

# Portfolio Choice with a Large Number of Assets: Jumps and Diversification

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- Portfolio problem when asset returns are generated by processes other than Brownian motion, such as stable processes or more general Lévy processes: see Han and Rachev (2000), Kallsen (2000), Choulli and Hurd (2001), Emmer and Kluppelberg (2004), Madan (2004).



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- So the existing papers have usually **solved numerically** the problem with a **single risky asset** (e.g., Liu, Longstaff and Pan (2003), Das and Uppal (2004)).

- But we show that by selecting a **well-chosen basis** dependent upon the structure of the jump process and the covariance matrix of returns, we can obtain closed-form solutions up to a constant.

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  - A misspecified problem where jumps are mistakenly lumped together with Brownian volatility.
- Example: worldwide asset allocation

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- The available investment opportunities consist of a riskless asset with price  $S_{0,t}$  and  $n$  risky assets with prices  $\mathbf{S}_t = [S_{1,t}, \dots, S_{n,t}]'$  with dynamics

$$\frac{dS_{0,t}}{S_{0,t}} = r dt,$$

$$\frac{dS_{i,t}}{S_{i,t-}} = (r + R_i) dt + \sum_{j=1}^n \sigma_{i,j} dW_{j,t} + J_i Z_t dN_t$$

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- We write  $\mathbf{R} = [R_1, \dots, R_n]'$ ,  $\mathbf{J} = [J_1, \dots, J_n]'$ , and

$$\sigma = \begin{pmatrix} \sigma_{1,1} & \cdots & \sigma_{1,n} \\ \vdots & \ddots & \vdots \\ \sigma_{n,1} & \cdots & \sigma_{n,n} \end{pmatrix}$$

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- $\sigma$  is assumed to be nonsingular.

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- In the absence of any other source of income, the investor's wealth, starting with the initial endowment  $X_0$ , is  $X_t = \sum_{i=0}^n \pi_{i,t}$ .
- The investor's **wealth dynamics** are given by

$$\begin{aligned} dX_t &= \pi_{0,t} \frac{dS_{0,t}}{S_{0,t-}} + \sum_{i=1}^n \pi_{i,t} \frac{dS_{i,t}}{S_{i,t-}} \\ &= (X_t - \pi_t' \mathbf{1}) r dt + \sum_{i=1}^n \pi_{i,t} \frac{dS_{i,t}}{S_{i,t-}}. \end{aligned}$$



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- The **discounted wealth dynamics** are given by

$$dY_t = \omega'_t \mathbf{R} dt + \omega'_t \sigma d\mathbf{W}_t + \omega'_t \mathbf{J} Z_t dN_t.$$

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- The investor's problem at time  $t$  is to pick the portfolio vector process  $\{\omega_s\}_{t \leq s \leq T}$  which maximizes the expected utility of terminal wealth,  $V(Y_t, t)$ , subject to the dynamics of his discounted wealth.

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- Using stochastic dynamic programming and the appropriate form of Itô's lemma for semi-martingale processes, the **Hamilton-Jacobi-Bellman** equation characterizing the optimal solution to the investor's problem is:

$$0 = \frac{\partial V(Y_t, t)}{\partial t} + \max_{\{\omega_t\}} \left\{ \frac{\partial V(Y_t, t)}{\partial Y} \omega_t' \mathbf{R} + \frac{1}{2} \frac{\partial^2 V(Y_t, t)}{\partial Y^2} \omega_t' \Sigma \omega_t + \lambda \int [V(Y_t + \omega_t' \mathbf{J} z, t) - V(Y_t, t)] \nu(dz) \right\}.$$

where  $\Sigma = \sigma \sigma'$ .

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- We can look for a solution in the form

$$V(y, t) = -e^{K(T-t)} e^{-\gamma y}$$

where  $K$  is constant. In which case, the HJB equation becomes

$$0 = -K + \min_{\{\omega_t\}} \{g(\omega_t)\}.$$

- The objective function

$$g(\omega) = -\gamma\omega'\mathbf{R} + \frac{1}{2}\gamma^2\omega'\Sigma\omega + \lambda \int_0^\infty [e^{-\gamma\omega'\mathbf{J}z} - \mathbf{1}] v(dz),$$

is time independent, strictly convex, goes to  $+\infty$  in all directions, and hence always has a **unique time independent minimizer** which is proportional to  $\gamma^{-1}$ :

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- The issue is that with  $n$  assets, this is an  $n$ -dimensional equation to be minimized.

- In the **pure diffusive case**,  $\lambda = 0$  and we obtain of course the familiar solution

$$\omega^* = \frac{1}{\gamma} \Sigma^{-1} \mathbf{R}.$$

- So far, this is pretty much standard material.

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- To begin, we consider the simplest possible case, where the  $n$  risky assets have the **same jump size and expected excess return** characteristics

$$\mathbf{J} = \bar{J}\mathbf{1} \quad \text{and} \quad \mathbf{R} = \bar{R}\mathbf{1}$$

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- To fix ideas, let us assume that  $\bar{J} < 0$  in order to capture the **downward risk** inherent in the types of jumps we are concerned about.

- As for  $\Sigma$ , we assume the **one factor structure**

$$\Sigma = v^2 \begin{pmatrix} 1 & \rho & \cdots \\ \rho & \cdots & \rho \\ \cdots & \rho & 1 \end{pmatrix}$$

where  $v^2 > 0$  is the **variance** of the returns generated by the diffusive risk, and  $0 < \rho < 1$  their common **correlation** coefficient.

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- The key to characterizing the optimal portfolio solution in this simple situation is to **decompose both  $\Sigma$  and  $\omega$  on a well chosen basis**, consisting in this case of the  $n$ -vector  $\mathbf{1}$  and its orthogonal hyperplane.

- Specifically, the spectral decomposition of the  $\Sigma$  matrix is:

$$\Sigma = \underbrace{\kappa_1 \frac{1}{n} \mathbf{1}\mathbf{1}'}_{\bar{\Sigma}} + \underbrace{\kappa_2 \left( \mathbf{I} - \frac{1}{n} \mathbf{1}\mathbf{1}' \right)}_{\Sigma^\perp}$$

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- $\kappa_1$  has multiplicity 1 and eigenvector  $\mathbf{1}$  and  $\kappa_2$  has multiplicity  $n - 1$ .

- Let us decompose the **portfolio weight vector**  $\omega$  according to

$$\omega = \bar{\omega}\mathbf{1} + \omega^\perp$$

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- Then, from above, the optimal  $\bar{\omega}^*$  and  $\omega^{\perp*}$  must satisfy

$$\begin{aligned} (\omega^{\perp*}, \bar{\omega}^*) = \arg \min_{\{\omega^\perp, \bar{\omega}\}} & \left\{ -\gamma n \bar{\omega} \bar{R} + \frac{1}{2} \gamma^2 n \bar{\omega}^2 \kappa_1 \right. \\ & \left. + \lambda \int_0^\infty \left[ e^{-\gamma n \bar{\omega} \bar{J} z} - 1 \right] v(dz) + \frac{1}{2} \gamma^2 \omega^{\perp'} \Sigma^\perp \omega^\perp \right\}. \end{aligned}$$

- This separates into an optimization problem for  $\omega^\perp$  and a separate one for the scalar  $\bar{\omega}$  :

$$(\omega^{\perp*}, \bar{\omega}^*) = \arg \min_{\{\omega^\perp, \bar{\omega}\}} \{g^\perp(\omega^\perp) + \bar{g}(\bar{\omega})\}$$

where

$$g^\perp(\omega^\perp) = \frac{1}{2}\gamma^2\omega^{\perp'}\kappa_2\left(\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}'\right)\omega^\perp$$

$$\bar{g}(\bar{\omega}) = -\gamma n\bar{\omega}\bar{R} + \frac{1}{2}\gamma^2 n^2\bar{\omega}^2\kappa_1\frac{1}{n}$$

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- The optimal solution for  $\omega^\perp$  in this case is obviously  $\omega^{\perp*} = \mathbf{0}$ .

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- With the change of variable  $\varpi_n = n\bar{\omega}$ , we see that

$$\varpi_n^* = \arg \min_{\{\varpi\}} \left\{ -\gamma\varpi\bar{R} + \frac{1}{2}\gamma\varpi^2\kappa_1/n + \lambda \int_0^\infty \left[ e^{-\gamma\varpi\bar{J}z} - 1 \right] v(dz) \right\}.$$

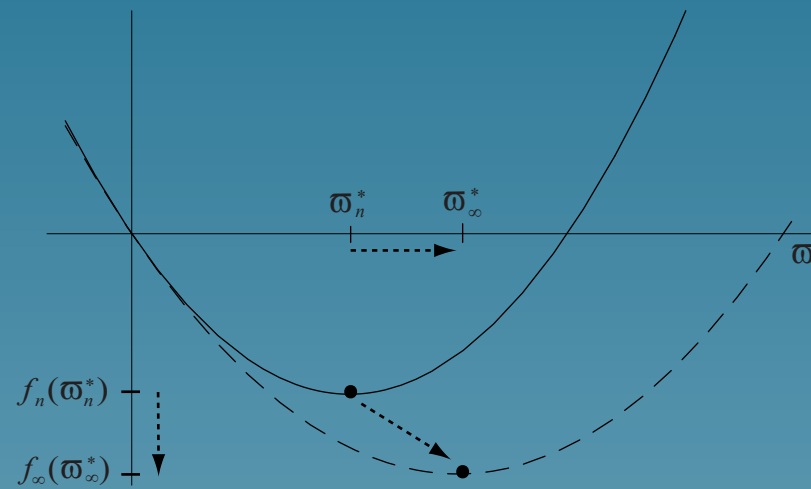
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- Letting  $n \rightarrow \infty$ , we have that  $\kappa_1/n \rightarrow v^2\rho$  and so  $\varpi_n^* \rightarrow \varpi_\infty^*$  where

$$\varpi_\infty^* = \arg \min_{\{\varpi\}} \left\{ -\gamma\varpi\bar{R} + \frac{1}{2}\gamma^2\varpi^2v^2\rho + \lambda \int_0^\infty \left[ e^{-\gamma\varpi\bar{J}z} - 1 \right] v(dz) \right\}.$$



- The optimal portfolio choice is characterized by

$$\omega^* = \bar{\omega}^* \mathbf{1} = \varpi_n^* \mathbf{1}/n$$

- So  $Y_t$  follows an **arithmetic Lévy process** with

$$b = \varpi_n^* \bar{R}$$

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- Decomposing on the same basis as above, let

$$\mathbf{R} = \bar{R}\mathbf{1} + \mathbf{R}^\perp$$

with  $\mathbf{R}^{\perp'}\mathbf{R}^\perp = O(n)$ , while  $\mathbf{J}^\perp = \mathbf{0}$ , so that  $\mathbf{J} = \bar{J}\mathbf{1}$ .

- The optimal portfolio solution, as above, can be decomposed as  $\omega = \bar{\omega}\mathbf{1} + \omega^\perp$  so that the optimization problem for  $\omega$  again separates into  $g^\perp(\omega^\perp)$  and  $\bar{g}(\bar{\omega})$ .

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- As before, in the limit where the number of assets  $n \rightarrow \infty$ , we have  $\varpi_n^* \rightarrow \varpi_\infty^*$ .

- Here, since  $\mathbf{R}^\perp/\mathbf{R}^\perp = O(n)$  and  $\kappa_1 = O(n)$  while  $\kappa_2 = O(1)$ , we have that  $b$  and  $c$  are  $O(n)$ , while the Lévy measure remains  $O(1)$  as  $n \rightarrow \infty$ .

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- Thus, compared to the case where  $\mathbf{R}^\perp = \mathbf{0}$ , the presence of  $\mathbf{R}^\perp$  with  $\mathbf{R}^{\perp'}\mathbf{R}^\perp = O(n)$  allows the CARA investor to optimize in such a way as to increase their expected return at the expense of increased variance resulting from the diffusive risk, while keeping the exposure to jumps fixed.

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- Because the investor dumps all the risky assets and goes long on the riskless asset.

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## 4. Optimal Portfolio in the Presence of Jumps in a Multi-Sector Economy

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- We are interested in the situation where  $m$  is fixed and  $k$  goes to infinity with  $n$ .

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$$\Sigma_{n \times n} = \begin{pmatrix} \Sigma_1 & 0 & \cdots \\ 0 & \ddots & 0 \\ \cdots & 0 & \Sigma_m \end{pmatrix}$$

is a block diagonal matrix with blocks

$$\Sigma_l_{k \times k} = v_l^2 \begin{pmatrix} 1 & \rho_l & \cdots \\ \rho_l & \ddots & \rho_l \\ \cdots & \rho_l & 1 \end{pmatrix}$$

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$$\begin{aligned}\Sigma &= \sum_{l=1}^m \kappa_{1l} \frac{1}{k} \mathbf{1}_l \mathbf{1}_l' + \sum_{l=1}^m \kappa_{2l} \left( \mathbf{F}_l - \frac{1}{k} \mathbf{1}_l \mathbf{1}_l' \right) \\ &= \bar{\Sigma} + \Sigma^\perp\end{aligned}$$

where

$$\begin{aligned}\kappa_{1l} &= v_l^2 + v_l^2 (k - 1) \rho_l \\ \kappa_{2l} &= v_l^2 (1 - \rho_l)\end{aligned}$$

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- The multiplicity of each  $\kappa_{1l}$  is 1, and the multiplicity of each  $\kappa_{2l}$  is  $k - 1$ . The eigenvector for  $\kappa_{1l}$  is  $\mathbf{1}_l$ , the  $n$ -vector with ones placed in the rows corresponding to the  $l$ -block and zeros everywhere else, that is

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- Corresponding to the above spectral structure, we have the **orthogonal decomposition**  $\mathbb{R}^n = \bar{V} \oplus V^\perp$  where  $\bar{V}$  is the span of  $\{\kappa_{1l}\}_{l=1, \dots, m}$ .

- As for the jump vector  $\mathbf{J}$  in our  $m$ -sector economy, we assume that  $\mathbf{J} \in \bar{V}$ :

$$\mathbf{J} = \sum_{l=1}^m j_l \mathbf{1}_l = [\underbrace{j_1, \dots, j_1}_{\text{sector 1}}, \underbrace{j_2, \dots, j_2}_{\text{sector 2}}, \dots, \underbrace{j_m, \dots, j_m}_{\text{sector } m}]'$$

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meaning that **firms within a given sector have the same response to the arrival of a Poisson jump**, i.e., to a change in  $N_t$ .

- But the **proportional response** of firms of **different sectors** to the arrival of a jump can be **different**.
- Finally, we assume that the vector of expected excess returns has the form

$$\mathbf{R} = \sum_{l=1}^m r_l \mathbf{1}_l + \mathbf{R}^\perp = \bar{\mathbf{R}} + \mathbf{R}^\perp.$$



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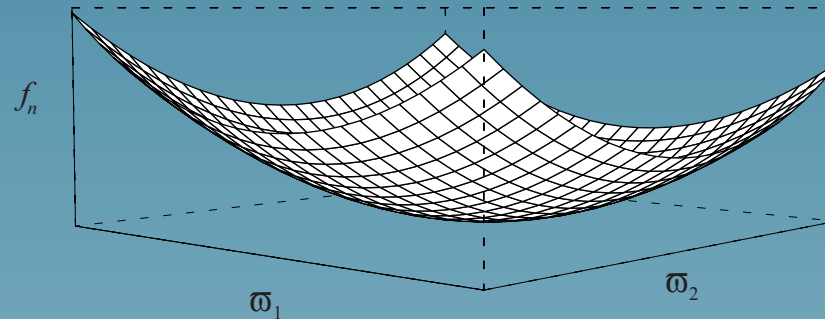
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- The optimal solution for  $\omega^{\perp*}$  has components

$$\omega_l^{\perp*} = \frac{1}{\gamma \kappa_{2l}} \mathbf{R}_l^\perp.$$

- The optimal solution for  $\bar{\omega}$ , with the change of variable  $\varpi_n = k\bar{\omega}$  satisfies

$$\varpi_n^* = \arg \min_{\{\varpi\}} \left\{ -\gamma \sum_{l=1}^m \varpi_l r_l + \frac{1}{2} \gamma^2 \sum_{l=1}^m \varpi_l^2 \kappa_{1l} / k + \lambda \int_0^\infty \left[ e^{-\gamma z} \sum_{l=1}^m \varpi_l j_l - 1 \right] v(dz) \right\}$$

- Letting  $k \rightarrow \infty$ , we get  $\varpi_n^* \rightarrow \varpi_\infty^*$ .



- With  $\mathbf{R}_l^\perp / \mathbf{R}_l^\perp = O(k)$ , the optimal portfolio can achieve expected gains at the expense of variance which both **grow approximately linearly with  $k$**
- While **keeping the exposure to jumps bounded** because the increasing fraction of assets is allocated to the component of the expected returns that lies in the subspace orthogonal to the vectors  $\mathbf{1}_l$ .

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- This  $\Sigma$  matrix has three distinct eigenvalues

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$$\kappa_2 = v^2 - v^2 k \rho_0 + v^2 (k - 1) \rho$$

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- This  $\Sigma$  matrix has three distinct eigenvalues

$$\kappa_1 = v^2 + v^2 k (m - 1) \rho_0 + v^2 (k - 1) \rho$$

$$\kappa_2 = v^2 - v^2 k \rho_0 + v^2 (k - 1) \rho$$

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- With  $\mathbf{R}_l^\perp / \mathbf{R}_l^\perp = O(k)$ , the optimal portfolio can achieve expected gains at the expense of variance which both **grow approximately linearly with  $k$**
- While **keeping the exposure to jumps bounded** because the increasing fraction of assets is allocated to the subspace orthogonal to the vectors  $\mathbf{1}_l$ .

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- Suppose now that the true model is as before but **the investor thinks instead that the risky assets are Gaussian processes** (with no jumps):

$$\frac{dS_{i,t}}{S_{i,t-}} = (r + \hat{R}_i) dt + \sum_{j=1}^n \hat{\sigma}_{i,j} dW_{j,t}, i = 1, \dots, n.$$

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- In this case the **total expected excess returns** are

$$\hat{R}_i dt = E_t \left[ \frac{dS_{i,t}}{S_{i,t-}} \right] - r dt = \left( R_i + J_i \lambda \int z v(dz) \right) dt$$

and the **total variance-covariance matrix**  $\hat{\Sigma} = \hat{\sigma} \hat{\sigma}'$  is given by

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- Now the investor thinks that  $\omega_t^{NJ*}$  has to satisfy:

$$\omega^{NJ*} = \arg \min_{\{\omega^{NJ}\}} g(\omega^{NJ})$$

where

$$g(\omega^{NJ}) = -\gamma (\omega^{NJ})' \hat{\mathbf{R}} + \frac{1}{2} \gamma^2 (\omega^{NJ})' \hat{\Sigma} \omega^{NJ}$$



- We assume a one sector economy with different expected excess returns, then

$$\begin{aligned}\hat{\mathbf{R}} &= \left( \bar{R} + \lambda \bar{J} \int z v(dz) \right) \mathbf{1} + \mathbf{R}^\perp, \\ \hat{\Sigma} &= \left( \bar{\Sigma} + \lambda \bar{J}^2 \int z^2 v(dz) \right) \mathbf{1}\mathbf{1}' + \Sigma^\perp, \\ \omega^{NJ} &= \bar{\omega}^{NJ} \mathbf{1} + \omega^\perp,\end{aligned}$$

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- The minimization problem becomes

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- Thus an investor who recognizes the presence of jumps will leave larger amounts in the riskless asset than one who simply adjusts his returns vector and variance-covariance matrix.
- This is a **higher moment effect**.

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- Parameter values (annualized):  $r = 0.06$ ,  $\rho = 0.5$ ,  $\rho_0 = 0.3$ ,  $v = 0.3$ ,  $\gamma = 2$ .

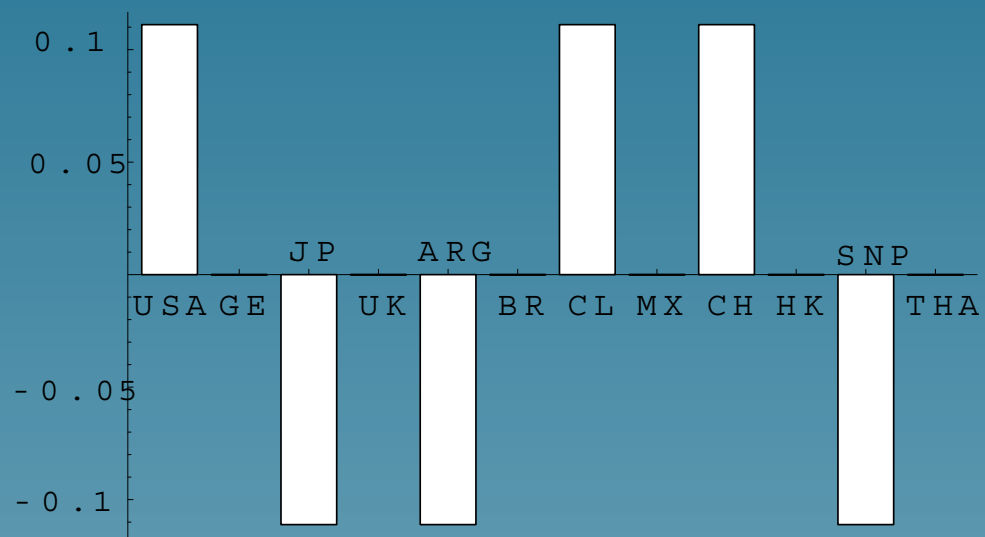
Region 1	USA	GE	JP	UK
$R_i$	0.04	0.03	0.02	0.03
$J_i$	-1.00			
Region 2	ARG	BR	CL	MX
$R_i$	0.06	0.07	0.08	0.07
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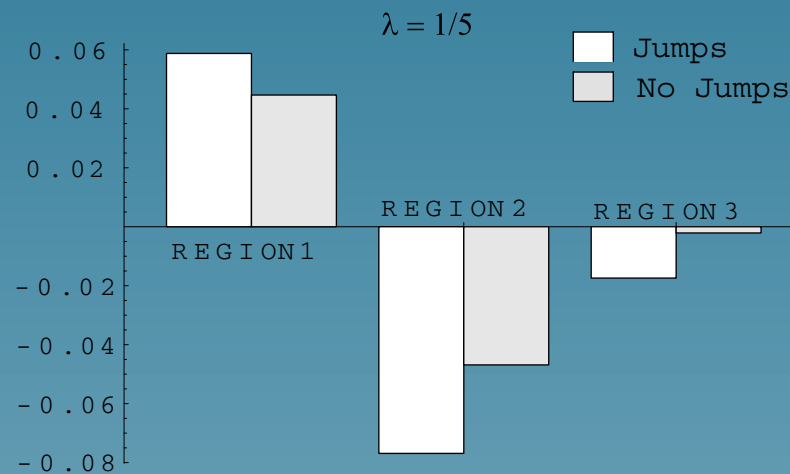
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- $\omega^{\perp*}$  captures only diversification within regions:



- $\bar{\omega}^* = (\bar{\omega}_1^*, \bar{\omega}_2^*, \bar{\omega}_3^*)$  and  $\bar{\omega}^{NJ*} = (\bar{\omega}^{NJ*}, \bar{\omega}^{NJ*}, \bar{\omega}^{NJ*})$  for  $\lambda = 1/5$



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- By decomposing the two types of risks on a well-chosen basis, we provide a new methodology for **determining the optimal solution in closed form, up to a constant.**

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