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A simplified exposition of smooth pasting

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Abstract

The decision on when to make an irreversible investment is considered as a trade-off between the instantaneous size of the net benefit and the time at which it is obtained. The benefit can be larger by waiting longer, but then it will also have to be more discounted. Smooth pasting arises as a first-order condition for maximum expected profit. The relationship to the standard approach is illustrated by a geometric Brownian price process. © 1998 Elsevier Science S.A.

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1. Introduction

Consider the basic problem when to invest a constant C to obtain (once) a revenue V that is fluctuating according to a continuous Markov process $\{V_t\}$. McDonald and Siegel (1986) looked upon this investment problem as a perpetual call option, involving a right but no obligation to invest. The optimal rule, which will be constant as long as the discount rate is constant and the process is characterized by first-order stochastic dominance in the sense that a higher current value shifts the distribution to the right, is to invest the first time V reaches a specific $V^* > C$. The markup from C to V^* reflects the value of the opportunity to wait.

The standard approach to solving irreversible investment problems of this kind consists of calculating option values, determining optimal decisions by the familiar value matching and smooth pasting conditions. Denoting the option value by $\bar{F}(C,V)$ and its derivative with respect to the second argument by $\bar{F}_V(C,V)$, the conditions here are

$$\bar{F}(C,V^*) = V^* - C, \tag{1}$$

and

$$\bar{F}_V(C,V^*) = 1. \tag{2}$$

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The value matching condition (1) reflects an intuitive requirement for continuity at the optimal exercise point V^*. Further, it is well-known that the smooth pasting condition (2) is a first-order condition for optimum, as already proposed by Samuelson (1965). Under weak conditions it can also be shown to be sufficient; see Brekke and Øksendal (1991). However, the general theory underlying such results is rigorous and hardly accessible to many economists. Even simplified results showing the necessity of smooth pasting, as in Dixit and Pindyck (Dixit and Pindyck, 1994:130–132), are based on rather technical arbitrage arguments considering what would happen if $\bar{F}(C,V)$ had a kink at V^*, not leaving much room for basic intuition.

The new approach to irreversible investment proposed by Dixit et al. (1997) can be used to simplify the treatment of smooth pasting. They regard the investment decision as a trade-off between the size of the net benefit $V-C$ and the effect of discounting. If assuming that the current value of the process is some low V_0, and that investment takes place when it reaches an arbitrary $V>V_0$, the expected and discounted profit becomes:

$$E[e^{-\rho T}](V-C). \tag{3}$$

Here E is the expectations operator, ρ is the discount rate, and T is the first hitting time from V_0 to V. Since the process is continuous, the expected discount factor will be strictly increasing in V_0 and decreasing in V, so it can be described equivalently as a function $D=D(V_0,V)$. Thus the expected profit from a decision to invest when the price has increased to V for the first time, can be stated as

$$D(V_0,V)(V-C), \tag{4}$$

which is to be maximized with respect to V. The first order condition for optimum becomes

$$D_2(V_0,V^*)\cdot V^* + D(V_0,V^*) = D_2(V_0,V^*)\cdot C, \tag{5}$$

where D_2 is the derivative of D with respect to the second argument. Alternatively, we have

$$\frac{V^*-C}{V^*} = \frac{1}{\epsilon_D}, \tag{6}$$

where ϵ_D is the elasticity of the discount factor with respect to the investment threshold:

$$\epsilon_D \equiv -\frac{V^*\cdot D_2(V_0,V^*)}{D(V_0,V^*)}. \tag{7}$$

Eq. (6) is analogous to the markup pricing rule in a static model with a downward sloping demand curve D depending on a price V, regarding V_0 as a constant. The elasticity of the discount factor is analogous to a price elasticity of demand, V^* to an optimal price, and C to a constant marginal cost. It can be shown that ϵ_D does not depend on V_0. This ensures that the optimal investment rule is not affected by changes in V_0.

As D is strictly decreasing in V, the inverse function $V=V(V_0,D)$ could alternatively be used to maximize (4) with respect to D. That would yield the following revised version of Eq. (5) for an arbitrary $V>V_0$:

$$V + D(V_0,V)/D_2(V_0,V) = C. \tag{8}$$

The left-hand side of Eq. (8) is analogous to a static marginal revenue function. Optimum V is found by setting the marginal revenue equal to the investment cost.

2. Smooth pasting

Let us reconsider the investment problem above, denoting the net benefit from investing at a general $V>V_0$ by:

$$F=V-C. \tag{9}$$

Since C is constant, the process for the benefit $\{F_t\}$ will share basic properties with $\{V_t\}$; and the optimal rule will be to exercise the option when a specific $F^*>0$ is reached for the first time. The expected profit can be expressed as a function

$$\Phi(F_0,F)=Q(F_0,F)\cdot F, \tag{10}$$

where Q is the expected discount factor similar to D of the previous section. Thus Φ is the expected and discounted profit from exercising the option when the benefit has increased to $F>F_0$, instead of doing it right away and obtaining F_0. As the benefit F obtained by waiting arises in the future, it must be discounted by the appropriate factor $Q(F_0,F)$. The first order condition for maximum can be stated as

$$\epsilon_Q=1, \tag{11}$$

where

$$\epsilon_Q\equiv-\frac{Q_2(F_0,F^*)\cdot F^*}{Q(F_0,F^*)}. \tag{12}$$

Eq. (11) establishes the smooth pasting condition for this problem. It simply says that in optimum, the marginal cost of discounting equals the marginal net benefit from further waiting. As ϵ_Q is independent of F_0, optimum arises at a unique F^*. Further, the initial slope of Φ when evaluated as a function of F becomes:

$$\Phi_2(F_0,F_0)=Q_2(F_0,F_0)\cdot F_0+1. \tag{13}$$

Since $Q_2<0$, $\Phi_2(F_0,F_0)$ is larger or smaller than one, depending on whether F_0 is smaller or larger than zero. The reason is that if $F_0<0$, the marginal effect of waiting will be to discount a loss, while if $F_0>0$ it will be to discount a benefit. It is also observed that if $F_0\leq 0$, the curve passes through origo. Using this information, Fig. 1 plots Φ as a function of F for four initial values of the benefit, $F_0(\text{i})\ldots F_0(\text{iv})$.

All curves start from the 45 degree line as $\Phi(F_0,F_0)=F_0$, and they all obtain maximum at the unique $F^*>0$ at which it is optimal to exercise the option. However, the maximum value is increasing in F_0, because it takes longer to reach F^* the smaller the initial value.

Curve (i) starts from a negative benefit $F_0(\text{i})$. Initially, the slope is larger than unity as the marginal effect of discounting is positive. However, the slope decreases to zero as F approaches F^*. Curve (ii)

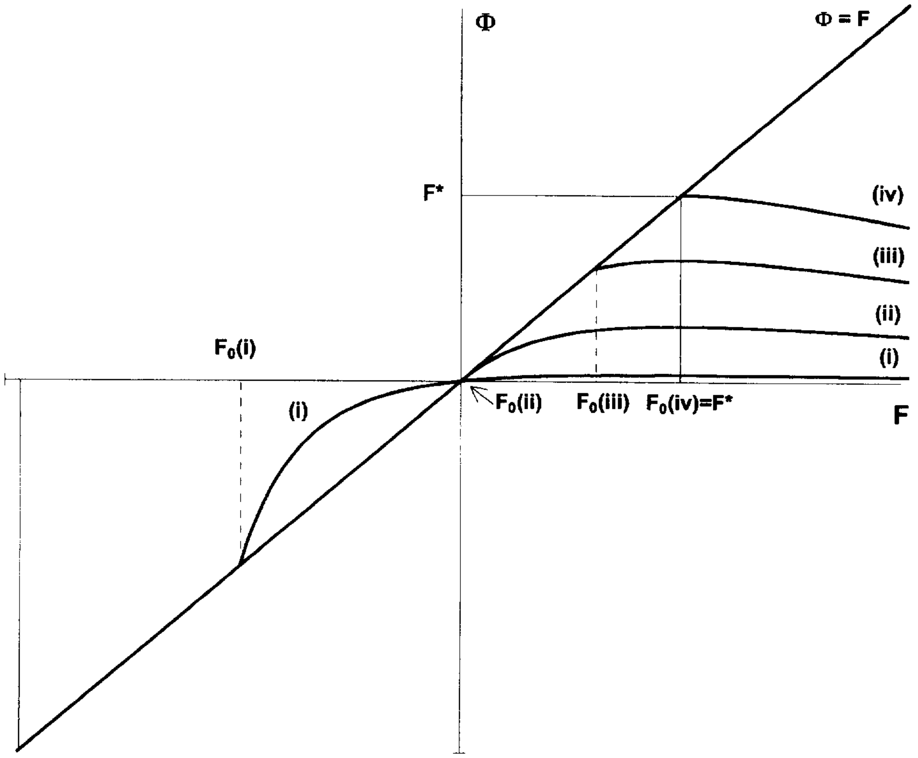

Fig. 1. Expected profit functions.

takes off from origo, as the initial benefit is zero when $F_0(\text{ii})=0$. Since the marginal effect of discounting is also zero, the curve is tangential to $\Phi=F$ at the initial point. Curve (iii) assumes a positive starting value smaller than F^*. The initial slope is positive as there is a value from waiting, but less than unity due to discounting. Curve (iv) assumes an initial value $F_0(\text{iv})$ that is equal to F^*. That is right where the marginal value of waiting is zero, so the initial slope is zero. (The leftmost vertical line in Fig. 1 is explained below.)

The standard forms of value matching and smooth pasting follow by assuming that the option is optimally exercised. Thus for a general initial $F<F^*$ with a corresponding $V<V^*$, the option value function is given by:

$$\bar{F}(C,V)\equiv\Phi(F,F^*). \tag{14}$$

By letting $F=F^*$, this gives the standard value matching condition (1) directly, as $Q(F^*,F^*)$. Using the fact that $Q_1(F^*,F^*)=-Q_2(F^*,F^*)$, the standard smooth pasting condition (2) is also easily obtained from Eq. (11).

3. Relationship between the two approaches

The approaches to irreversible investment which have been discussed are related by two elasticities that coincide in optimum. To see this, define the elasticity of the option value as

$$\epsilon_F \equiv \frac{V^* \cdot \bar{F}_V(C,V^*)}{\bar{F}(C,V^*)}. \tag{15}$$

By combining Eqs. (1), (2), (6), we have:

$$\epsilon_D = \epsilon_F. \tag{16}$$

The relationship is illustrated in Fig. 2. On the left-hand side, $\bar{F}(C,V)$ is plotted as an increasing function of V. At the optimal V^*, the option value function hits the "profit line" $V-C$ tangentially, according to the value matching and smooth pasting conditions. To the far right, $D(V_0,V)$ is plotted as a downward sloping demand function. The steep curve closer to origo represents a part of the marginal revenue function given by the left-hand side of Eq. (8). The optimal discount factor D^* is found where marginal revenue equals the investment cost, with a corresponding threshold V^*. Observe that D^* is smaller than D^N, that would apply by a simple net present value rule.

The profit line $V-C$ connects the two approaches. If the investment cost is increased, the line shifts vertically upwards. On the right-hand side, V^* increases via the marginal revenue function. On the

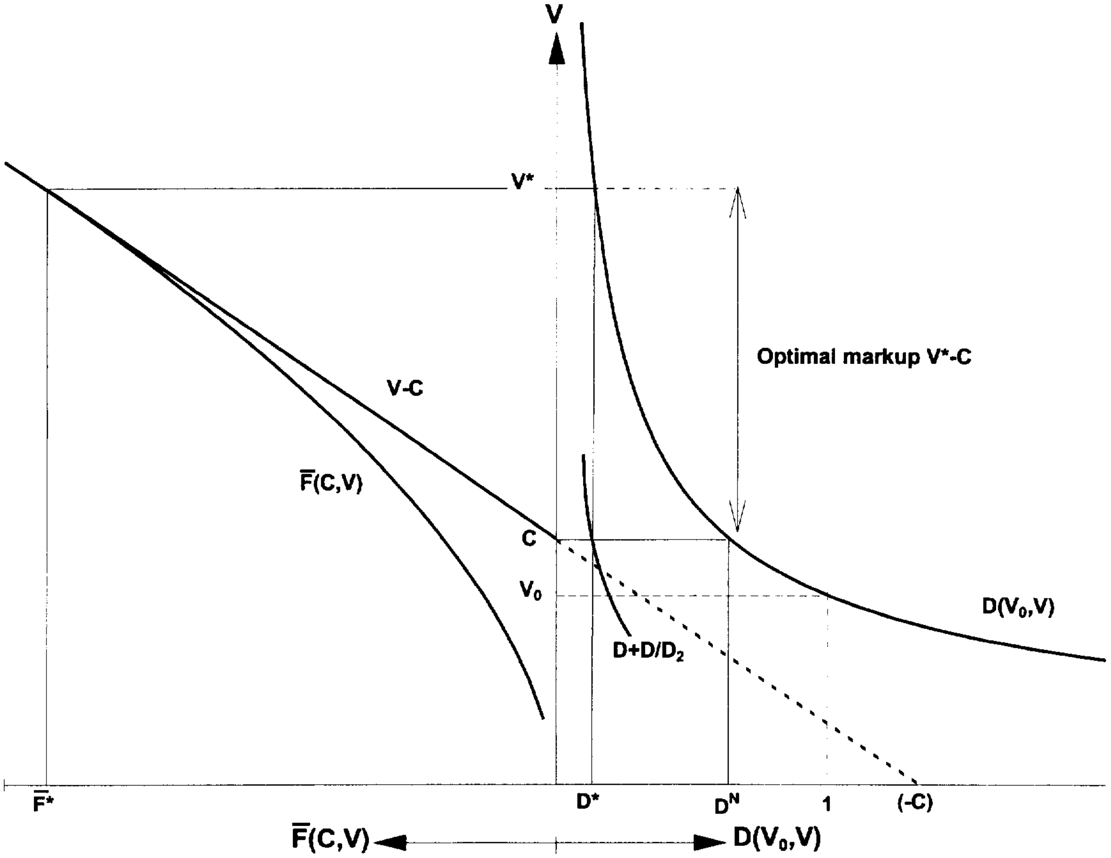

Fig. 2. Optimal investment rules.

left-hand side, the entire option value function shifts down (i.e. closer to the vertical axis), hitting the new profit line tangentially for a higher V^*.

4. Example – a geometric Brownian price process

Assume a constant investment cost C and a geometric Brownian price

$$dV = \mu V dt + \sigma V dz, \tag{17}$$

where μ is the trend and σ is the volatility. By Ito's lemma, the process for the benefit is given by

$$dF = \mu(F + C)dt + \sigma(F + C)dz. \tag{18}$$

Following the approach described by Dixit et al. (1997) to finding expected discount factors, we have

$$Q(F_0,F) = \left(\frac{F_0 + C}{F + C}\right)^{\beta}, \tag{19}$$

where β is the positive root of the following quadratic equation in x:

$$\tfrac{1}{2}\sigma^2 x(x - 1) + \mu x - \rho = 0. \tag{20}$$

The elasticity of the discount factor becomes

$$\epsilon_Q = \beta \frac{F}{F + C}, \tag{21}$$

and setting this equal to unity according to Eq. (11), we have:

$$F^* = \frac{C}{\beta - 1}. \tag{22}$$

By inserting $F^* = V^* - C$, we obtain the familiar expression of the optimal investment rule for this problem:

$$V^* = \frac{\beta}{\beta - 1} C. \tag{23}$$

For convergence, we need $\beta > 1$, implying $\mu < \rho$. The option value follows by inserting optimal values into Eq. (14), yielding:

$$\bar{F}(C,V) = \frac{(\beta - 1)^{\beta - 1}}{\beta^{\beta} C^{\beta - 1}} V^{\beta}. \tag{24}$$

Thus the option value is an upward sloping convex function in V starting from origo. As the identity $Q(F_0,F) \equiv D(V_0,V)$ must hold, we also have:

$$D(V_0,V) = \left(\frac{V_0}{V}\right)^{\beta}. \tag{25}$$

Hence the geometric Brownian price process corresponds to an isoelastic demand function with elasticity β. Eq. (16) is also easily verified. In fact, Figs. 1 and 2 correspond to a set of numbers for this process, although their general shape applies in a wider context as well. In Fig. 1, the leftmost vertical line corresponds to $V_0=0$, implying $F_0=-C$. If F_0(iv) were moved closer to $-C$, the initial slope would approach infinity, since $dV=0$ at $V=0$. For the same reason, Φ would go to zero at $F=F^*$ in that case.

5. Final remarks

The smooth pasting condition has been derived by optimization, considering an irreversible investment as a trade-off between the size of the net benefit by investing now, and the effect of discounting by waiting further. Smooth pasting turned out as the first-order condition that must hold to ensure maximum expected and discounted profit. Finally, it should be noted that the interpretation of the expected discount factor as a dynamic measure of quantity can be generalized beyond the level of a demand function. For example, it is straightforward to apply a similar approach to the related investment problem in which V is constant and C is fluctuating. If the cost process has the same properties as those assumed for V in the previous sections, then the discount factor will be analogous to a supply function.

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