Risk Minimizing Static-Dynamic Hedges for Exotic Options

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Static-Dynamic Hedging

- Suppose we have written an exotic derivative with payoff G^e
- Want to minimize risk from possible loss at time T
- Find optimal hedging strategy for dynamically trading in stocks and statically trading in options

$$-G^e + V_0 + \int_0^T \theta \cdot dS + \lambda \cdot G - \lambda \cdot p$$

- Justification: options have higher transaction costs.
- Only meaningful if option cannot be replicated by dynamic trading strategy ($\Rightarrow$ market incomplete).
- Alternatively maximize utility.

What is meant by "utility" or "risk"?

- "Utility" is measured as expected utility from wealth. Utility function $U(v)$ is concave in v. Examples:
  - $U(v) = -\exp(-\gamma v)$, $\gamma > 0$
  - $U(v) = v^\beta$, $0 < \beta < 1$, $v \geq 0$
  - $U(v) = \log v$, $v > 0$
- $\mathcal{X}$ = vector space of bounded ($\mathcal{F}_T$-measurable) random variables (financial position at time T)
- "Risk" is measured by a convex risk measure. A convex risk measure is a function $\rho : \mathcal{X} \to \mathbb{R}$ satisfying the following axioms:
  - *monotonicity*:

$$X \geq Y \quad \Rightarrow \quad \rho(X) \leq \rho(Y).$$

– *convexity*: if $0 < \lambda < 1$, then

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y).$$

– *monetary invariance*: if $m \in \mathbb{R}$, then

$$\rho(X + m) = \rho(X) - m.$$

- A convex risk measure is coherent if

$$\rho(tX) = t\rho(X) \quad \text{for } t \geq 0.$$

- A convex risk measure ρ is determined by its acceptance set:

$$A_\rho := \{X \mid \rho(X) \leq 0\}$$

- "utility" is measured by $-\{$convex risk measure$\}$, "utility" does not need to satisfy monetary invariance.

Examples of Risk Measures

- *Value at Risk.*

$$\mathrm{VaR}_\lambda(X) = \min\{m \in \mathbb{R} \mid P\{X + m < 0\} \leq \lambda\}$$

does *not* define a convex risk measure in general.

- *Average Value at Risk.*

$$\mathrm{AVaR}_\lambda(X) = \frac{1}{\lambda} \int_0^\lambda \mathrm{VaR}_\mu(X)\, d\mu$$

does define a (coherent) convex risk measure.

- Any convex subset $\mathcal{A}$ of X satisfying

$$X \in \mathcal{A}, Y \geq X \quad \Rightarrow \quad Y \in \mathcal{A}$$

gives rise to a convex risk measure.

$$\rho_{\mathcal{A}}(X) = \inf\{m \in \mathbb{R} \mid m + X \in \mathcal{A}\}$$

- *Exponential Utility or Entropic risk measure.*

$$\rho(X) = \gamma^{-1} \log E^P\{e^{-\gamma X}\}$$

- *Shortfall risk.* If $l : \mathbb{R} \to \mathbb{R}$ is a convex increasing function and P is a prior probability measure then

$$\mathcal{A} = \{X \mid E^P\{l(-X)\} \leq l_0\}$$

is the acceptance set of a convex risk measure.
Example: $l(x) = x^p/p$, for $x \geq 0$, and $l(x) = 0$ otherwise, where $p > 1$.
Example: $l(x) = e^{\gamma x}$ gives exponential utility.

References

- Coherent risk measures: Artzner, Delbaen, Eber, Heath (1999).
- Convex risk measures: Föllmer, Schied (2002).

Convex Duality

- Convex risk measures can be understood through convex duality:
$$\rho(X) = \sup_Q \left(E^Q\{-X\} - \alpha(Q)\right),$$
where Q runs over all probability measures and α is a penalty function (the convex dual of ρ).

- Strict convexity of ρ corresponds to differentiability of α and vice versa.

- Example: $\rho(X) = \log E\{e^{-\gamma X}\}$ gives entropic penalty $\alpha(Q) = H(Q|P) = E\left\{\frac{dQ}{dP}\log\frac{dQ}{dP}\right\}$.

- Example: shortfall with loss function $l(x) = x^p/p$ gives $\alpha(Q) = E\left\{\left(\frac{dQ}{dP}\right)^q\right\}^{1/q}$, $p^{-1} + q^{-1} = 1$.

Convolution of Risk Measures

- Want to optimize over static *and* dynamic strategies.
- Can incorporate dynamic part of optimization problem into the risk measure by inf-convolution.
- Given the convex risk measure ρ define new convex risk measure $\rho^{\mathcal{H}}$ by

$$\rho^{\mathcal{H}}(X) = \inf_{\theta \in \mathcal{H}} \rho\left(X + V_T^{\theta}\right).$$

- The convolution in going from ρ to $\rho^{\mathcal{H}}$ corresponds to going from α to $\alpha^{\mathcal{H}} = \alpha + \infty \cdot \mathbf{1}_{\mathcal{M}}$, where $\mathcal{M}$ is the set of martingale measures for S.
- Reference: Barrieu, El Karoui (2004).

Indifference Pricing

- Recall the static hedging problem:

$$\lambda^* = \arg\min_{\lambda} \rho^{\mathcal{H}}(\lambda \cdot G - G^e) + \lambda p.$$

- Define a new convex risk measure, the indifference price given an initial liability G^e, by

$$\rho^{e,\mathcal{H}}(\lambda \cdot G) = \rho^{e,\mathcal{H}}(\lambda \cdot G - G^e) - \rho^{\mathcal{H}}(-G^e).$$

- Recasting the static hedging problem:

$$\lambda^* = \arg\min_{\lambda} \rho^{e,\mathcal{H}}(\lambda \cdot G) + \lambda p.$$

- Reference: Mingxin Xu (2005).

Optimal strategies in Finite State Spaces

- **Theorem.** Assume that
  (i) the convex risk measure ρ is *differentiable* and *strictly convex* on $\mathcal{X}$,
  (ii) no nontrivial linear combination $\lambda \cdot G$ of the derivatives can be replicated by a dynamic trading strategy in the stocks and bond, then

$$\begin{aligned} \lambda \mapsto \rho^{\mathcal{H}}(\lambda \cdot G - G^e) &= \inf_{\theta} \rho(V^{\theta} + \lambda \cdot G - G^e) \\ &= \sup_{Q \in \mathcal{M}} \{E^Q\{-\lambda \cdot G + G^e\} - \alpha(Q)\} \end{aligned}$$

is strictly concave.

- Proof: if the function was affine on the segment joining λ_1, λ_2, then the optimal measures are the same, $Q_1^* = Q_2^*$, and the option $(\lambda_2 - \lambda_1) \cdot G$ is redundant.
- Therefore, as long as p is a no-arbitrage price vector for G, there exists a unique minimizer $\lambda^* = \lambda^*(p)$ minimizing

$$\rho^{\mathcal{H}}(\lambda \cdot G - G^e) + \lambda \cdot p.$$

Exponential Utility

- Assume stock price S is a locally bounded semimartingale in the "usual" continuous-time setup, option payoffs are bounded, and there exists a martingale measure $Q \sim P$ of finite relative entropy.
- Let $\mathcal{P}_f$ be the set of martingale measures for S with finite relative entropy with respect to P.
- **Theorem**.

$$\inf_{\theta \in \mathcal{H}} \log E\{e^{-\gamma(G+V_T^{\theta})}\} = \sup_{Q \in \mathcal{P}_f} E^Q\{-G\} - \frac{1}{\gamma} H(Q|P),$$

where the supremum and infimum are achieved by Q^* and θ^* such that

$$\frac{\mathrm{d}Q^*}{\mathrm{d}P} = ce^{-\gamma(V_T^{\theta^*}+G)}.$$

DGRSSS (2002).

◇ Let

$$b(\lambda) = \inf_{\theta \in \mathcal{H}} \log E \left\{ e^{-\gamma\left(\int_0^T \theta \, dS + \lambda \cdot G - G^e\right)} \right\}.$$

Lemma. Set of no-arbitrage option prices= $\{\nabla b(\lambda) | \lambda \in \mathbb{R}^N\}$.

◇ **Theorem**. If no nontrivial linear combination $\lambda \cdot G$ of the derivatives is replicable by a dynamic trading strategy in the stocks and bond, then, for any no-arbitrage option price p, there exists a unique static options position $\lambda^*(p)$ minimizing

$$b(\lambda) + \lambda \cdot p.$$

◇ Can include semi-static hedging in options, i.e., rolling over option positions.

◇ Extension to other risk measures (e.g. shortfall risk) is current work.

Implementation

- Would like to find the optimal strategies in an efficient way.
- Seems to be a nontrivial problem in general.
- Can try tree methods or PDE methods by exploiting convex duality.

Trees

- Special recursive structure available in exponential utility case. Essentially stems from, for a suitable measure $\tilde{P} \sim P$,

$$H(Q|P) + E^Q[G] = H(Q|\tilde{P}) \quad \text{and} \quad H(Q|P) = \sum_t H_t(Q|P).$$

  - Musiela, Zariphopoulou (2004)
  - Lim (2005)
- Recursive structure not present in general.

Expected Shortfall

- Non-traded asset model

$$\begin{aligned} \mathrm{d}S_t &= \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t^1, \quad S_0 = S, \\ \mathrm{d}Y_t &= bY_t \mathrm{d}t + aY_t(\rho \mathrm{d}W_t^1 + \sqrt{1-\rho^2}\mathrm{d}W_t^2), \quad Y_0 = y. \end{aligned}$$

- Shorfall risk with short derivative position $g(Y_T)$

$$\begin{aligned} \rho^{\mathcal{H}}(g(Y_T)) &= \inf_{\theta\in\mathcal{H}} \inf\{m \in \mathbb{R} \,|\, E\{\ell(-m + g(Y_T) - V_T^\theta - v)\} \leq \rho_0\}, \\ &= \inf\{m \in \mathbb{R} \,|\, \inf_{\theta\in\mathcal{H}} E\{\ell(g(Y_T) - m - V_T^\theta - v)\} \leq \rho_0\}, \\ &= \inf\{m \in \mathbb{R} \,|\, \inf_{\theta\in\mathcal{H}} E\{\ell(g(Y_T) - V_T^\theta - m)\} \leq \rho_0\} - v. \end{aligned}$$

- Take $l(x) = x^p/p$ for $x \geq 0$, where $p > 1$.
- Introduce the value function

$$H(t,v,y) = \inf_{\theta} \frac{1}{p} \mathbb{E}\left\{((g(Y_T) - V_T^{\theta})^+)^p | V_t = v, Y_t = y\right\},$$

- the "inverse" of H

$$H(t, u(t,v,y,\xi), y) = \xi,$$

- and the Fenchel-Legendre transform of u

$$\hat{u}(t,y,z) = \inf_{\xi \geq 0} \left(u(t,y,\xi) + \xi z\right).$$

- The PDE that $\hat{u}$ solves:

$$\hat{u}_t+\mathcal{L}_y\hat{u}+\frac{\rho\mu a}{\sigma}(z\hat{u}_{zy}-\hat{u}_y)+\frac{\mu^2}{2\sigma^2}z^2\hat{u}_{zz}-\frac{1}{2}a^2(1-\rho^2)\frac{(z\hat{u}_{zy}-\hat{u}_y)^2}{z^2u_{zz}}=0,$$

with

$$\hat{u}(t,y,z)=g(y)-\left(\frac{p-1}{p}\right)z^{1/(1-p)}.$$

- This is the HJB equation related to another stochastic control problem:

$$\sup_{\gamma}E^{Q^{\gamma}}\left\{g(Y_T)-\left(\frac{p-1}{p}\right)Z_T^{1/1-p}|Y_t=y,Z_t=z\right\},$$

where under Q^{γ}

$$\begin{aligned}
\mathrm{d}Z_t &= Z_t\left(\frac{\mu}{\sigma}\mathrm{d}W_t^{\gamma,1}+\gamma_t\mathrm{d}W_t^{\gamma,2}\right),\quad Z_t=z,\\
\mathrm{d}Y_t &= \left(b-\rho\frac{\mu}{\sigma}a-\sqrt{1-\rho^2}a\gamma_t\right)Y_t\mathrm{d}t+aY_t(\rho\mathrm{d}W_t^{\gamma,1}+\sqrt{1-\rho^2}\mathrm{d}W_t^{\gamma,2}).
\end{aligned}$$

- ⋆ For exponential utility, the nonlinear PDE that we need to solve:

$$\begin{aligned} \phi_t + \mathcal{L}_y^0 \phi + \frac{1}{2}(1-\rho^2)a^2 y^2 (\phi_y)^2 &= \frac{\mu^2}{2\gamma\sigma^2}, \\ \phi(T,x,y) &= g(y). \end{aligned}$$

- ⋆ Long-Maturity Call and Put Example:

$$\begin{aligned} G^e &= -(Y_{T_c} - K_c)^+, \\ G &= -(K_p - Y_{T_p})^+. \end{aligned}$$

Parameters for numerics: $K_c = K_p = 100$, $\mu = 0.075$, $\sigma = 0.22$, $a = 0.45$, $b = \mu a/\sigma$, $\rho = 0.5$, $T_p = 1$, $T_c = 3$, $\gamma = 1$.

Numerical Solution

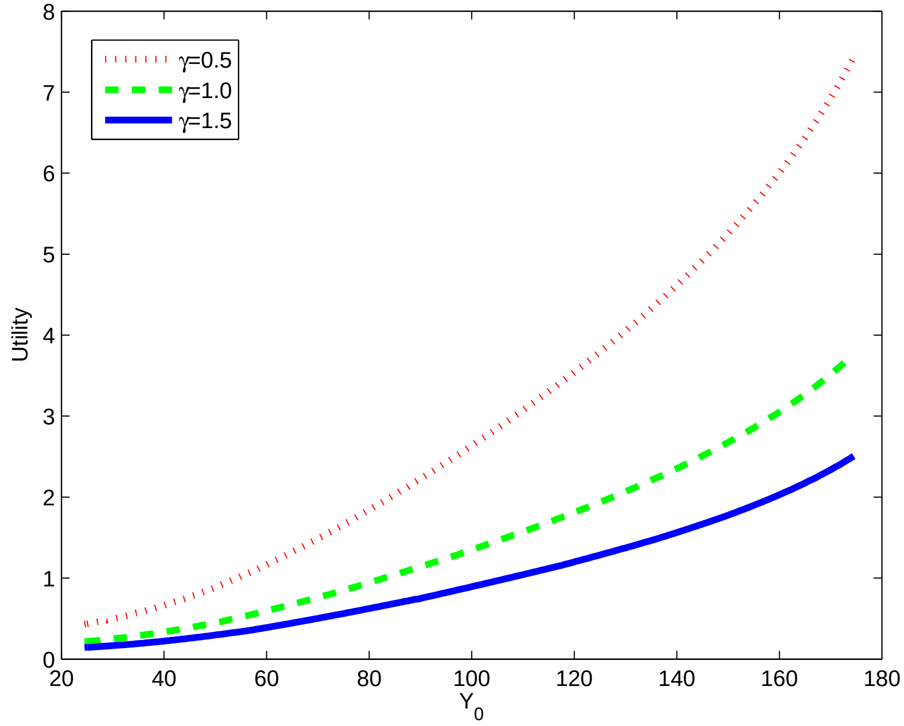

Figure 1: Negative of the risk measure with optimal static hedging in the puts.

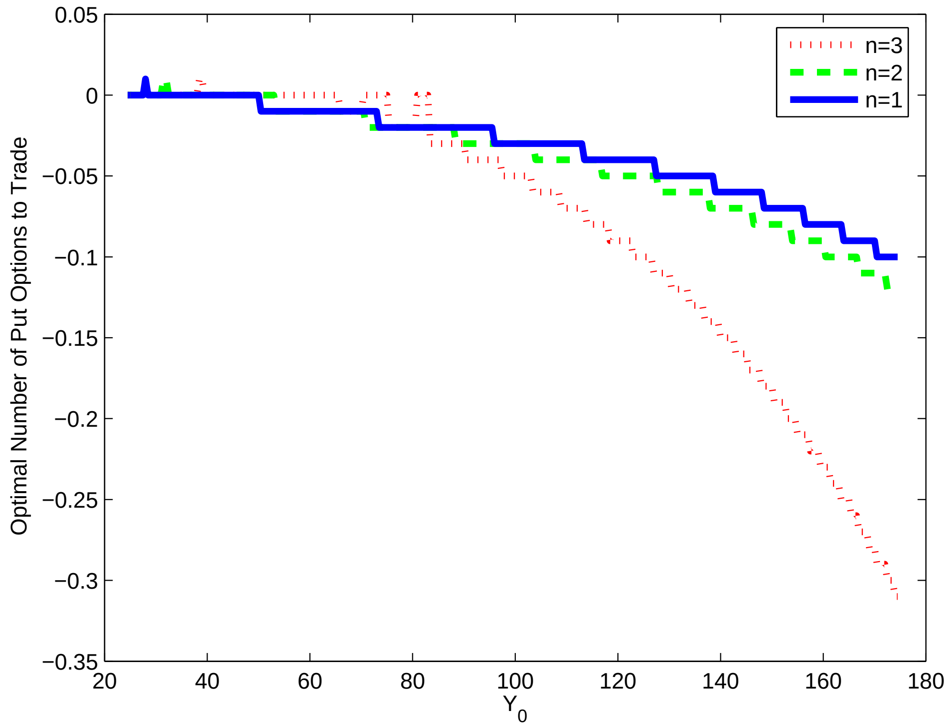

Figure 2: Optimal number of one year put options to trade each year.

Conclusion

- ↪ Given a risk measure ρ, try to find optimal static-dynamic trading (or hedging) strategies.
- ↪ Results in:
  - finite state spaces
  - continuous time with exponential utility
- ↪ Current work:
  - continuous time and other risk measures
  - Efficient numerical implementation.