The relation between conditionally heteroskedastic factor models and factor GARCH models

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Summary   The factor GARCH model of Engle (1987) and the latent factor ARCH model of Diebold and Nerlove (1989) have become rather popular multivariate volatility parametrizations due to their parsimony, and the commonality in volatility movements across different financial series. Nevertheless, there is some confusion in the literature between them. The purpose of this paper is to make clear their similarities and differences by providing a formal nesting of the two models, which can be exploited to analyse their statistical features in a more general context. At the same time, their differences may be important in the interpretation of empirical results.

Keywords: *Asset pricing, Factor models, Multivariate ARCH, Volatility.*

1. INTRODUCTION

Most of the econometric and finance literature on time-varying volatility models is concerned with univariate time series. However, many issues in finance can only be fully addressed within a multivariate framework. Although multivariate generalizations of the existing univariate models are straightforward in theory, their empirical applications have been hampered by the sheer number of parameters involved. Motivated by the commonality in volatility movements across different financial time series, several parsimonious alternatives have been proposed. Within the ARCH class, two such parametrizations have become rather popular, namely the factor GARCH model of Engle (1987) and the conditionally heteroskedastic latent factor model introduced by Diebold and Nerlove (1989) and extended by King *et al.* (1994). Broadly speaking, Engle's factor GARCH model specifies the time-varying part of the covariance matrix as a function of a few linear combinations of the observed random variables, but leaves its constant part fully unrestricted. On the other hand, Diebold and Nerlove's model is a traditional statistical factor analysis model, with a diagonal idiosyncratic covariance matrix, in which the variances of the common factors are parametrized as univariate ARCH models. Given that the two models are closely related, it is perhaps not surprising that there is some confusion in the literature on the similarities and differences between them, and in some cases they are treated as equivalent. Nevertheless, there are at least two important differences that distinguish them. First, while the covariance matrix of a factor GARCH model is measurable by construction with respect to an information set that contains only past values of the observed variables, this is not the case for Diebold and Nerlove's

original model, which in fact can be regarded as a stochastic volatility model (see, for example, Andersen 1992, Shephard 1996). A second, less well-known distinctive feature, is that the implicit definition of the factors is significantly different between the two. While in Diebold and Nerlove's model the factors capture the co-movements between the observed series, in Engle's model they are directly related to those linear combinations of the series which summarize the co-movements in their conditional variances. The purpose of this paper is to make clear their similarities and differences by providing a formal nesting of the two models. This is important for at least two reasons. First, while many properties of these models have already been studied in detail, it is not always entirely clear whether they are specific to the model analysed, or apply to a broader class. Second, as we shall see, in some cases of substantive interest the interpretation of the empirical results may be sensitive to the specific nature of the model used. The rest of the paper is organized as follows. We define the general class of conditionally heteroskedastic models in Section 2, and discuss some of their properties. Then in Section 3, we introduce the factor ARCH model, summarize its main properties, and present our main result. Finally, Section 4 contains a discussion of our results in relation to empirical tests of multi-beta asset-pricing theories. Two auxiliary lemmas are included in the appendix.

2. CONDITIONALLY HETEROSKEDASTIC FACTOR MODELS

Consider the following multivariate model:

$$x_t = C f_t + w_t \tag{1}$$

$$\begin{pmatrix} f_t \\ w_t \end{pmatrix} | I_{t-1} \sim \mathrm{D} \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Lambda_t & 0 \\ 0 & \Gamma \end{pmatrix} \right\} \tag{2}$$

where x_t is a $N \times 1$ vector of observable random variables, f_t is a $k \times 1$ vector of unobserved common factors, C is the $N \times k$ matrix of associated factor loadings, with $N \geq k$ and $\mathrm{rank}(C) = k$, w_t is a $N \times 1$ vector of idiosyncratic noises, which are conditionally orthogonal to f_t, Γ is a $N \times N$ positive semidefinite (p.s.d.) matrix of constant idiosyncratic variances, Λ_t is a $k \times k$ positive definite (p.d.) matrix of time-varying factor variances, and I_{t-1} is an information set that contains the values of x_t and f_t up to, and including $t - 1$. Our assumptions imply that the distribution of x_t conditional on I_{t-1} has zero mean, and covariance matrix $\Sigma_t = C \Lambda_t C' + \Gamma$. For this reason, we shall refer to the data-generation process specified by (1) and (2) as a multivariate conditionally heteroskedastic factor model. Such a formulation nests several models widely used in the empirical literature, which typically assume that the unobserved factors follow dynamic heteroskedastic processes, but differ in the exact parametrization of Λ_t and Γ. For instance, in the latent factor model with ARCH effects on the underlying factors introduced by Diebold and Nerlove (1989), the conditional variances of the common factors are parametrized as univariate *strong* ARCH models, in the sense of Drost and Nijman (1993). In particular, for a GARCH(p, q) formulation,

$$\lambda_{jjt} = V(f_{jt} | I_{t-1}) = \phi_j + \sum_{s=1}^{q} \alpha_{js} f_{jt-s}^2 + \sum_{r=1}^{q} \beta_{jr} \lambda_{jjt-r} \qquad (j = 1, \ldots, k). \tag{3}$$

In this context, it is important to distinguish between I_{t-1}, which includes past values of x_t and f_t, and the econometrician's information set $X_{t-1} = \{x_{t-1}, x_{t-2}, \ldots\}$, which only includes

lagged values of x_t. Notice that in this case, $\lambda_{jjt} \in F_{t-1}$, where $F_{t-1} = \{f_{t-1}, f_{t-2}, \ldots\}$, but $\lambda_{jjt} \notin X_{t-1}$, which has important inference implications (see Harvey *et al*. 1992, Shephard 1996, and Fiorentini *et al*. 1998). Notice also that a significant characteristic of a model like (1)–(3) is that it is dimension independent, in the sense that it remains valid for any subset of x_t. Another frequently made assumption is that the common factors represent (conditionally) orthogonal influences, which implies that Λ_t is diagonal. Otherwise, we say that the factor model is oblique. Similarly, it is often assumed that Γ is diagonal, in which case the factor structure is termed strict or exact. However, in some applications, diagonality of Γ may be thought to be too restrictive. For instance, it will not be preserved by linear transformations. For that reason Chamberlain and Rothschild (1983) introduced the concept of approximate factor structures, in which the idiosyncratic terms may be correlated, but only up to a certain degree. Alternatively, one could assume that Γ has reduced rank. In fact, the rank of Γ is related to the observability of the factors. If rank$(\Gamma) = N - k$, the factors would be fully revealed by the x_t variables, otherwise they are only partially revealed (see King *et al*. 1994). A non-trivial advantage of these models is that they automatically guarantee a p.s.d. covariance matrix for x_t once we ensure that the covariance matrix of the factors is p.s.d. Moreover, using Lemma 1 in the Appendix, it is straightforward to derive necessary and sufficient conditions that guarantee that Σ_t is actually p.d. even when Γ is of reduced rank. But the most distinctive feature of factor models is that they provide a parsimonious specification of the dynamic and cross-sectional dependence of a vector of observable random variables. In this case in particular, the factor structure, together with the constancy of Γ, implies that the time variation of Σ_t is of reduced rank (see Engle *et al*. 1990). More formally, if $\ddot{E}$ denotes a $N \times (N-k)$ matrix of full column rank such that $\ddot{E}'C = 0$, and $\bar{E} = (\dot{E}, \ddot{E})$ is a $N \times N$ matrix of full rank, with $\dot{E}$ arbitrary, then, the only time-varying component in the covariance matrix of $\bar{x}_t' = x_t'\bar{E} = (x_t'\dot{E}, x_t'\ddot{E}) = (\dot{x}_t', \ddot{x}_t')$ is the covariance matrix of $\dot{x}_t$. In this respect, Gouriéroux *et al*. (1991) provide a particularly attractive choice of $\bar{E}$, which makes $\dot{x}_t$ and $\ddot{x}_t$ orthogonal, and allows us to express any conditionally heteroskedastic factor model as an oblique factor model with time-varying conditional variances, and a singular idiosyncratic covariance matrix. In particular, if we assume for simplicity that Γ is non-singular, we can write:

$$x_t = Cf_t^G + w_t^G \tag{4}$$

$$\begin{pmatrix} f_t^G \\ w_t^G \end{pmatrix} | I_{t-1} \sim \mathrm{D}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Lambda_t + (C'\Gamma^{-1}C)^{-1} & 0 \\ 0 & \Gamma - C(C'\Gamma^{-1}C)^{-1}C' \end{pmatrix} \right\}$$

where

$$f_t^G = (C'\Gamma^{-1}C)^{-1}C'\Gamma^{-1}x_t = f_t + (C'\Gamma^{-1}C)^{-1}C'\Gamma^{-1}w_t \tag{5}$$

are the generalized least squares (GLS) estimates of the common factors and $w_t^G = \{I - C(C'\Gamma^{-1}C)^{-1}C'\Gamma^{-1}\}x_t$. Notice that when Λ_t is diagonal, the elements of f_t^G are contemporaneously correlated, unless $(C'\Gamma^{-1}C)$ is itself diagonal, but with constant conditional covariances. Also note that if Λ_t is given by a strong GARCH model such as (3), the results in Nijman and Sentana (1996) imply that these factor scores will follow weak GARCH processes. Finally, notice that if f_t is conditionally homoskedastic, the model in (1), (2) reduces to the standard (i.e. static) factor-analysis model (see, for example, Lawley and Maxwell 1971). But even if f_t is conditionally heteroskedastic, provided that it is covariance stationary, it also implies an unconditional k factor structure for x_t. That is, the unconditional covariance matrix, Σ, can be written as:

$$\Sigma = C\Lambda C' + \Gamma \tag{6}$$

where $V(f_t) = E(\Lambda_t) = \Lambda$. This property makes the models considered here compatible with traditional factor analysis.

3. FACTOR GARCH MODELS AS CONDITIONALLY HETEROSKEDASTIC FACTOR MODELS

The factor GARCH model was originally introduced as a parsimonious special case of the multivariate GARCH(p, q) model considered in Bollerslev *et al.* (1988):

$$x_t | X_{t-1} \sim \mathrm{D}(0, \Sigma_t)$$

$$\mathrm{vech}(\Sigma_t) = \Phi + \sum_{s=1}^{q} A_s \, \mathrm{vech}(x_{t-s} x_{t-s}') + \sum_{r=1}^{q} B_r \, \mathrm{vech}(\Sigma_{t-r})$$

where A_s and B_r are square matrices of order $m(m+1)/2$, and Φ is a vector of the same order. In fact, it could be regarded as a prototype of the positive definite BEEK parametrization in Engle and Kroner (1995), which is the most general covariance specification proposed so far within this class (see Kroner 1987, Lin 1992). Formally, Engle's (1987) k factor GARCH(p, q) model is characterized by the following conditional covariance matrix specification:

$$\Sigma_t = \Psi + \sum_{j=1}^{k} c_j c_j' \left\{ \sum_{s=1}^{q} \alpha_{js} (d_j' x_{t-s})^2 + \sum_{r=1}^{p} \beta_{jr} (d_j' \Sigma_{t-r} d_j) \right\}$$

$$= \Psi + \sum_{j=1}^{k} c_j c_j' \{ \alpha_j(L)(d_j' x_t)^2 + \beta_j(L)(d_j' \Sigma_t d_j) \} \tag{7}$$

where Ψ is a $N \times N$ symmetric p.s.d. matrix of rank $N_1 \leq N$, $C = (c_1, \ldots, c_k)$ and $D = (d_1, \ldots, d_k)$ are $N \times k$ full column rank coefficient matrices satisfying $D'C = I_k$, with each column d_j (or c_j) normalized to avoid scale indeterminacy, and $\alpha_j(L) = \sum_{s=1}^{q} \alpha_{js} L^s$, $\beta_j(L) = \sum_{r=1}^{p} \beta_{jr} L^r$ are polynomials in the lag operator, with the roots of $1 - \beta_j(L)$ outside the unit circle. The name factor GARCH stems from the fact that the time variation in the conditional variance can be summarized by k linear combinations of x_t which are univariate strong GARCH. In this respect, it can be regarded as the conditional variance counterpart to reduced rank vector autoregressive models. These linear combinations, $d_j' x_t$, referred to as 'factor representing portfolios' by Ng *et al.* (1992), are univariate strong GARCH(p, q) processes with conditional variance

$$\delta_{jjt} = \vartheta_{jj} + \sum_{s=1}^{q} \alpha_{js} (d_j' x_{t-s})^2 + \sum_{r=1}^{p} \beta_{jr} \delta_{jjt-r} = \frac{\vartheta_{jj}}{1 - \beta_j(1)} + \frac{\alpha_j(L)}{1 - \beta_j(L)} (d_j' x_t)^2$$

where $\vartheta_{jj} = d_j' \Psi d_j$. Note, however, that in general $\mathrm{cov}_{t-1}(d_i' x_t, d_j' x_t) = d_i' \Psi d_j = \vartheta_{ij} \neq 0$ for $i \neq j$, so that the constant part of $V_{t-1}(D' x_t)$ is not usually diagonal (see Lin 1992). In order to guarantee that $\delta_{jjt} \geq 0 \; \forall t$, it is often assumed that $\vartheta_{jj} \geq 0$ and the coefficients in the power expansion of $\alpha_j(L)/\{1 - \beta_j(L)\}$ are non-negative (see Nelson and Cao 1992, Drost and Nijman 1993). In practice, it is also convenient to rule out cases in which d_j is in the nullspace of Ψ, i.e. $\vartheta_{jj} = 0$, since otherwise δ_{jjt}, and indeed $d_j' x_t$, will converge to zero with probability 1 (cf. Nelson 1990). Given that the only restriction on Ψ is positive semi-definiteness, it is not surprising that

the factor GARCH structure (7) is preserved under full-rank linear transformations of x_t (see Lin 1992). However, these models are not generally closed with respect to (block) marginalization. More formally, if we call X_{1t-1} an information set that only contains the past values of a subvector of x_t, in most cases $\delta_{jjt} \notin X_{1t-1}$, so that the relevant block of Σ_t becomes non-measurable with respect to the smaller information set (see the discussion after expression (3)). In this respect, Nijman and Sentana (1996) derive the weak GARCH representations of the individual components of x_t for the bivariate case. Bollerslev and Engle (1993) show that x_t is covariance stationary if and only if $\alpha_j(1) + \beta_j(1) < 1 \; \forall j$. In that case, its unconditional covariance matrix is given by the following expression:

$$\Sigma = \Psi + \sum_{j=1}^{k} c_j c_j' \vartheta_{jj} \frac{\alpha_j(1) + \beta_j(1)}{1 - \alpha_j(1) - \beta_j(1)}.$$

For our purposes, it is more convenient to write Σ_t as

$$\Sigma_t = \Psi - \sum_{j=1}^{k} c_j c_j' \vartheta_{jj} + \sum_{j=1}^{k} c_j c_j' \delta_{jjt} = \Psi + \sum_{j=1}^{k} c_j c_j' \left\{ \frac{\vartheta_{jj}\beta_j(1)}{1 - \beta_j(1)} + \frac{\alpha_j(L)}{1 - \beta_j(L)} (d_j' x_t)^2 \right\}$$

or in matrix form

$$\Sigma_t = \Psi + C\{\Pi + \Lambda_t(0)\}C' = \bar{\Psi} + C\Lambda_t(0)C' \tag{8}$$

where

$$\Lambda_t(0) = \mathrm{diag}\left\{ \frac{\alpha_1(L)}{1 - \beta_1(L)} (d_1' x_t)^2, \ldots, \frac{\alpha_k(L)}{1 - \beta_k(L)} (d_k' x_t)^2 \right\}$$

$$\Pi = \mathrm{diag}\left\{ \frac{\vartheta_{11}\beta_1(1)}{1 - \beta_1(1)}, \ldots, \frac{\vartheta_{kk}\beta_k(1)}{1 - \beta_k(1)} \right\}$$

and $\bar{\Psi} = \Psi + C\Pi C'$. Equation (8) clearly suggests that Engle's model must be related to the conditionally heteroskedastic factor model in (1), (2). In fact, as the following proposition shows, the factor GARCH model is observationally equivalent to a whole family of oblique factor models with constant conditional covariances, whose limiting cases are an orthogonal factor model in which the variances of the factors are given by $\Lambda_t(0)$, and an oblique factor model like (4) with a singular idiosyncratic covariance matrix:

Proposition 1. *Assume that Σ_t is p.d., and furthermore that $\bar{\Psi}$ has full rank. Let Ω be any $k \times k$ p.s.d. matrix such that the eigenvalues of $\Omega C' \bar{\Psi}^{-1} C$ are less than or equal to 1. Then, the k factor GARCH(p, q) model in (7) is observationally equivalent (up to conditional second moments) to any of the following conditionally heteroskedastic factor models:*

$$x_t = C f_t(\Omega) + w_t(\Omega)$$

where

$$V_{t-1}\{f_t(\Omega)\} = \Lambda_t(\Omega) = \Omega + \Lambda_t(0)$$
$$V_{t-1}\{w_t(\Omega)\} = \Gamma(\Omega) = \bar{\Psi} - C\Omega C'$$
$$\mathrm{cov}_{t-1}\{f_t(\Omega), w_t(\Omega)\} = 0.$$

Proof. Since $V_{t-1}(x_t) = \Gamma(\Omega) + C\Lambda_t(\Omega)C' = \bar{\Psi} + C\Lambda_t(0)C' = \Sigma_t$, we only need to check that $\Gamma(\Omega)$ and $\Lambda_t(\Omega)$ are p.s.d. Given that $\bar{\Psi}$ is p.d., the restrictions on Ω guarantee that $\Gamma(\Omega)$ is p.s.d. in view of Lemma 2. Similarly, since $\delta_{jjt} \geq \vartheta_{jj}/[1 - \beta_j(1)] \, \forall t$, $\lambda_{jjt}(\Omega) \geq \lambda_{jjt}(0) = \delta_{jjt} - \vartheta_{jj}/[1 - \beta_j(1)] \geq 0.$ □

It is also straightforward to check that the GLS factor representing portfolios of the above models, $f_t^G(\Omega) = \{C'\Gamma^{-1}(\Omega)C\}^{-1}C'\Gamma^{-1}(\Omega)x_t$ do not depend on Ω, and coincide with $f_t\{(C'\bar{\Psi}^{-1}C)^{-1}\} = (C'\bar{\Psi}^{-1}C)^{-1}C'\bar{\Psi}^{-1}x_t$. In this respect, it is worth noting that the 'largest' Ω compatible with $\Gamma(\Omega)$ p.s.d. is precisely $(C'\bar{\Psi}^{-1}C)^{-1}$. Therefore, if we call $(C'\bar{\Psi}^{-1}C)^{jj}$ the jth diagonal element of $(C'\bar{\Psi}^{-1}C)^{-1}$, $\lambda_{jjt}(\Omega) \leq \lambda_{jjt}\{(C'\bar{\Psi}^{-1}C)^{-1}\} = \delta_{jjt} + (C'\bar{\Psi}^{-1}C)^{jj} - d_j'\bar{\Psi}d_j \leq \delta_{jjt}$ by the Cauchy–Schwartz inequality, with equality if and only if $D' = (C'\bar{\Psi}^{-1}C)^{-1}C'\bar{\Psi}^{-1}$. Except in this unlikely case, $\bar{\Psi} * D'\bar{\Psi}D$ is an indefinite matrix, and we cannot interpret $D'x_t$ as common factors. As a result, $f_t(\Omega)$ will generally follow weak, rather than strong, GARCH processes, since $D'x_t \neq f_t(\Omega)$ for all admissible Ω. In any case, we can replace the infinite distributed lags in $\Lambda_t(\Omega)$ by the recursions

$$\lambda_{jjt}(\Omega) = \{1 - \beta_j(1)\}\omega_{jj} + \alpha_j(L)(d_j'x_t)^2 + \beta_j(L)\lambda_{jjt}(\Omega).$$

Finally, note that $\text{rank}(\bar{\Psi}) = N$ is very slightly stronger than required, as it only excludes those cases in which $\text{rank}(U_2'C\Pi^{1/2}) < N - N_1$ but $\text{rank}(U_2'C) = \text{rank}[U_2'C\{\Pi + \Lambda_t(0)\}^{1/2}] = N - N_1$, where the columns of U_2 constitute a basis of the nullspace of Ψ. Nevertheless, Proposition 1 can be tediously extended on the basis of Lemma 2b in the Appendix to those cases in which Σ_t is p.d. but $\bar{\Psi}$ is not, at the cost of making the range of admissible Ω's depend on k and the rank of $\bar{\Psi}$. For instance, if $k = 1$, Ω can be any non-negative scalar such that $\Omega C'\bar{\Psi}^{-1}C \leq 1$ if $\text{rank}(\bar{\Psi}) = N$, and 0 if $\text{rank}(\bar{\Psi}) = N - 1$.

4. SUMMARY AND DISCUSSION

In this paper, we discuss the relation between Engle's (1987) factor GARCH model, and a general class of conditionally heteroskedastic factor models, which includes the latent factor ARCH model of Diebold and Nerlove (1989) as a special case. We formally introduce both models, discuss some of their properties, and provide a precise nesting of the two. Such a nesting can be fruitfully exploited to analyse the statistical features of the factor GARCH model in the more general context of model (1), (2). For example, Demos and Sentana (1998) exploit this relationship to adapt the EM algorithm to this class of models, while Sentana and Fiorentini (1997) analyse the identifiability of the factor loadings matrix C within a unified framework. Similarly, it can also be employed to extend some of the inference procedures developed for the factor GARCH model to the general class of models in (1), (2). This would be the case, for instance, of the common features test in Engle and Kozicki (1993), under our maintained assumption of constant idiosyncratic variances. Nevertheless, one should always remain aware of their differences, especially in empirical applications. For instance, an implication of our main result is that if we interpret the factor GARCH model as a traditional factor model in which the variances of the factors change over time, we would not generally be able to separately identify the contributions to the unconditional covariance matrix of the idiosyncratic terms and common factors, as we can transfer variance between them (see Sentana and Fiorentini 1997). This problem is particularly relevant in testing some of the implications of multi-beta asset-pricing

theories, such as the integration of financial markets (see King *et al*. 1994). Similarly, even if we avoid the aforementioned observational equivalence, we will not be able to differentiate the contribution to the covariance matrix of conditionally homoskedastic common factors (see Engle *et al*. 1990). For instance, if for identification purposes we go to the extreme, and choose the parameters so that the idiosyncratic covariance matrix has rank $N - k$, then model (7) can also be written as a model with N common factors and no idiosyncratic noise, in which the first k factors have time-varying variances but constant covariances, and the remaining $N - k$ factors are conditionally homoskedastic, as well as orthogonal to each other and to the first k ones. Obviously, the corresponding asset-pricing implications are rather different.

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APPENDIX

The following lemmas are useful to derive the relationship between the k factor GARCH in (7), and conditionally heteroskedastic factor models in (1), (2).

Lemma 1. *Let Z be a $N \times k$ matrix of* rank k, Ω *a* $k \times p$ *p.s.d. matrix with* rank $k_1 \leq k$, *and* Ψ *a* $N \times N$ *p.s.d. matrix with* rank $N_1 \leq N$, *where the spectral decompositions of Ω and Ψ are given by*

$$\Omega = V \Phi V' = (V_1 \quad V_2) \begin{pmatrix} \Phi_1 & \\ & 0 \end{pmatrix} \begin{pmatrix} V_1' \\ V_2' \end{pmatrix} = V_1 \Phi_1 V_1'$$

$$\Psi = U \Delta U' = (U_1 \quad U_2) \begin{pmatrix} \Delta_1 & \\ & 0 \end{pmatrix} \begin{pmatrix} U_1' \\ U_2' \end{pmatrix}$$

$$= (U_1 \Delta_1^{1/2} \quad U_2) \begin{pmatrix} I_{N_1} & \\ & 0 \end{pmatrix} \begin{pmatrix} \Delta_1^{1/2} U_1' \\ U_2' \end{pmatrix} = \tilde{U} \tilde{\Delta} \tilde{U}'$$

with $\Phi_1, \Delta_1 > 0$, rank$(\Phi_1) = k_1$ *and* rank$(\Delta_1) = N_1$, *so that* $\Psi^+ = U_1 \Delta_1^{-1} U_1'$ *is the Moore–Penrose inverse of Ψ.* (a) *If* rank$(\Psi) = N$, *then* rank$(\Psi + Z\Omega Z') =$ rank(Ψ). (b) *If* rank$(\Psi) < N$, *then* rank$(\Psi + Z\Omega Z') =$ rank$(\Psi) +$ rank$(U_2' Z V_1)$

Proof. We only prove (b), as (a) is trivial. Given that U is orthogonal, we can write

$$\Psi + Z\Omega Z' = U(\Delta + U'ZV_1\Phi_1 V_1'Z'U)U' = U\Upsilon U'$$

with

$$\Upsilon = \begin{pmatrix} \Upsilon_{11} & \Upsilon_{12} \\ \Upsilon_{12}' & \Upsilon_{22} \end{pmatrix} = \begin{pmatrix} \Delta_1 + U_1'ZV_1\Phi_1 V_1'Z'U_1 & U_1'ZV_1\Phi_1 V_1'Z'U_2 \\ U_2'ZV_1\Phi_1 V_1'Z'U_1 & U_2'ZV_1\Phi_1 V_1'Z'U_2 \end{pmatrix}$$

so that rank$(\Psi + Z\Omega Z') =$ rank(Υ). But since rank$(\Upsilon_{11}) = N_1$, Υ has the same rank as

$$\begin{pmatrix} I_{N_1} & 0 \\ -\Upsilon_{12}'\Upsilon_{11}^{-1} & I_{N-N_1} \end{pmatrix} \Upsilon \begin{pmatrix} I_{N_1} & -\Upsilon_{11}^{-1}\Upsilon_{12} \\ 0 & I_{N-N_1} \end{pmatrix} = \begin{pmatrix} \Upsilon_{11} & 0 \\ 0 & \Upsilon_{22} - \Upsilon_{12}'\Upsilon_{11}^{-1}\Upsilon_{12} \end{pmatrix}.$$

Hence, $\text{rank}(\Upsilon) = \text{rank}(\Upsilon_{11}) + \text{rank}(\Upsilon_{22} - \Upsilon'_{12}\Upsilon_{11}^{-1}\Upsilon_{12})$. Finally, repeated application of the Woodbury formula for Υ_{11} (see e.g. Householder 1964) gives after some simplification

$$\Upsilon_{22} - \Upsilon'_{12}\Upsilon_{11}^{-1}\Upsilon_{12} = U'_2 Z V_1 (\Phi_1^{-1} + V'_1 Z' \Psi^+ Z V_1)^{-1} V'_1 Z' U_2.$$

$\square$

As a simple example, consider the case of $\text{rank}(\Psi) < N$ and $k = 1$, so that $V_1 = 1$ if $\Omega \neq 0$. Then $\text{rank}(\Psi + Z\Omega Z') = \text{rank}(\Psi) + 1$ unless $U'_2 Z = 0$, in which case Z is in the column space of U_1.

Lemma 2. *Let Z, Ω and Ψ be defined as in Lemma 1. a) If $\text{rank}(\Psi) = N$, then $\Psi - Z\Omega Z'$ remains p.s.d. if and only if the eigenvalues of $\Omega(Z'\Psi^{-1}Z)$ are all less than or equal to 1. b) If $\text{rank}(\Psi) < N$, then $\Psi - Z\Omega Z'$ remains p.s.d. if and only if $U'_2 Z V_1 = 0$ and the eigenvalues of $\Omega(Z'\Psi^+ Z)$ are all less than or equal to 1.*

Proof. We only prove (b), as (a) is proved along similar lines. Since $\tilde{U}$ has full rank, we can always write

$$\Psi - Z\Omega Z' = \tilde{U}(\tilde{\Delta} - \tilde{U}^{-1} Z\Omega Z' \tilde{U}'^{-1})\tilde{U}' = \tilde{U}\tilde{\Upsilon}\tilde{U}'.$$

Therefore, the definiteness of $\Psi - Z\Omega Z'$ is the same as the definiteness of $\tilde{\Upsilon}$. In this case, $\tilde{\Upsilon}$ is

$$\begin{pmatrix} I_{N_1} - \Delta_1^{-1/2} U'_1 Z V_1 \Phi_1 V'_1 Z' U_1 \Delta_1^{-1/2} & -\Delta_1^{-1/2} U'_1 Z V_1 \Phi_1 V'_1 Z' U_2 \\ U'_2 Z V_1 \Phi_1 V'_1 Z' U_1 \Delta_1^{-1/2} & -U'_2 Z V_1 \Phi_1 V'_1 Z' U_2 \end{pmatrix}$$

which is p.s.d. if and only if $U'_2 Z V_1 = 0$ and $I_{N_1} - \Delta_1^{-1/2} U'_1 Z V_1 \Phi_1 V'_1 Z' U_1 \Delta_1^{-1/2}$ is p.s.d. But this matrix has $N_1 - k_1$ unit eigenvalues, plus another k_1 eigenvalues which are 1 minus the eigenvalues of $\Omega Z' U_1 \Delta_1^{-1/2} \Delta_1^{-1/2} U'_1 Z = \Omega Z' \Psi^+ Z$ $\square$

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