# *A Consistent Instrumental Variable Estimator for Errors in Covariates in Limited Dependent Variable Models*

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# Outline of Presentation

- Overview & other approaches
- Problem of instrumental variable (IV) approach
- A Consistent IV Estimator
- Asymptotic covariance matrix
- Simulation
- Conclusions

# Overview and approaches

- Ordinal or dichotomous dependent variable
- If covariate available, straightforward probit

$$y^* = \alpha_1 + \alpha_2 \xi_1 + \epsilon \qquad \epsilon \sim N(0,1)$$

$$y = I(y^* \geq 0)$$

# Overview and approaches

- Only proxy covariate ( $x_1$ ) available

$$x_1 = \xi_1 + \delta_1$$

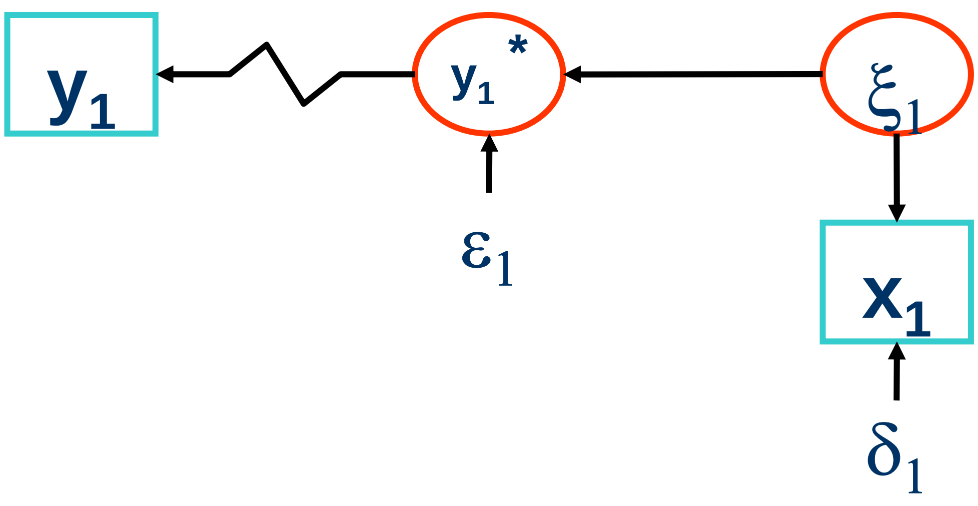
$$\xi_1 = x_1 - \delta_1$$

$$y^* = \alpha_1 + \alpha_2 \xi_1 + \epsilon$$

$$y^* = \alpha_1 + \alpha_2 x_1 - \alpha_2 \delta_1 + \epsilon$$

# Overview and approaches

- Path diagram of equations



# Overview and approaches

- Problems with usual probit regression

$$y^* = \alpha_1 + \alpha_2 x_1 - \alpha_2 \delta_1 + \epsilon$$

$$V(-\alpha_2\delta_1 + \varepsilon) = \alpha_2^2 V(\delta_1) + V(\varepsilon)$$ Not equal to 1

$$C(\delta_1, x_1) \neq 0$$

*plim* $(\widehat{\alpha}_2) \neq \alpha_2$

# Overview and approaches

- Approaches
  - Simulation extrapolation (e.g., Carroll et al., 1996)
  - Regression calibration (e.g., Carroll and Stefanski, 1990; Gleser, 1990)
  - Known reliability of covariate
  - Instrumental variable estimator
  - Above approximately consistent

# Problem of Instrumental Variable (IV) approach

- Suppose we have IV $x_2$ correlated with $\xi_1$ independent of $\delta_1$ and $\epsilon$
- Form $\hat{x}_1 = \beta_1 + \beta_2 x_2$

  and $y^* = \alpha_1 + \alpha_2 \hat{x}_1 + u$ $u = \alpha_2(x_1 - \hat{x}_1) - \alpha_2 \delta_1 + \epsilon$

  - $\hat{x}_1$ and $u$ uncorrelated (solves one problem)

  BUT $V(u) = V(\epsilon) + \alpha_2^2 \left[ \frac{V(\xi_1)V(x_2) - C^2(\xi_1, x_2)}{V(x_2)} \right]$

Not equal to 1, so inconsistent estimator

# A Consistent IV Estimator

- Generalize to multiple covariates, proxies, IVs:

$$y^* = \alpha_1 + \boldsymbol{\alpha}_2'\boldsymbol{\xi} + \varepsilon,$$

$$\mathbf{x} = \boldsymbol{\xi} + \boldsymbol{\delta},$$

$$\mathbf{z} \in R^q$$ IVs

- Usual IV estimator to predict covariates

$$\beta_2 = \Sigma_{\mathbf{zz}}^{-1}\Sigma_{\mathbf{zx}} \qquad \beta_1 = \mu_x - \beta_2'\mu_{\mathbf{z}}$$

# A Consistent IV Estimator

Define

$$\mathbf{u} = \boldsymbol{\xi} - \boldsymbol{\beta}_1 - \boldsymbol{\beta}_2'\mathbf{z}$$

Implies

$$\xi = \beta_1 + \beta_2'\mathbf{z} + \mathbf{u}$$

Substitute into

$$y^* = \alpha_1 + \boldsymbol{\alpha}_2'\boldsymbol{\xi} + \varepsilon,$$

Leading to

$$y^* = \gamma_1 + \boldsymbol{\gamma}_2'\mathbf{z} + v,$$

# A Consistent IV Estimator

$$y^* = \gamma_1 + \boldsymbol{\gamma}_2'\mathbf{z} + v,$$

where $\gamma_1 = \alpha_1 + \boldsymbol{\alpha}_2\boldsymbol{\beta}_1$ $\gamma_2 = \beta_2\boldsymbol{\alpha}_2$ $v = \varepsilon + \boldsymbol{\alpha}_2'\mathbf{u}$

Apply probit to equation at top: $\tilde{\gamma}_1 = \gamma_1/\sigma_v$ $\tilde{\boldsymbol{\gamma}}_2 = \boldsymbol{\gamma}_2/\sigma_v$

$\boldsymbol{\alpha}_2 = \mathbf{M}\boldsymbol{\gamma}_2 = \sigma_v\mathbf{M}\tilde{\boldsymbol{\gamma}}_2$ where $\mathbf{M} = \left(\boldsymbol{\Sigma}_{\mathbf{xz}}\boldsymbol{\Sigma}_{\mathbf{zx}}\right)^{-1}\boldsymbol{\Sigma}_{\mathbf{xz}}\boldsymbol{\Sigma}_{\mathbf{zz}}$

# A Consistent IV Estimator

- We show that

$$\sigma_v^2 = \left(1 + \tilde{\boldsymbol{\gamma}}_2' \boldsymbol{\Sigma}_{zz} \tilde{\boldsymbol{\gamma}}_2 - \left[\sqrt{1 + \tilde{\boldsymbol{\gamma}}_2' \boldsymbol{\Sigma}_{zz} \tilde{\boldsymbol{\gamma}}_2}\,\right] \tilde{\boldsymbol{\gamma}}_2' \mathbf{M}' \boldsymbol{\Sigma}_{xy} / \varphi \right)^{-1}.$$

where $\varphi = \phi\left[\Phi^{-1}(\mu_y)\right]$

$$\alpha_1 = \sigma_v \tilde{\gamma}_1 - \sigma_v \boldsymbol{\beta}_1' \mathbf{M} \tilde{\boldsymbol{\gamma}}_2. \qquad \boldsymbol{\alpha}_2 = \sigma_v \mathbf{M} \tilde{\boldsymbol{\gamma}}_2$$

# Asymptotic covariance matrix

- Use delta method: $$\sqrt{n}(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}) \xrightarrow{L} N(0, \sigma_v^2 \mathbf{ABA}')$$

$\mathbf{B}$ usual asymp. cov. matrix of MLE $\tilde{\boldsymbol{\gamma}}$

$$\mathbf{A} = \begin{pmatrix} 1 & -\boldsymbol{\beta}_1'\mathbf{M} + (\tilde{\gamma}_1 - \boldsymbol{\beta}_1'\mathbf{M}\tilde{\boldsymbol{\gamma}}_2)\mathbf{D}/\sigma_v \\ 0 & \mathbf{M} + \mathbf{M}\tilde{\boldsymbol{\gamma}}_2\mathbf{D}/\sigma_v \end{pmatrix}.$$

$$\mathbf{D} = -\frac{1}{2\sigma_v^3}\left(2\tilde{\gamma}_2'\Sigma_{zz} - \left[\sqrt{1+\tilde{\gamma}_2'\Sigma_{zz}\tilde{\gamma}_2}\right]\frac{\Sigma_{xy}'\mathbf{M}}{\varphi} - \frac{\Sigma_{xy}'\mathbf{M}\tilde{\gamma}_2\tilde{\gamma}_2'\Sigma_{zz}}{\varphi\sqrt{1+\tilde{\gamma}_2'\Sigma_{zz}\tilde{\gamma}_2}}\right).$$

# SIMULATION

## Model

$$y^* = \alpha + \beta_1\xi_1 + \beta_2\xi_2 + \beta_3\xi_3 + \zeta$$

$$\xi_i \sim N(0, 1), \quad i = 1, 2, 3, \qquad \zeta \sim N(0, 1)$$

where $\alpha = 0, \beta_1 = \beta_2 = \beta_3 = 1/sqrt(3)$,

and $\zeta$ and $\xi$ 's are independent.

Hence model CD = 0.5

# SIMULATION

Fallible measures of the ξ's

$$x_i = \xi_i + \delta_i , \qquad i = 1, 2, 3.$$

Instrumental variables

$$z_i = \xi_1 + \varepsilon_i , \qquad i = 1, 2$$

$$z_i = \xi_2 + \varepsilon_i , \qquad i = 3, 4$$

$$z_i = \xi_3 + \varepsilon_i , \qquad i = 4, 5$$

$\varepsilon$'s and $\delta$'s mutually independent, $N(0, 1)$, and independent of $\xi$'s and $\zeta$.

# SIMULATION

Bias

$$\overline{B}_i = \overline{\beta}_i - \sqrt{3}/3 \quad \text{where} \quad \overline{\beta}_i = \sum_{j=1}^{1000} \hat{\beta}_{ij} / 1000$$

Mean Bias

$$\overline{B} = \sum_{i=1}^{3} \overline{B}_i / 3$$

MC Standard Error

$$\sigma_{MC}(\overline{B}) = [\sqrt{(1' \hat{\Sigma}_\beta 1)/9}] / \sqrt{1000}$$

# SIMULATION

**3 latent predictors; model CD = 0.5; 3 indicators per latent predictor; CD = 0.5 for each indicator; 2 IV's per latent predictor.**

| Sample Size (N) | $\overline{B}$ | $\sigma_{MC}(\overline{B})$ | Bias Detected? $\lvert\overline{B}\rvert > 2\sigma_{MC}(\overline{B})$ |
|---|---|---|---|
| 100 | 5.3% | 1.5% | Yes |
| 200 | 3.2% | 0.8% | Yes |
| 500 | 1.6% | 0.5% | Yes |
| 1000 | 0.3% | 0.3% | No |

**Mean bias for intercept undetectable even for N = 100**

# Conclusions

- Consistent IV estimator possible
  - No need to settle for "approximately consistent"
- Asymptotic covariance matrix presented
- Simulation illustration shows maximum finite sample bias at small N (=100 , 5%) and minimum at large N (=1000, 0.3%)
- Simulations under broader range of conditions & comparison to other estimators needed