear solve

Smoothing Newton ala Qi & Liao

Let

$$g(x, \lambda) = \lambda \ln \left(\sum_{i=1}^{n} e^{x_i/\lambda} \right)$$

The entropy function g is a smooth approximation of the min function:
$\min(x_1, \ldots, x_n) = \lim_{\lambda \searrow 0} g(x, \lambda)$

Thus we can solve $0 = \min(x_1, \ldots, x_n)$ by solving $0 = g(x, \lambda)$ and letting $\lambda > 0$ go to 0

Qi & Liao propose an algorithm based on Newton's method (in x and λ) modified to prevent negative λ

Easy to modify the algorithm to include a finite termination criteria

Let $z = \max_i x_i$

$$g(x, \lambda) = z + \lambda \ln \left(\sum_{i=1}^{n} e^{(x_i - z)/\lambda} \right)$$

This form prevents overflow problems

New Approaches - Radial basis functions

$$V(S) \approx \sum_{i=1}^{n} \eta(\|S - s_i\|)c_i$$

Pros:

very flexible node placement

good approximation qualities

Cons:

Solving linear system is non trivial

New Approaches - Sparse Grids with Hierarchical Basis Functions

Sparse tensor products do not use products for which the sum of the levels exceeds a specified value

Pros:
- good approximation qualities
- interpolation problem is simple

Cons:
- node placement fixed
  - (adaptive schemes are under development)
- linear solve for PDEs is non-trivial

1-D

level	point
0	1/2
1	0
	1
2	1/4
	3/4

2-D

level	points	
0	1/2	1/2
1	0	1/2
	1	1/2
	1/2	0
	1/2	1
2	1/4	1/2
	3/4	1/2
	0	0
	1	0
	0	1
	1	1
	1/2	1/4
	1/2	3/4

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