1 Epstein-Zin-GARCH Asset Pricing Problem

Let C_t denote the monthly consumption endowment, let $c_t = \log C_t$, and let $\Delta c_t = c_t - c_{t-1}$. Similarly, let D_t denote the monthly dividend on a stock, let $d_t = \log D_t$, and let $\Delta d_t = d_t - d_{t-1}$. The driving processes are the consumption growth and dividend growth processes $\{\Delta c_t\}$ and $\{\Delta d_t\}$; collect them in the column vector $y_t = (\Delta c_t, \Delta d_t)'$. The location function of the driving processes is a VAR(1):

$$\mu_{t-1} = b_0 + B y_{t-1}.$$

The scale function is a BEKK(1,1):

$$\Sigma_{t-1} = R_0 R_0' + Q\Sigma_{t-2}Q' + P(y_{t-1} - \mu_{t-2})(y_{t-1} - \mu_{t-2})'P'.$$

Above, R_0 is an upper triangular matrix. The scale function is factored as $\Sigma_{t-1} = R_{t-1}R_{t-1}'$ where R_{t-1} is upper triangular. The driving process is, then,

$$y_t = \mu_{t-1} + R_{t-1}e_t.$$

We shall take the errors to be independent multivariate normal $N_2(0, I)$. Evidently, the state vector is

$$S_t = (y_t, y_{t-1}, \Sigma_{t-1}).$$

The parameter values are

$$b_0 = \begin{pmatrix} 0.0019012 \\ 0.0012394 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.030469 & -0.0048240 \\ 0.30314 & 0.061984 \end{pmatrix}$$

$$R_0 = \begin{pmatrix} 0.0010742 & 0.00075859 \\ 0 & 0.0012917 \end{pmatrix}$$

$$P = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$$

$$Q = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.8 \end{pmatrix}.$$

The Epstein-Zin-Weil utility function is

$$U_t = \left[(1-\delta)C_t^{\frac{1-\gamma}{\theta}} + \delta(\mathcal{E}_t U_{t+1}^{1-\gamma})^{\frac{1}{\theta}}\right]^{\frac{\theta}{1-\gamma}}$$

where γ is the coefficient of risk aversion,

$$\theta = \frac{1-\gamma}{1-1/\psi},$$

and ψ is the elasticity of intertemporal substitution. The values of these parameters are

$$\begin{aligned}\delta &= 0.999566 \\ \theta &= -12.2843 \\ \psi &= 2.0.\end{aligned}$$

Let P_{ct} denote the price of an asset that pays the consumption endowment and let

$$V_c(S_t) = \frac{P_{ct}}{C_t}$$

denote the corresponding price dividend ratio. The pricing function $V_c(S_t)$ is the solution to the nonlinear conditional integral equation

$$V_c(S_t) = \mathcal{E}_t\Big\{\delta^\theta \exp[-(\theta/\psi)\Delta c_{t+1} + (\theta-1)r_{c,t+1}]\,[1+V_c(S_{t+1})]\,\exp(\Delta c_{t+1})\Big\} \tag{1}$$

where

$$r_{c,t+1} = \log\left[\frac{1+V_c(S_{t+1})}{V_c(S_t)}\,\exp(\Delta c_{t+1})\right] \tag{2}$$

is the geometric return on the asset and $\mathcal{E}_t(\cdot) = \mathcal{E}(\cdot|S_t)$. Evidently, the marginal rate of substitution is

$$M_{t,t+1} = \delta^\theta \exp[-(\theta/\psi)\Delta c_{t+1} + (\theta-1)r_{c,t+1}].$$

The price dividend ratio $V_d(S_t) = P_{dt}/D_t$ on the asset that pays D_t is the solution to

$$V_d(S_t) = \mathcal{E}_t\Big\{\delta^\theta \exp\left[-(\theta/\psi)\Delta c_{t+1} + (\theta-1)r_{c,t+1}\right][1+V_d(S_{t+1})]\,\exp(\Delta d_{t+1})]\Big\}. \tag{3}$$

The price of the asset that pays \$1 with certainty is the solution to

$$V_f(S_t) = \mathcal{E}_t\Big\{\delta^\theta \exp\left[-(\theta/\psi)\Delta c_{t+1} + (\theta-1)r_{c,t+1}\right]\Big\}. \tag{4}$$

Once the pricing functions $V_c(S_t)$, $V_d(S_t)$, and $V_f(S_t)$ have been determined, the geometric stock return and the geometric risk free rate can be determined from

$$\begin{aligned} r_{dt} &= \log\left[\frac{1+V_d(S_t)}{V_d(S_{t-1})}\exp(\Delta d_t)\right] \\ r_{ft} &= -\log[V_f(S_t)]. \end{aligned}$$

The problem is to compute the pricing functions $V_c(S_t)$, $V_d(S_t)$, and $V_f(S_t)$ by solving the conditional integral equations (1) together with (2), (3), and (4).

The solutions are to be placed in the inner loop of an MCMC computation and must therefore be fast. They must also accurately track any curvature in the pricing functions because the objective of the overall exercise is to determine how complicated the driving process must be to accurately mimic the conditional moments of observed data. If the putative solution is nearly linear when the correct solution is not, then results will be biased toward finding complicated driving processes.

The approach used to date has been the following. Substitute (2) into (1) and denote the resulting conditional integral equation by

$$\mathcal{E}_t\{g[\Delta c_{t+1}, V_c(S_{t+1}), V_c(S_t)]\} = 0$$

Choose some basis functions such as the Hermite functions to the second order, arrange them in the column vector $b(S)$, approximate as $V_c(S) \doteq a'b(S)$, and substitute to obtain

$$\mathcal{E}_t\{g[\Delta c_{t+1}, a'b(S_{t+1}), a'b(S_t)]\} = 0 \tag{5}$$

The conditional integral equation (5) implies the system of unconditional integral equations

$$\mathcal{E}\{b(S_t)g[\Delta c_{t+1}, a'b(S_{t+1}), a'b(S_t)]\} = 0$$

which can be approximated using a long simulation of the driving process as

$$\frac{1}{N}\sum_{t=1}^{N} b(S_t)g[\Delta c_{t+1}, a'b(S_{t+1}), a'b(S_t)] = 0 \tag{6}$$

Typically N is on the order of 50,000. If the vector of basis functions has dimension d then (6) is a system of d nonlinear equations in the d unknowns $a = (a_0, a_1, \ldots, a_d)$. One can use a nonlinear equation solver to compute these coefficients. One can solve equations (3) and

(4) analogously. These two conditional integral equations are much simpler to solve because (2) is known once (1) together with (2) has been solved.

This approach has been sufficiently fast and accurate when the driving process has no GARCH terms (i.e. $P = Q = 0$). But GARCH terms expand the support of the distribution of the driving processes and this method has been unable to obtain accurate, stable solutions over the expanded support.

A reference is Bansal, Ravi, A. Ronald Gallant, and George Tauchen (2004) "Rational Pessimism, Rational Exuberance, and Markets for Macro Risks," Manuscript, Fuqua School of Business, Duke University, ftp://ftp.econ.duke.edu/pub/arg/papers/ms.pdf. This problem and solution method is described in more detail therein.

2 Log-Linear Stochastic Volatility Problem

Price a European option on the process labeled LL2VF in Chernov, Mikhail, A. Ronald Gallant, Eric Ghysels, and George Tauchen (2003), "Alternative Models for Stock Price Dynamics," *Journal of Econometrics* 116, 225–257, at the parameter settings given in Table 4 of that reference, which is repeated as Table 2 here.

Briefly, the LL2VF model is

$$\begin{pmatrix} \frac{dP_t}{P_t} \\ dU_{2t} \\ dU_{3t} \\ dU_{4t} \end{pmatrix} = \begin{pmatrix} \alpha_{10} + \alpha_{12} U_{2t} \\ \alpha_{22} U_{2t} \\ \alpha_{33} U_{3t} \\ \alpha_{44} U_{4t} \end{pmatrix} dt \quad +$$

$$\begin{pmatrix} e^{\beta_{10} + \beta_{13} U_{3t} + \beta_{14} U_{4t}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 + \beta_{33} U_{3t} & 0 \\ 0 & 0 & 0 & 1 + \beta_{44} U_{4t} \end{pmatrix} \bullet$$

$$\begin{pmatrix} \sqrt{1 - \psi_{13}^2 - \psi_{14}^2} & 0 & \psi_{13} & \psi_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dW_1 \\ dW_2 \\ dW_3 \\ dW_4 \end{pmatrix}$$

with parameter values for it and various simplifications given in Table 2 below. In Table 2 and in the article itself, the following conventions are observed:

- Reporting Conventions:

  Results are stated in annual time for the returns process

$$\frac{dP_t}{P_t} = \mu_t dt + \sigma_t dW_t$$

- Simulation Conventions:

  What is actually simulated is the log price process $U_t = \log(P_t)$, where

$$dU_t = \left(\mu_t - \frac{1}{2} \sigma_t^2 \right) dt + \sigma_t dW_t,$$

Table 2. Parameter Estimates and Standard Errors for Logarithmic Models

	LL1V		LL1VF		LL2V		LL2VI		LL2VF	
	Est	SE	Est	SE	Est	SE	Est	SE	Est	SE
α_{10}	0.0831	0.0200	0.0841	0.0195	0.0780	0.0337	0.0589	0.0225	0.0674	0.0279
α_{12}	0.6787	0.0513	0.8870	0.1397	0.9833	0.4820	1.1670	0.7872	2.1927	0.7281
α_{22}	-0.6087	0.8540	-1.0531	0.3425	-1.1219	1.0043	-1.1701	2.5137	-7.0195	5.9997
α_{33}	-6.3778	0.9315	-4.3016	0.3601	-0.0041	0.0291	-0.0512	0.0410	-0.1203	0.1227
α_{44}					-74.7610	10.7994	-52.6673	3.1107	-51.3082	8.2119
β_{10}	-2.2882	0.0320	-2.2585	0.0253	-2.0659	1.0294	-2.1969	0.0414	-2.2143	0.0486
β_{13}	1.3708	0.1059	1.2051	0.0410	0.0367	0.0261	0.0863	0.0400	0.1348	0.0695
β_{14}					3.5477	0.5162	2.7688	0.2597	2.7442	0.3130
β_{33}			0.5342	0.1168					0.0408	0.3672
β_{44}							1.9228	0.2260	2.2169	0.3655
ψ_{13}	-0.6482	0.0216	-0.6365	0.0107	-0.3382	0.3285	-0.2966	0.0240	-0.3403	0.1077
ψ_{14}					-0.3538	0.0793	-0.2915	0.0408	-0.2804	0.0564

Simulation is in annual time using an Euler scheme with 24 steps per day assuming 252 trading dates per year; i.e. $\Delta t = 1/6048$.

- Simulated Daily Returns Process:

$$y_s = 100(U_t - U_{t-24\Delta t})$$

where $t = s(24\Delta t),\ s = 1, 2, \ldots, N$.