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Tutorial on Financial Mathematics

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Financial Mathematics/Engineering

- Use of *stochastic models* to quantify uncertainty in prices and other economic variables.
- Often departs from classical economics by modeling at a *phenomenological* level.
- Tools derived from probability theory, differential equations, functional analysis, among others.
- Education :
  - Dozens of Master's programs, plus PhD and undergraduate programs in Math, OR, Statistics departments.
  - Great demand for Masters/PhD students with quantitative training in financial math. Demand increases in economic downtimes.
  - Enormous textbook industry.

Relation to Practice

- Interaction with *financial services industry* has gone both ways:
  - Beauty of the Black-Scholes theory (1973) spurred growth of options markets.
  - Development of "structured products" (esp. in credit markets - CDOs, CDO^2s) currently way ahead of mathematical technology.
- Very rapid transition from academic theory to (at least) testing in investment houses.

Relation to Academia

- A fair dose of *give and take* ...
- Early on (Harrison-Pliska, 1981) : *Thus the parts of probability theory most relevant to the general question [of market completeness] are those results, usually abstract in appearance and French in origin, which are invariant under substitution of an equivalent measure.*
- Last 25 years : Many new mathematical and computational challenges created from financial applications.

Outline

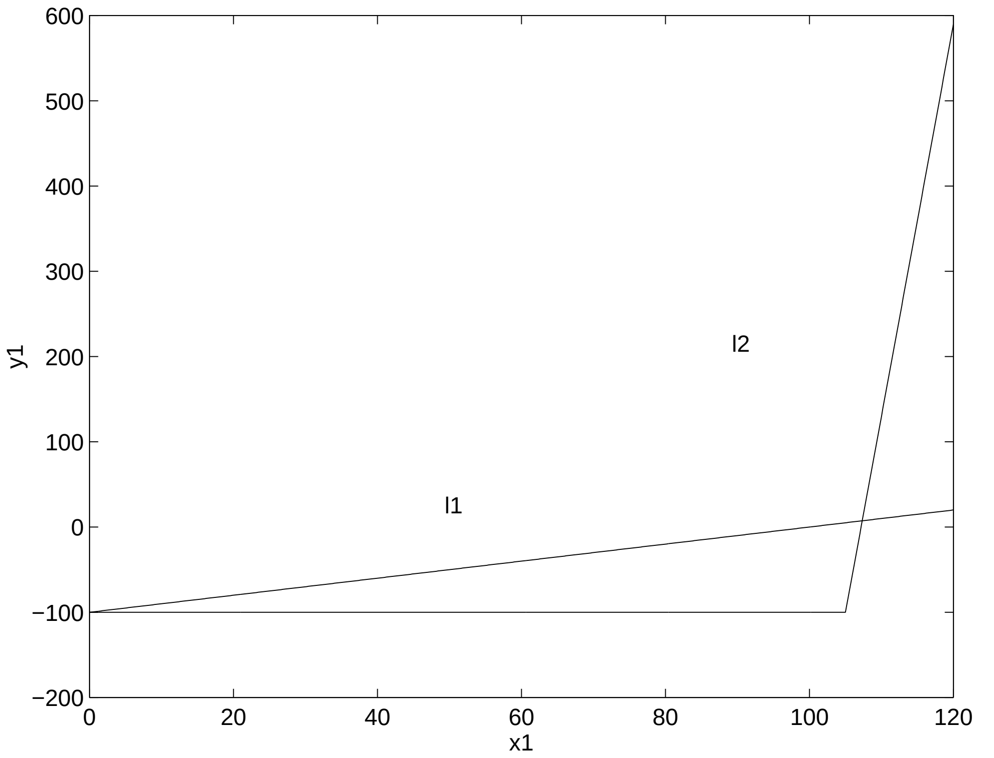
- Option Pricing
  - Simple binomial tree model
  - Continuous-time Black-Scholes theory
- Portfolio Optimization - Merton problem.
- Derivatives + Portfolio Optimization : utility indifference pricing, convex risk measures.
- Credit Risk .

Options + Derivative Securities

- Call Option on a Stock: Contract giving the holder the right, but not the obligation to buy one share on expiration date T for the strike price $\$K$.
- Investor is betting that the stock price will exceed $\$K$ by date T.
- Large profits if correct, but he/she loses everything if wrong.

Example

- Investing in stocks vs. investing in options.
- $K = \$105$, $T = 6$ months, today's stock price $= \$100$.
- With \$1000, can buy 10 shares or 461 call options.



One-Period Binomial Tree Model

- One time period of length T years.
- Current stock price S_0 . Stock goes up to uS_0 with probability p , or down to dS_0 with probability $1 - p$.

$$uS_0$$

$$S_0$$

$$dS_0$$

Option Payoffs

- Let $h(S_T)$ denote the payoff of the (European) option.

$$\text{Call} : \; h(S_T) = (S_T - K)^+ \quad \text{Put} : \; h(S_T) = (K - S_T)^+$$

- Tree for the option:

$$h(uS_0)$$

$$P_0?$$

$$h(dS_0)$$

- What is the fair price P_0?

$$P_0 \neq e^{-rT} \left(ph(uS_0) + (1-p)h(dS_0) \right).$$

Replicating Strategy

- Look for a *replicating portfolio* whose payoff at maturity is identical to (replicates) the option's in both up and down states .
- The portfolio is a strategy or investment which involves buying a stocks and investing $\$b$ in the bank at time zero.
- If a or b are negative the stock is short-sold or the money is borrowed from the bank. There are no further adjustments to the portfolio till date T.
- The cost of setting up the portfolio at time zero is Π_0 and

$$\Pi_0 = aS_0 + b.$$

Replication Conditions

- Tree for the replicating portfolio tree is

$$auS_0 + be^{rT}$$

$$aS_0 + b$$

$$adS_0 + be^{rT}$$

- For replication, we must solve

$$\begin{aligned} auS_0 + be^{rT} &= h(uS_0) \\ adS_0 + be^{rT} &= h(dS_0) \end{aligned}$$

for a and b.

No Arbitrage Condition

- We get

$$a = \frac{h(uS_0) - h(dS_0)}{(u-d)S_0},$$

$$b = \frac{uh(dS_0) - dh(uS_0)}{e^{rT}(u-d)}.$$

- If there is not to be an arbitrage opportunity , the price of the option P_0 must equal the cost of setting up the portfolio that replicates it.

- Therefore,

$$P_0 = \frac{1 - de^{-rT}}{u-d} h(uS_0) + \frac{ue^{-rT} - 1}{u-d} h(dS_0).$$

Observations

- Hedging ratio given by

$$a = \frac{h(uS_0) - h(dS_0)}{uS_0 - dS_0} \approx \frac{\partial P}{\partial S}?$$

- Option price is determined (uniquely) by enforcing *no arbitrage.*

- The probability p played no role .

Risk-Neutral Probability

- Recall that P_0 is *not* given by

$$e^{-rT}\left(ph(uS_0) + (1-p)h(dS_0)\right),$$

where p is *our* specified probability (belief).

- Re-write the option pricing formula as

$$P_0 = e^{-rT}\left(\frac{e^{rT}-d}{u-d}h(uS_0) + \frac{u-e^{rT}}{u-d}h(dS_0)\right).$$

- Define

$$q = \frac{e^{rT}-d}{u-d},$$

and notice then that

$$P_0 = e^{-rT}\left(qh(uS_0) + (1-q)h(dS_0)\right),$$

because

$$\frac{e^{rT} - d}{u - d} + \frac{u - e^{rT}}{u - d} = 1.$$

- q is a probability as long as $d < e^{rT} < u$.

To cut a long story short ...

- Under very general conditions (S is a semi-martingale ...), absence of arbitrage is equivalent to the existence of a risk-neutral (equivalent martingale) probability measure Q, under which the discounted price of any traded security is a martingale.

- Consequence : The price P_0 of a claim paying the random amount G on date T is given by

$$P_0 = \mathbb{E}^Q\{e^{-rT}G\}.$$

- Complete Market: Q is unique .

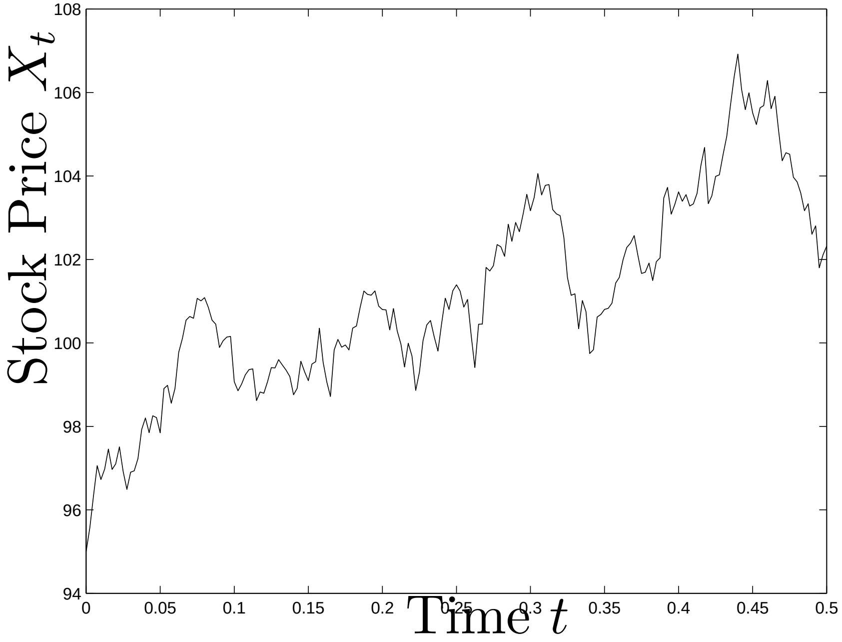
- Otherwise (more common): many EMMs Q and market is incomplete .

Some references

- Binomial tree & Risk-neutral measure: Cox-Ross-Rubinstein (1979)
- Discrete & Continuous time: Harrison-Kreps (1979); Harrison-Pliska (1981).
- General semi-martingale theory: Delbaen-Schachermayer (1994-98) .

Samuelson Geometric Brownian Motion Model

Stock price random walk model X_t



$$\frac{dX_t}{X_t} = \mu\, dt + \sigma\, dW_t,$$

where σ = volatility ($CONSTANT$).

Black-Scholes Argument for Option Pricing

- Portfolio: buy one option P_t and sell Δ_t stocks

$$\Pi_t = P_t - \Delta_t X_t.$$

- Choose Δ_t to exactly balance the risks.
- If the combined portfolio can be made riskless , then the market should price the option so that this investment yields exactly the same as putting the money in the bank instead: *no arbitrage* .

Incrementally ...

- Portfolio is self-financing

$$d\Pi_t = dP_t - \Delta_t dX_t.$$

- Assume $P_t = P(t, X_t)$. Then, by Itô's Formula

$$dP_t = \left(\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 P}{\partial x^2} \right) dt + \frac{\partial P}{\partial x}\, dX_t.$$

- Therefore

$$d\Pi_t = \left(\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 P}{\partial x^2} \right) dt + \left(\frac{\partial P}{\partial x} - \Delta_t \right) dX_t.$$

- Choose

$$\Delta_t = \frac{\partial P}{\partial x}(t, X_t)$$

to exactly balance the risks.

No Arbitrage Argument

- With this choice, portfolio is perfectly hedged (over the infinitesimal time period).
- To exclude the possibility of an arbitrage opportunity , the riskless portfolio must grow as if we had invested the amount $\$\Pi_t$ in the bank.
- It must grow at the (risk-free) interest rate r: $d\Pi_t = r\Pi_t \, dt$.
- Since

$$\Pi_t = P - \Delta_t X_t = P - X_t \frac{\partial P}{\partial x},$$

this gives

$$\left(\frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2 X_t^2 \frac{\partial^2 P}{\partial x^2} \right) dt = r \left(P - X_t \frac{\partial P}{\partial x} \right) dt.$$

- Result: Price of option $P_t = P(t, x)$ at time t when $X_t = x$ is given by a formula.

- The pricing function $P(t, x)$ is found by solving a *partial differential equation*:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 P}{\partial x^2} + r\left(x\frac{\partial P}{\partial x} - P\right) = 0,$$

with terminal condition $P(T, x) = (x - K)^+$.

- Hedging Strategy: Sell $\Delta_t = \frac{\partial P}{\partial x}(t, X_t)$ shares at time t.
ELIMINATES RISK

- Need only estimate historical volatility σ from past price data.

Success of Black-Scholes

- Simple Black-Scholes pricing formula for European call option.
- Parameter estimation simple

$$\text{Model } (\mu, \sigma) \longrightarrow \text{Need } \sigma \ .$$

The μ has vanished. Projections of μ highly variable and reflect modeller's expectations of the stock. Projections of σ (probably) less subjective: take the historical estimate.

- Modern viewpoint : European option prices are set by the market and are observed data .
- Theory/modelling is used to price other exotic derivative securities consistent with these "vanilla " contracts.

Implied Volatility

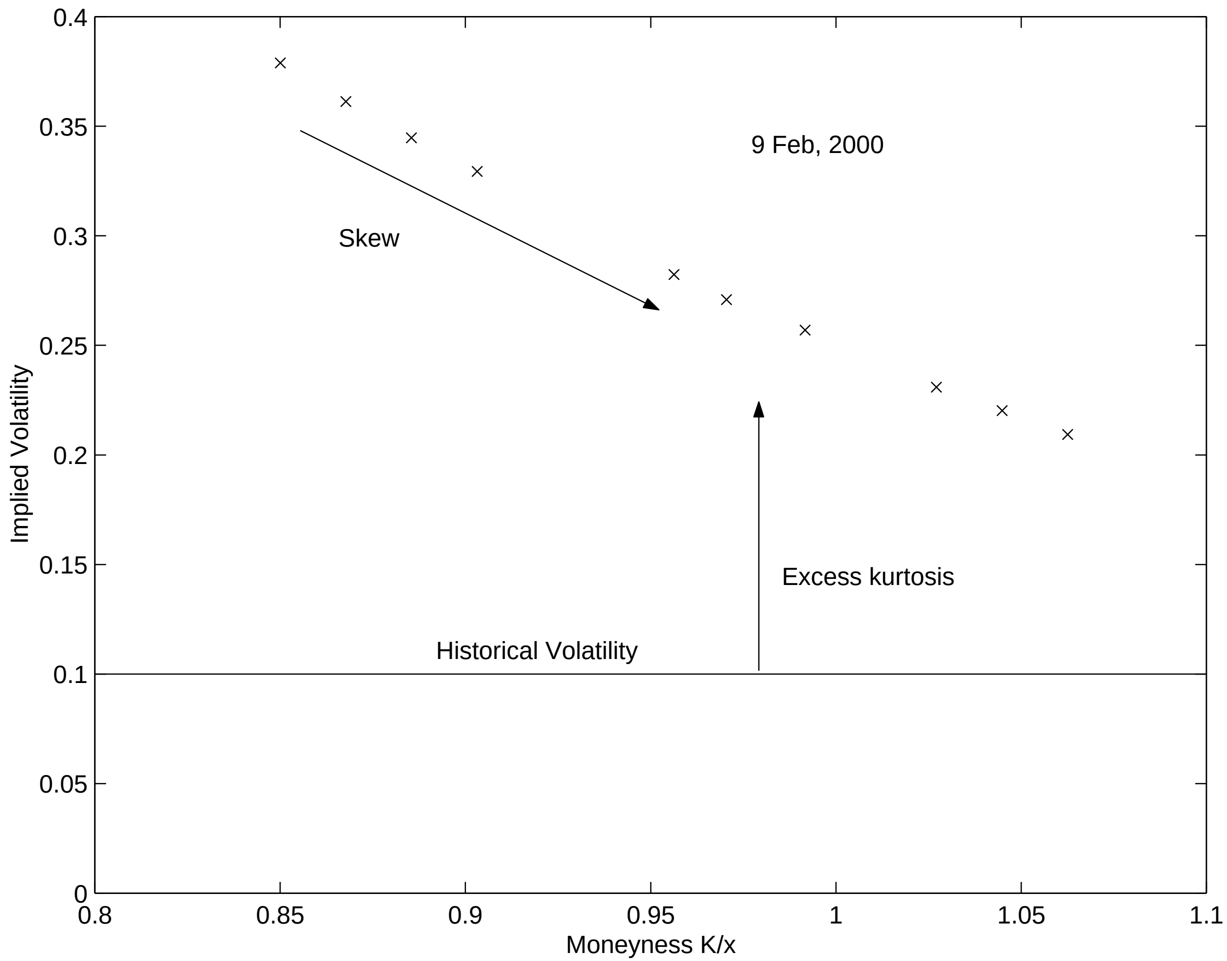
- Observed European option prices are usually quoted in terms of implied volatility I

$$P^{\text{obs}} = P(I).$$

Note: Easy to compute I because of explicit formula for P.

- If market actually priced according to Black-Scholes theory, then we would get $I \equiv \sigma$, the constant *historical* volatility from options of all strikes and expiration dates.

Implied Volatility Smile Curve/Skew



- Upwards shift from historical volatility and skewed .

Stochastic Volatility

- Motivation
  1. Estimates of historical volatility are not constant, have "random" characteristics.
  2. Implied volatility skew or smile.
  3. Heavy-tailed and skewed returns distributions.
  4. Other market frictions.
- Volatility σ_t is a stochastic process

$$dX_t = \mu X_t \, dt + \sigma_t X_t \, dW_t.$$

Stochastic Volatility Models

- Volatility σ_t is a stochastic process

$$dX_t = \mu X_t \, dt + \sigma_t X_t \, dW_t,$$

introduced by Hull-White, Wiggins, Scott 1987, typically a Markovian Itô process: $\sigma_t = f(Y_t)$,

$$dY_t = \alpha(Y_t)dt + \beta(Y_t)d\hat{Z}_t,$$

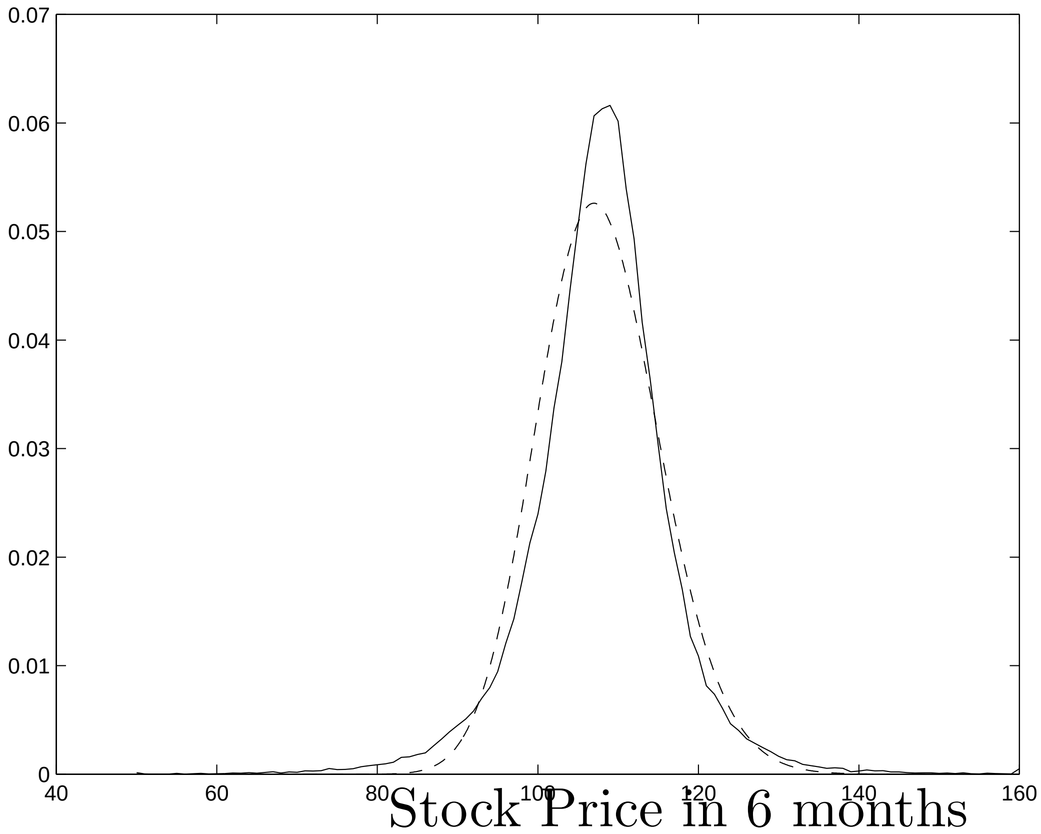
$(\hat{Z}_t)$ =Brownian motion, correlation ρ

$$\mathbb{E}\{dW_t \, d\hat{Z}_t\} = \rho \, dt.$$

- Problems
  - How to model the volatility process: α, β, f?
  - Derivative prices $P(t, X_t, Y_t)$ need today's volatility (unobserved).
  - Estimation of parameters of a *hidden* process.

Generic Models

For a generic stochastic volatility model we obtain excess kurtosis (peakedness) in stock price distribution. With correlation $\rho < 0$ obtain heavy left-tail over lognormal model.



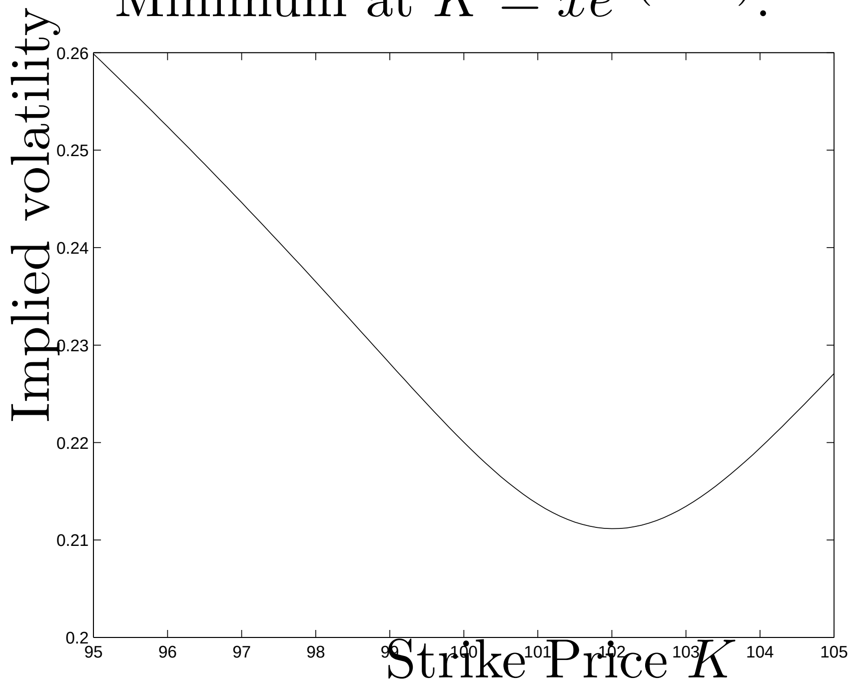
Robust Results

- Renault-Touzi (1992)

Stochastic Volatility $\Longrightarrow$ *SMILE*

$\rho = 0$, subordination

Minimum at $K = xe^{r(T-t)}$.



- Genuine *smiles* were typically observed before the 1987 crash .

Skews & NonZero Correlation

- Numerical simulations & small-fluctuation asymptotic results suggest

$$\rho < 0 \Longrightarrow \text{Downward sloping implied volatility}$$
$$\rho > 0 \Longrightarrow \text{Upward sloping implied volatility} \ .$$

- Robust to specific modeling of the volatility.

- Other approaches: Jumps.

Overview

- Merton portfolio optimization problem.
- Utility indifference pricing mechanism.
- Optimal investment with derivative securities. Static-dynamic hedging.

Some History

- Samuelson 1960s: Brownian motion based models in economics.
- Merton 1969: utility maximization problem; explicit solution in special cases (fixed mix investments).
- Black-Scholes 1973: solution of the option pricing problem (under constant volatility); *perfect* replication/hedging of options.
- Karatzas *et al.* 1986: mathematical study of optimization problems in finance via convex duality.
- Kramkov & Schachermayer 1999: quite general duality theory for semimartingale models.
- Hodges & Neuberger 1989: utility indifference pricing mechanism.

Merton Problem (1969/1971)

- How to optimally invest capital between a risky stock and a riskless bank account?
- Objective function: expected utility of terminal wealth .
- Let X_T be the portfolio value at time T (fixed). Want to maximize

$$\mathbb{E}\{U(X_T)\},$$

where U is an *increasing* and *concave* function. E.g.:

  - $U(x) = x^p/p$, $p < 1, p \neq 0$, $x \in \mathbb{R}^+$.
  - $U(x) = \log x$, $x \in \mathbb{R}^+$.
  - $U(x) = -e^{-\gamma x}$, $x \in \mathbb{R}$.

Control

- The control (π_t) is a real-valued *non-anticipating* process representing the dollar amount held in stock . (The original Merton papers include a controlled continuous *consumption rate*, which we ignore here).
- Geometric Brownian motion model for stock price (S_t) :

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t,$$

where (W_t) = Brownian motion; $\sigma =$ volatility.

- Let X_t be the value of the portfolio (or wealth).

$$dX_t = \frac{\pi_t}{S_t} dS_t + r(X_t - \pi_t)\, dt.$$

$$\rightarrow dX_t = (rX_t + \pi_t(\mu - r))\, dt + \sigma \pi_t\, dW_t.$$

Objective

- Want to maximize $\mathbb{E}\{U(X_T)\}$.
- Introduce value function

$$M(t,x) = \sup_{\pi} \mathbb{E}\left\{U(X_T) \mid X_t = x\right\}.$$

- Consider the associated *Hamilton-Jacobi-Bellman* (HJB) equation

$$M_t + rxM_x + \sup_{\pi}\left(\frac{1}{2}\sigma^2\pi^2 M_{xx} + \pi(\mu - r)M_x\right) = 0,$$

in $t < T$, with terminal condition $M(T,x) = U(x)$.

- The optimization of the quadratic in $\pi \in \mathbb{R}$ gives (in the absence of constraints)

$$\pi^* = -\frac{(\mu - r)}{\sigma^2} \frac{M_x}{M_{xx}}$$

and

$$\max = -\frac{(\mu - r)^2}{2\sigma^2} \frac{M_x^2}{M_{xx}},$$

so the HJB becomes

$$M_t + rxM_x - \frac{(\mu - r)^2}{2\sigma^2} \frac{M_x^2}{M_{xx}} = 0.$$

Power Utility

- Work with
$$U(x) = \frac{x^p}{p}, p < 1, x \in I\!R^+.$$
Admissible strategies such that
$$X_t \geq 0 \qquad \text{a.s. for } t \in [0, T].$$
- Now have terminal condition
$$M(T, x) = \frac{x^p}{p}.$$
Look for a separable solution of the form
$$M(t, x) = \frac{x^p}{p} g(t).$$

- Substituting the *ansatz* into the HJB gives

$$\frac{x^p}{p}\left(g' + rpg - \frac{(\mu - r)^2 p}{2\sigma^2(p-1)} g\right) = 0,$$

with $g(T) = 1$.

- Final value function is

$$M(t, x) = \frac{x^p}{p} \exp\left(\left(r + \frac{(\mu - r)^2}{2\sigma^2(1-p)} p(T - t)\right)\right).$$

- More importantly

$$\pi_t^* = \frac{(\mu - r)}{\sigma^2(1-p)} X_t.$$

The optimal strategy is to hold the fixed fraction

$$\frac{(\mu - r)}{\sigma^2(1-p)}$$

(the *Merton ratio*) of wealth in the stock.

Exponential Utility

- Work with

$$U(x) = -e^{-\gamma x}, x \in I\!R.$$

- Recall

$$dX_t = (rX_t + \pi_t(\mu - r))\, dt + \sigma \pi_t \, dW_t.$$

Let $\hat{X}_t = X_t e^{-rt}$and call $\hat{\pi}_t = \pi_t e^{-rt}$. Then

$$\begin{aligned} d\hat{X}_t &= -r\hat{X}_t\, dt + e^{-rt} dX_t \\ &= \hat{\pi}_t \hat{\mu}\, dt + \sigma \hat{\pi}_t \, dW_t, \end{aligned}$$

where $\hat{\mu} = \mu - r$. W.l.o.g. we consider $r = 0$ case.

- Now have terminal condition

$$M(T, x) = -e^{-\gamma x}.$$

- Look for a separable solution of the form

$$M(t, x) = -e^{-\gamma x} g(t).$$

- Substituting the *ansatz* into the HJB gives

$$-e^{-\gamma x} \left(g' - \frac{\mu^2}{2\sigma^2} g \right) = 0,$$

with $g(T) = 1$.

- Final value function is

$$M(t, x) = -e^{-\gamma x} \exp\left(-\frac{\mu^2}{2\sigma^2} (T - t) \right).$$

- More importantly

$$\pi_t^* = \frac{\mu}{\gamma\sigma^2}.$$

The optimal strategy is to hold the fixed amount

$$\frac{\mu}{\gamma\sigma^2}$$

(the *Merton ratio*) in the stock.

Summary

- Solution to Merton optimal investment problem is explicit in certain cases. Having explicit formulas makes verification relatively straightforward.
- Generalizes to cases of multiple stocks with different rates of return (a vector μ) and a variance-covariance matrix Σ. The μ/σ^2 in the Merton ratios is replaced by $(\Sigma\Sigma^T)^{-1}\mu$.
- General *complete markets* theory using replicability of $\mathcal{F}_T-$measurable claims G by dynamic trading strategies

$$G = x + \int_0^T \pi_t \frac{dS_t}{S_t}$$

  is well-established (see *e.g.* book by Karatzas & Shreve (1998)).
- Interesting problems: adding derivative securities in *incomplete markets*.

Derivative Pricing in Incomplete Markets

- Stock price process

$$dS_t = \mu S_t \, dt + \sigma(Y_t) S_t \, dW_t$$

- Volatility-driving process

$$dY_t = b(Y_t)\, dt + a(Y_t) \left(\rho \, dW_t + \rho' dZ_t\right).$$

  (W_t) and (Z_t) are *independent* Brownian motions on a probability space $(\Omega, \mathcal{F}, I\!P)$ and $\rho' = \sqrt{1-\rho^2}$.

- Volatility is *not* a traded asset, and the market is said to be incomplete.

- A (European) derivative security pays $g(S_T)$ on expiration date T in the future.

- Consequence of incompleteness: this payoff *cannot* be replicated by trading the stock.
- Black-Scholes case: σ =const. and there is a *unique* equivalent martingale measure $I\!P^\star$ under which the traded asset is a martingale:

$$dS_t = \sigma S_t \, dW_t^\star,$$

with $(W_t^\star)$ a $I\!P^\star-$Brownian motion.

- The derivative price is

$$h(S,t) = I\!E^\star\{g(S_T) \mid S_t = S\},$$

and this function solves the Black-Scholes PDE problem

$$\begin{aligned} h_t + \frac{1}{2}\sigma^2 S^2 h_{SS} &= 0, \\ h(S,T) &= g(S). \end{aligned}$$

Volatility Risk-Premium & *No-arbitrage* Pricing

- Basic theorem in finance: for there to be no arbitrage opportunities, there must exist some equivalent probability measure under which prices of traded securities are *martingales*.
- In a diffusion model, equivalent measures $I\!\!P^\star$ are "generated" by a Girsanov transformation: add a drift to the Brownian motions

$$dW_t \mapsto dW_t^\star - c_t\,dt \qquad dZ_t \mapsto dZ_t^\star - \lambda_t\,dt.$$

- For (S_t) to be a $I\!P^{\star}$−martingale, $c_t = -\mu/\sigma(Y_t)$, but λ_t is arbitrary and called the *volatility risk premium*. It parameterizes the set of equivalent martingale measures.

- Under $I\!P^{\star(\lambda)}$,

$$
\begin{aligned}
dS_t &= \sigma(Y_t) S_t \, dW_t^{\star} \\
dY_t &= \left[b(Y_t) - \rho\mu \frac{a(Y_t)}{\sigma(Y_t)} - \rho' a(Y_t)\lambda_t \right] dt \\
&\quad + a(Y_t) \left(\rho \, dW_t^{\star} + \rho' dZ_t^{\star} \right).
\end{aligned}
$$

Utility-Indifference Pricing

- Investor's wealth process (X_t)

$$dX_t = \pi_t \frac{dS_t}{S_t} = \mu \pi_t \, dt + \sigma(Y_t) \pi_t \, dW_t,$$

where π_t is the amount held in the stock at time t.

- Utility function $U(x) = -e^{-\gamma x}$: increasing, concave; defined on $x \in \mathbb{R}$; $\gamma > 0$ is called the *risk-aversion coefficient*.

- Derivative writer's problem : maximize expected *utility* of wealth at time T after paying out to derivative holder

$$V(x, S, y, t) = \sup_{\pi} \mathbb{E} \left\{ -e^{-\gamma(X_T - g(S_T))} \mid X_t = x, S_t = S, Y_t = y \right\}.$$

The starting wealth is denoted x.

- Classical Merton problem: maximum utility from trading the stock (no derivative liability).

$$M(x, y, t) = \sup_{\pi} I\!E \left\{ -e^{-\gamma X_T} \mid X_t = x, Y_t = y \right\}.$$

- Utility-indifference (writer's) price $h(x, S, y, t)$ of the derivative is defined by

$$M(x, y, t) = V(x + h(x, S, y, t), S, y, t),$$

the compensation to the derivative writer such that he/she is *indifferent* in terms of maximum expected utility to the liability from the short position.

- References: Hodges-Neuberger (1989), Davis-Panas-Zariphopoulou (1990).

- In the constant volatility *complete* case, this recovers the Black-Scholes price (in fact for *any* utility function).

Indifference Pricing PDE

- If the utility is exponential, h is independent of initial wealth x: $h = h(S, y, t)$. (This is essentially if and only if).
- From the Hamilton-Jacobi-Bellman (HJB) equation for V and the HJB equation for M, we get a (quasilinear) PDE problem for h:

$$\begin{aligned} h_t + \widetilde{\mathcal{L}}_{S,y} h + \frac{1}{2} a(y)^2 (1-\rho^2)\gamma h_y^2 &= 0, \\ h(S,y,T) &= g(S), \end{aligned}$$

$$\begin{aligned} \widetilde{\mathcal{L}}_{S,y} = {} & \frac{1}{2}\sigma(y)^2 S^2 \frac{\partial^2}{\partial S^2} + \rho\sigma(y)a(y)S\frac{\partial^2}{\partial S \partial y} + \frac{1}{2}a(y)^2\frac{\partial^2}{\partial y^2} \\ & + \left[b(y) - \rho\mu\frac{a(y)}{\sigma(y)} + a(y)^2\frac{m_y}{m}\right]\frac{\partial}{\partial y}, \end{aligned}$$

the infinitesimal generator of (S_t, Y_t) under the martingale measure $\mathbb{P}_m^\star$.

- Under the martingale measure $I\!P^\star_m$, (S_t, Y_t) follows

$$
\begin{aligned}
dS_t &= \sigma(Y_t) S_t \, dW_t^\star \\
dY_t &= \left[b(Y_t) - \rho\mu \frac{a(Y_t)}{\sigma(Y_t)} + a(Y_t)^2 \frac{m_y}{m} \right] dt \\
&\quad + a(Y_t) \left(\rho \, dW_t^\star + \rho' dZ_t^\star \right).
\end{aligned}
$$

- The function $m(y,t)$ in the coefficient comes from the value function of the Merton problem

$$
M(x,y,t) = -e^{-\gamma x} m(y,t)^{1/(1-\rho^2)}.
$$

It satisfies a *linear* PDE and does not depend on γ:

$$
m_t + \frac{1}{2} a(y)^2 m_{yy} + \left[b(y) - \rho\mu \frac{a(y)}{\sigma(y)} \right] m_y - \frac{\mu^2 (1-\rho^2)}{2\sigma(y)^2} m = 0,
$$

with $m(y,T) = 1$.

Option on Non-Traded Asset

- Suppose the option is on the volatility: $g = g(Y_T)$. The interpretation is that Y is a non-traded asset like temperature on which we have a weather derivative which we try and hedge with a correlated asset S like electricity.

- Now S disappears from the problem and $h = h(y,t)$. The indifference pricing PDE is just

$$\begin{aligned} h_t + \widetilde{\mathcal{L}}_y h + \frac{1}{2}a(y)^2(1-\rho^2)\gamma h_y^2 &= 0, \\ h(y,T) &= g(y), \end{aligned}$$

$$\widetilde{\mathcal{L}}_y = \frac{1}{2}a(y)^2\frac{\partial^2}{\partial y^2} + \left[b(y) - \rho\mu\frac{a(y)}{\sigma(y)} + a(y)^2\frac{m_y}{m}\right]\frac{\partial}{\partial y}.$$

Transformation to a Linear PDE

- The reduction in dimension allows a Hopf-Cole-type transformation $h = k \log \phi$ so that

$$h_y = k\frac{\phi_y}{\phi}, \qquad h_{yy} = k\left(\frac{\phi_{yy}}{\phi} - \frac{\phi_y^2}{\phi^2}\right),$$

so the PDE becomes

$$\frac{k}{\phi}\left(\phi_t + \widetilde{\mathcal{L}}_y\phi\right) - k\frac{1}{2}a(y)^2\frac{\phi_y^2}{\phi^2} + \frac{1}{2}a(y)^2\gamma k^2\frac{\phi_y^2}{\phi^2} = 0.$$

- Therefore, choosing

$$k = \frac{1}{\gamma(1-\rho^2)},$$

gives the linear PDE $\phi_t + \widetilde{\mathcal{L}}_y\phi = 0$, with $\phi(y,T) = \exp(\gamma(1-\rho^2)g(y)$.

Probabilistic representation

$$h(y,t) = \frac{1}{\gamma(1-\rho^2)} \log \mathbb{E}_{t,y}^{\mathbb{P}_m^\star} \left\{ e^{\gamma(1-\rho^2) g(Y_T)} \right\}.$$

Summary

- In Markovian models, indifference price is typically characterized by quasilinear PDE problem. [This can be transformed to a linear problem when there is only one space variable *e.g.* non-traded asset case.]
- Indifference pricing seems a reasonable preference-based valuation mechanism in illiquid, OTC markets *e.g.* for some credit derivatives . Perhaps less so in liquid equity markets where *no arbitrage* valuation is possible.
- However, the original investment problem with derivatives and stocks is of interest in such cases, and its solution has an interpretation in terms of the indifference price.

Problem Statement

- In an incomplete market, an investor would like to use derivatives to indirectly trade untradeable risks.
- Example: Using straddles to be “long volatility”.
- How many derivative contracts to buy to *maximize expected utility*?
- Or how many vanilla options to optimally *hedge* an exotic options position?
- Tractable under exponential utility.

Simplest Setting

- Investor has initial capital $\$x$.
- He/she can invest *dynamically* in a stock (and bank acct.) and *statically* in a single derivative security that pays G on date T.
- Market price of the derivative is $\$p$.
- Investor buys and holds λ derivatives and trades his/her remaining $\$(x - \lambda p)$ continuously in the Merton portfolio (stock & bank account).

Notation

- Value of the Merton portfolio is $(X_t)_{0\leq t\leq T}$, and (π_t) is the amount held in the stock (the dynamic control). Throughout, interest rate $r = 0$.
- Let
$$u(x;\lambda) = \sup_{\pi} \mathbb{E}\left\{-e^{-\gamma(X_T+\lambda G)} \mid X_0 = x\right\},$$
where $\gamma > 0$ is the risk-aversion parameter.
- Objective:
$$\max_{\lambda} u(x - \lambda p; \lambda).$$

Duality with Relative Entropy Minimization

If G is bounded and the price process S is locally bounded, then Delbaen, Grandits, Rheinlander, Samperi, Schweizer & Stricker (2002) show

$$u(x;\lambda) = -e^{-\gamma x} e^{-\gamma \inf_Q \left[\mathbb{E}^Q\{\lambda G\} + \frac{1}{\gamma} H(Q|\mathbb{P}) \right]},$$

where $\mathbb{P}$ is the subjective measure, $Q \in P_f$ with

$$P_f = \{\text{ALMMs with finite relative entropy}\},$$

and

$$H(Q \mid \mathbb{P}) = \mathbb{E}\left\{ \frac{dQ}{d\mathbb{P}} \log \frac{dQ}{d\mathbb{P}} \right\} = E^Q \left\{ \log \frac{dQ}{d\mathbb{P}} \right\}.$$

Utility-Indifference Price

- Let $b(\lambda)$ be the buyer's *utility-indifference price* of λ derivatives:

$$u(x;\lambda) = u(x + b(\lambda); 0).$$

- By duality,

$$u(x;\lambda) = -e^{-\gamma(x+b(\lambda))} e^{-\inf_Q H(Q|\mathbb{P})},$$

so that

$$u(x - \lambda p;\lambda) = -e^{-\gamma(x+b(\lambda)-\lambda p)} e^{-\inf_Q H(Q|\mathbb{P})}.$$

- The problem is reduced to

$$\max_{\lambda} b(\lambda) - \lambda p,$$

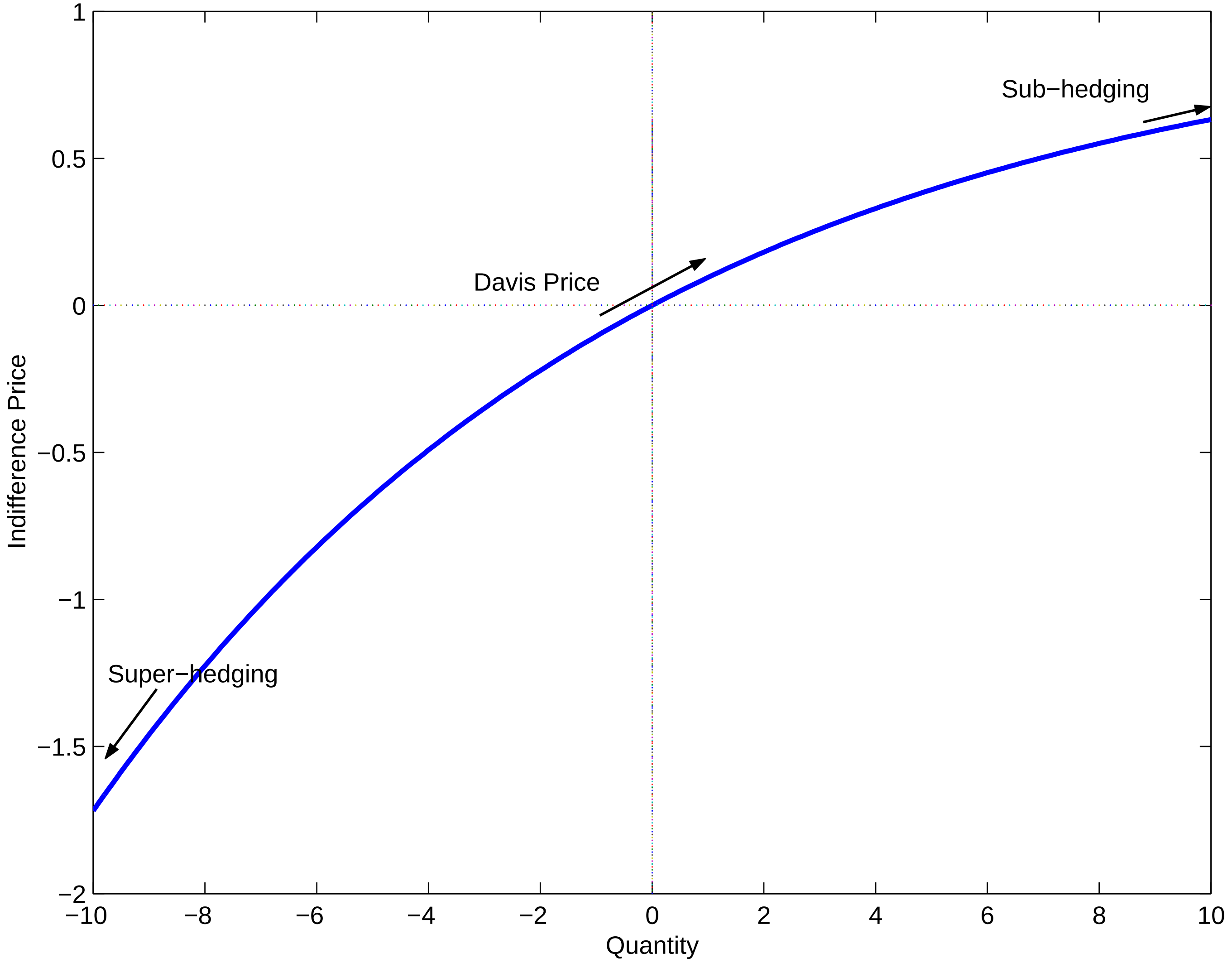
the *Fenchel-Legendre transform of the indifference price at the market price p.*

Characterization of the Indifference Price

- From the duality,

$$b(\lambda) = \inf_{Q} \left\{ \lambda \mathbb{E}^{Q}\{G\} + \frac{1}{\gamma} H(Q \mid \mathbb{P}) \right\} - \inf_{Q} \frac{1}{\gamma} H(Q \mid \mathbb{P}).$$

- As $b(\lambda)$ is the infimum of affine functions of λ, it is concave.
- Can show: differentiable and strictly concave.



Sub−hedging
Davis Price
Super−hedging
Indifference Price
Quantity
1
0.5
0
−0.5
−1
−1.5
−2
−10
−8
−6
−4
−2
0
2
4
6
8
10

Hedging Barrier Options

- Down-and-in call option; barrier at $B < S_0$; payoff

$$G_B = (S_T - K)^+ \mathbf{1}_{\tau_B \le T}; \qquad \tau_B = \inf\{t \ge 0 : S_t \le B\}.$$

- Use a vanilla put with strike $K' = B^2/K$ for a static hedge. Payoff is $G_P = (K' - S_T)^+$, market price is P.

- Given initial capital \$$v$, sell $\lambda \ge 0$ puts. Problem is

$$\max_{\lambda \ge 0} u(v + \lambda P; G_\lambda),$$

where

$$\begin{aligned} u(x; G_\lambda) &= \sup_{\pi} \mathbb{E}\left\{-e^{-\gamma(X_T - G_\lambda)} \mid X_0 = x\right\}, \\ G_\lambda &= \lambda G_P - G_B. \end{aligned}$$

Connection to Indifference Price

- Similar to before, reduces to

$$\max_{\lambda \geq 0} \lambda P - h(G_\lambda),$$

where $h(G_\lambda)$ is the (writer's) indifference price of the barrier option.

- In the stochastic volatility model, $h(t, S, y)$ solves for $t < T$, $S > B$:

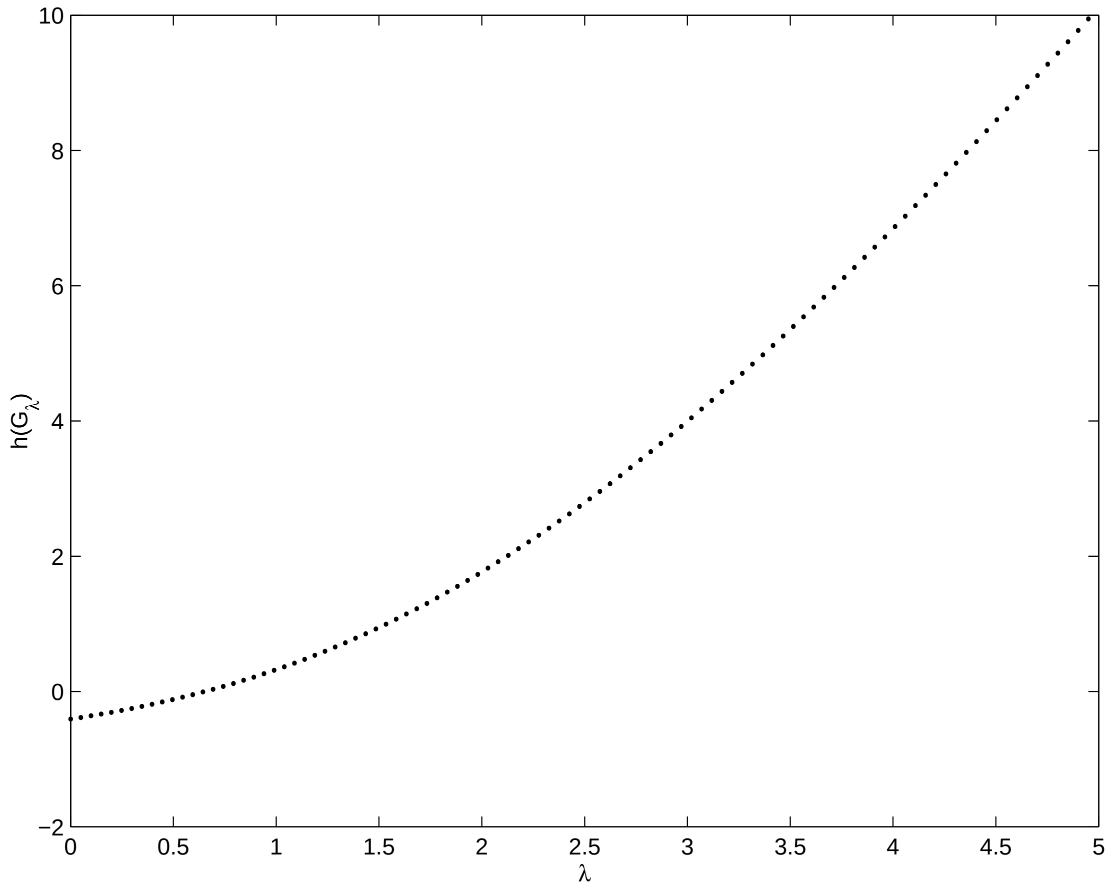
$$\begin{aligned} h_t + \mathcal{L}_{S,y} h + \frac{1}{2}\gamma(1-\rho^2)a(y)^2 h_y^2 &= 0 \\ h(T, S, y) &= \lambda(K' - S)^+ \\ h(t, B, y) &= h^\star(t, B, y) \end{aligned}$$

with $h^\star$ the indifference price of the European

$$\lambda(K' - S_T) - (S_T - K)^+.$$

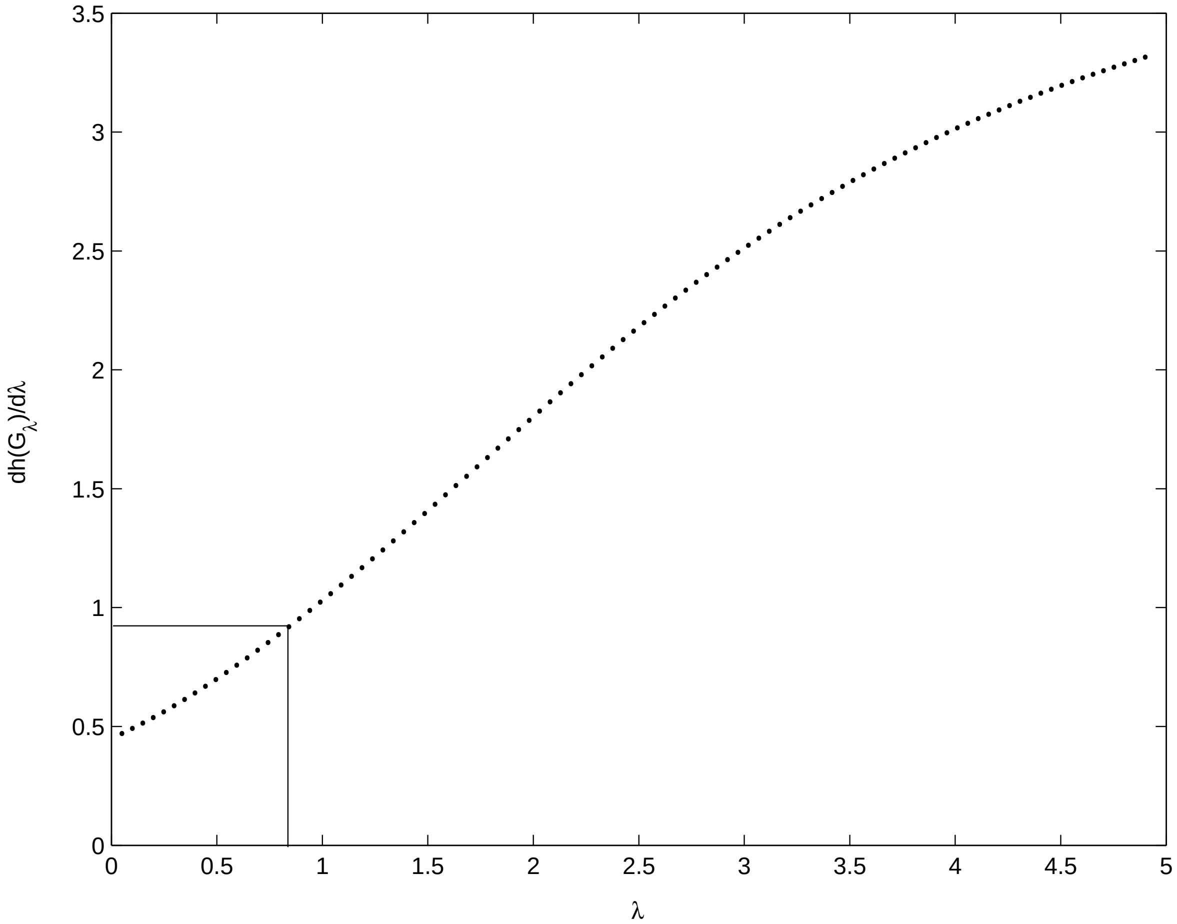
Barrier Indifference Price

$$B = 85 \qquad K = 100 \qquad T = 0.5 \qquad \gamma = 1.5.$$



Slope

$$dh(G_\lambda)/d\lambda$$



Risk Measures

The convenient properties of the exponential utility function have been axiomatized in a beautiful theory of risk measures.

Definition 1. *A mapping $\rho : \mathcal{X} \mapsto \mathbb{R}$ is called a convex measure of risk if it satisfies the following for all $X, Y \in \mathcal{X}$:*

- *Monotonicity: If $X \leq Y$, $\rho(X) \geq \rho(Y)$.*
- *Translation Invariance: If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.*
- *Convexity: $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$, for $0 \leq \lambda \leq 1$.*
- *If also: Positive Homogeneity: $\rho(\lambda X) = \lambda\rho(X), \forall\lambda \geq 0$, it is called a coherent measure of risk.*

Under positive homogeneity, convexity is equivalent to:

- Subadditivity : $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

- A classical example of a convex risk measure is related to exponential utility:

$$\forall X \in \mathcal{X}, \quad e_\gamma(X) = \frac{1}{\gamma} \log\left(\mathbb{E}\left\{e^{-\gamma X}\right\}\right).$$

- Any convex risk measure ρ on $\mathcal{X}$ is of the form

$$\rho(X) = \sup_{Q \in \mathcal{M}_{1,f}} \left(\mathbb{E}^Q\{-X\} - \alpha(Q)\right), \quad \forall X \in \mathcal{X}, \tag{1}$$

where the minimal penalty function α is given by

$$\alpha(Q) = \sup_{X \in \mathcal{X}} \left(\mathbb{E}^Q\{-X\} - \rho(X)\right), \quad \forall Q \in \mathcal{M}_{1,f}. \tag{2}$$

Moreover, the supremum in (1) is attained, and α is convex.

- If ρ were a coherent risk measure, in addition to being convex, the representation is

$$\rho(X) = \sup_{Q \in \mathcal{Q}} \mathbb{E}\{-X\}, \quad \forall X \in \mathcal{X}. \tag{3}$$

Comments and References

- Axiomatic study of risk measures introduced by Artzner, Delbaen, Eber, Heath (1999) gained a lot of attention due to the failure of a common risk measure, Value at Risk, to reward diversification.
- Subsequently, Follmer and Schied (2002) relaxed the positive homogeneitycondition.
- Many convex, but non-coherent, risk measures of the form

$$\rho(X) = \inf\{m \in \mathbb{R} \mid \mathbb{E}\{\ell(-X - m)\} \leq x_0\},$$

  for a convex *loss function* ℓ.
- Major research issue: construction and computation of dynamic risk measures.

Credit Risk

- Defaultable instruments, or credit-linked derivatives, are financial securities that pay their holders amounts that are contingent on the occurrence (or not) of a *default event* such as the bankruptcy of a firm, non-repayment of a loan or missing a mortgage payment.
- The market in credit-linked derivative products has grown more than seven-fold in recent years, from $170 billion outstanding notional in 1997, to almost $1400 billion through 2001.

- The primary problem is modeling of a random default time when a firm or obligor cannot or chooses not to meet a payment. This may come from a diffusion model of asset values hitting a fixed debt level, in which case we can, in some sense, see the default event coming; or it may be modeled as the jump time of some exogenous Poisson-type process (which does not have a direct economic interpretation), in which case the default comes as a surprise.

- The major challenge is extending useful single-name frameworks to the multi-name case, assessing accurately correlation of defaults across firms, and evaluating basket portfolios affected by many sources of credit risk.

Constant Volatility: Black-Cox Approach

$$\mathbb{E}^\star\left\{\mathbf{1}_{\{\inf_{t\le s\le T} X_s > B\}} \mid \mathcal{F}_t\right\}$$

$$= \mathbb{P}^\star\left\{\inf_{t\le s\le T}\left((r-\frac{\sigma^2}{2})(s-t)+\sigma(W_s^\star - W_t^\star)\right) > \log\left(\frac{B}{x}\right) \mid X_t = x\right\}$$

computed using distribution of minimum, or using PDE's:

$$\mathbb{E}^\star\left\{e^{-r(T-t)}\mathbf{1}_{\{\inf_{t\le s\le T} X_s > B\}} \mid \mathcal{F}_t\right\} \quad = \quad u(t, X_t)$$

where $u(t,x)$ is the solution of the following problem

$$\mathcal{L}_{BS}(\sigma)u = 0 \text{ on } x > B,\, t < T$$

$$u(t,B) = 0 \quad \text{for any } t \le T$$

$$u(T,x) = 1 \text{ for } x > B,$$

which is to be solved for $x > B$.

Yield Spread Curve

The *yield spread* $Y(0,T)$ at time zero is defined by

$$e^{-Y(0,T)T} = \frac{P^B(0,T)}{P(0,T)},$$

where $P(0,T)$ is the default free zero-coupon bond price given here, in the case of constant interest rate r, by $P(0,T) = e^{-rT}$, and $P^B(0,T) = u(0,x)$, leading to the formula

$$Y(0,T) = -\frac{1}{T}\log\left(N\left(d_2(T)\right) - \left(\frac{x}{B}\right)^{1-\frac{2r}{\sigma^2}} N\left(d_2^-(T)\right)\right)$$

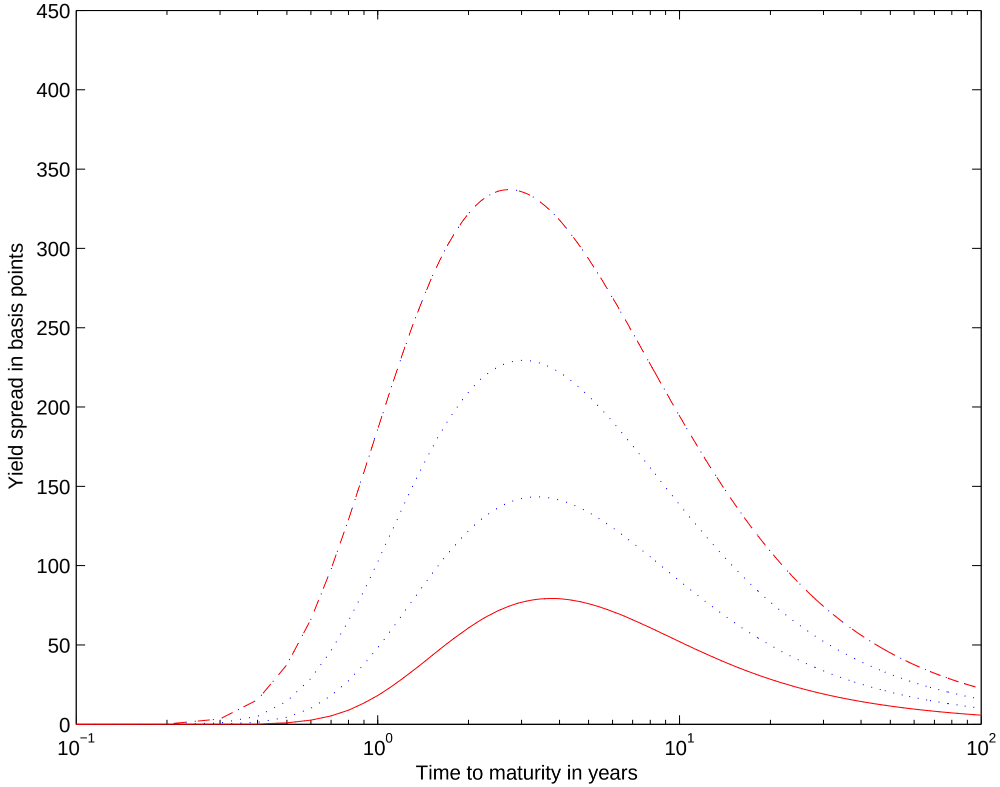


Figure 1: *The figure shows the sensitivity of the yield spread curve to the volatility level. The ratio of the initial value to the default level x/B is set to* 1.3*, the interest rate r is 6% and the curves increase with the values of σ: 10%, 11%, 12% and 13% (time to maturity in unit of years, plotted on the log scale; the yield spread is quoted in basis points)*

Challenge: Yields at Short Maturities

As stated by Eom et.al. (empirical analysis 2001), *the challenge for theoretical pricing models is to raise the average predicted spread relative to crude models such as the constant volatility model, without overstating the risks associated with volatility or leverage.*

Several approaches (**within structural models**) have been proposed to capture significant short-term spreads. These include

- Introduction of jumps (Zhou,...)
- Stochastic interest rate (Longstaff-Schwartz,...)
- Imperfect information (on X_t) (Duffie-Lando,...)
- Imperfect information (on B) (Giesecke)

Typical Single-Name Intensity Models

- All models under pricing measure $I\!P^\star$.
- Default time τ is first jump of a time-changed (standard) Poisson process:

$$N\left(\int_0^t \lambda_s \, ds\right),$$

where N and λ are independent.

- Draw $\xi \sim \text{EXP}(1)$, then

$$\tau = \inf\left\{t : \int_0^t \lambda_s \, ds = \xi\right\}.$$

- *E.g.*: λ is a diffusion (CIR).

Defaultable Bond Pricing

- Payoff $\mathbf{1}_{\{\tau > T\}}$.
- Price

$$\begin{aligned} P_0(T) &= \mathbb{E}^\star \left\{ e^{-rT} \mathbf{1}_{\{\tau > T\}} \right\} \\ &= e^{-rT} \mathbb{P}^\star \{\tau > T\} \\ &= \mathbb{E}^\star \left\{ \exp\left(-\int_0^T (r + \lambda_s)\, ds \right) \right\}. \end{aligned}$$

- Same structure as *short rate models*.
- Yield spread: $P_0(T) = \exp(-(r + Y(T))T)$:

$$Y(T) = -\frac{1}{T} \log\left(\frac{P_0(T)}{e^{-rT}} \right).$$

Issues

- Intensity models resolve a major shortcoming of (constant volatility) structural models: yield spreads not small at short maturities.
- *E.g.*: for λ constant, $Y(T) = \lambda$.
- Loss of economic intuition – why a default? No direct relation to firm's stock price.
- While single name default time models can be calibrated, how to deal with *joint distributions*?
- How to compute with ~ 300 names?

Complex Structured Products

- CDOs (Collateralized Debt Obligations) depend on the number of defaults over a fixed time of a number (~ 300) firms.
- Various slices of the loss distribution are sold as tranches , and these are sensitive to the correlation between default events.
- CDO^2's collate tranches of different CDOs. There are even CDO^3's !!!
- Only computationally tractable approach so far is through copulas – highly artificial "correlator".
- Frontpage Wall Street Journal article 9/12/05: How a formula ignited a market that burned investors .