Exoplanets Working Group

Models for Radial Velocities

Summarized from Eric Ford's paper "Quantifying the Uncertainty in the Orbits of Extrasolar Planets" (AJ 2005, 1706-1717), notes from Bill Jefferys, and from Danby's bible "Fundamentals of Celestial Mechanics".

Data: t_i, V_i, σ_i, for $i = 1 \ldots n$, where

$t_i =$ time of i^{th} observation, usually in days since an arbitrary date ("Julian days - XXX") or in Julian days (which are also days since and arbitrary date, namely noon on 1 January 4712 BCE).

$V_i =$ radial velocity of i^{th} observation, corrected so that they are line-of-sight velocities relative to the solar system center of mass; often with a large constant subtracted out so that velocities are actually deviations from a "template observation"; usually in ms^{-1}, sometimes km s^{-1}

$\sigma_i =$ i^{th} measurement error. These are known constants, usually measured in ms^{-1}, sometimes km s^{-1}

Model for a single planet on a Keplerian orbit:

$V_i \sim \mathsf{N}\left(C + \Delta V(t_i|\theta), \sigma_i^2 + s^2\right)$

Primary parameters:

$C, \theta = (K, P, e, \omega, M_0)$, and s, where

$C =$ a constant velocity, usually measured in ms^{-1}, sometimes km s^{-1}. C can be positive or negative. *Note:* If data on the same star are measured relative to different template observations (*e.g.*, at different observatories), then each one requires its own unique C_j, where j is an observatory index.

$K =$ velocity semi-amplitude, usually measured in ms^{-1}. K is, by convention, non-negative. (Allowing $K < 0$ would create an identifiability problem if M_0 and ω remained unrestricted.)

$P =$ period, usually measured in days.

$e =$ eccentricity. Unitless, $0 \leq e < 1$. $e = 0$ corresponds to a circular orbit. Larger values of e correspond to more eccentric orbits. For comparison, Earth's eccentricity is about 0.017. In our Solar system, Mercury ($e = 0.206$) and Pluto ($e = 0.248$) have the largest eccentricities.

$\omega =$ argument of periastron. (Periastron is the point at which the planet is closest to the star.) $0 \leq \omega < 2\pi$. Essentially, ω measures the angle at which we happen to be observing the elliptical orbit, and it is unintersesting from a physical point of view. ω is often measured in degrees rather than radians.

$M_0 =$ mean anomaly at time $t = 0$. $0 \leq M_0 < 2\pi$. (Mean anomaly is the angular distance of a planet from periastron—this will be clarified in a forthcoming version of this document.) Note that for near-circular orbits, both M_0 and ω are difficult to pin down, and they become unidentifiable at $e = 0$, a perfectly circular orbit, although M_0 then becomes a simple phase shift in the circular orbit model (see equation (5) on page 2).

$s^2 =$ variance of additional noise, or "stellar jitter"; s is usually measured in ms^{-1}.

Radial velocity function ΔV:

$$\Delta V(t|\theta) = K\left[\cos(\omega + T(t)) + e\cos\omega\right], \text{ where} \tag{1}$$

$T(t)$ is the *true anomaly* at time t, and is related to $E(t)$—the *eccentric anomaly* at time t—in the following way:

$$T(t) = 2\arctan\left[\tan\left(\frac{E(t)}{2}\right)\sqrt{\frac{1+e}{1-e}}\right] \tag{2}$$

The eccentric anomaly $E(t)$ is related to $M(t)$—the *mean anomaly* at time t—in the following way:

$$E(t) - e\sin(E(t)) = M(t) \tag{3}$$

And finally, the mean anomaly $M(t)$ is linearly related to t and is determined directly by the equation:

$$M(t) = \frac{2\pi}{P}t + M_0 \tag{4}$$

So at a given time t, from the parameters P and M_0 we obtain the mean anomaly $M(t)$; from $M(t)$ and e we obtain the eccentric anomaly $E(t)$; from $E(t)$ (and e again) we obtain the true anomaly $T(t)$, and from $T(t)$ and the other parameters we obtain the predicted radial velocity $V(t) = C + \Delta V(t|\theta)$.

Note that the equation (3) is transcendental, so a closed expression for $\Delta V(t|\theta)$ is not possible. There are a variety of approaches one may use to determine $E(t)$ from e and $M(t)$. One approach is to set $E(t) - e\sin E - M(t)$ equal to 0 and solve using Newton's method. Doing so with a quadratic term usually leads to a solution with at least ten decimal place accuracy in 3 or 4 iterations.

A note about symbols: The symbols for the parameters above vary among texts and published papers. Other symbols often used are given in the first of the two tables below. The second table shows other parameterizations sometimes used instead of the one used in this document.

parameter	**symbol in this document**	**other symbols**
constant velocity offset	C	V
period	P	T
true anomaly	T	υ, f
eccentric anomaly	E	u
mean anomaly	M	ℓ

parameter	**symbol**	**equivalence**
fraction of orbit between $t = 0$ and periastron	χ (Gregory)	$= \frac{1}{2\pi}M_0$
time of periastron crossing	T_0, t_0, T	$= \frac{M_0}{2\pi}P$

Model for a single planet on a circular orbit:

If $e = 0$ exactly, then equation (3) is no longer transcendental, ω becomes meaningless, and we have $T(t) = E(t) = M(t)$. In that case:

$$\Delta V(t|\theta) = K \cos\left(\frac{2\pi}{P}t + M_0\right), \text{ when } e = 0 \tag{5}$$

Null model (no planet):

If there is no planet orbiting a star, then its radial velocity function would be a contant:

$$\Delta V(t|\theta) = 0 \tag{6}$$

There would still be a constant velocity parameter and a noise parameter to estimate, so the probability model would be:

$V_i \sim \mathsf{N}\left(C, \sigma_i^2 + s^2\right)$

Model for multiple planets:

If there are multiple planets orbiting a single star and their masses are all quite small relative to the mass of the star, then the model for the star's radial velocity may be approximated as the sum of the contributing planetary components:

$V_i \sim \mathsf{N}\left(C + \Delta V(t_i|\theta_1) + \Delta V(t_i|\theta_2) + \ldots + \Delta V(t_i|\theta_{N_p}), \sigma_i^2 + s^2\right)$, where N_p is the number of planets in the model.

Priors:

A second document called "prior_statement.pdf" has been posted on the SAMSI Exoplanets Working Group web page that describes reasonable priors for the parameters in the model.

It may be reasonable to take the parameters to be independent a priori. (This is done in the document "prior_statement.pdf".) At least, it is computationally simpler than correlating them. In reality, they are probably correlated, but not enough is known about exoplanets yet to specify how, and it may not make a big difference in the posteriors anyway.

Suggested reparameterization for MCMC:

When using MCMC (in particular, Metropolis with independent parameter proposal transitions) to do parameter estimation, correlations in the parameters can be troublesome. The following reparameterizations can help reduce correlations between the parameters substantially, particularly for systems with small eccentricities:

- Translate the data t_i (essentially reparameterizing M_0) so that $t = 0$ occurs at the first observation or, better still, in the middle of the observations—perhaps at the weighted mean of the observation times, with the weights being inversely proportional to the measurement errors. This reduces correlations between M_0 and P.
- Use Poincaré variables $e \cos \omega$ and $e \sin \omega$ instead of e and ω to reduce (very strong) correlations between M_0 and ω. This is particularly important for low eccentricity orbits.
- Use $\omega + M_0$ instead of M_0 to further reduce correlations between these two parameters when $e \ll 1$; for larger e, it still doesn't hurt.

Eric Ford suggests other reparameterizations in a recent paper "Improving the Efficiency of Markov Chain Monte Carlo for Analyzing the Orbits of Extrasolar Planets", from *Astrophysics, 2005,* available on the web at arxiv.org/abs/astro-ph/0512634.